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Towards the geometry of double Hurwitz numbers☆

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Dedicated to Professor Michael Artin on the occasion of his seventieth birthday

Abstract

Double Hurwitz numbers count branched covers of $\mathbb{CP}^1$ with fixed branch points, with simple branching required over all but two points 0 and ∞, and the branching over 0 and ∞ specified by partitions of the degree (with m and n parts, respectively). Single Hurwitz numbers (or more usually, Hurwitz numbers) have a rich structure, explored by many authors in fields as diverse as algebraic geometry, symplectic geometry, combinatorics, representation theory, and mathematical physics. The remarkable ELSV formula relates single Hurwitz numbers to intersection theory on the moduli space of curves. This connection has led to many consequences, including Okounkov and Pandharipande's proof of Witten's conjecture.

In this paper, we determine the structure of double Hurwitz numbers using techniques from geometry, algebra, and representation theory. Our motivation is geometric: we give evidence that double Hurwitz numbers are top intersections on a moduli space of curves with a line bundle (a universal Picard variety). In particular, we prove a piecewise-polynomiality result analogous to that implied by the ELSV formula. In the case $m = 1$ (complete branching over one point) and n is arbitrary, we conjecture an ELSV-type formula, and show it to be true in

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genus 0 and 1. The corresponding Witten-type correlation function has a richer structure than that for single Hurwitz numbers, and we show that it satisfies many geometric properties, such as the string and dilaton equations, and an Itzykson–Zuber-style genus expansion ansatz. We give a symmetric function description of the double Hurwitz generating series, which leads to explicit formulae for double Hurwitz numbers with given m and n, as a function of genus. In the case where m is fixed but not necessarily 1, we prove a topological recursion on the corresponding generating series, which leads to closed-form expressions for double Hurwitz numbers and an analogue of the Goulden–Jackson polynomiality conjecture (an early conjectural variant of the ELSV formula). In a later paper (Faber's intersection number conjecture and genus 0 double Hurwitz numbers, 2005, in preparation), the formulae in genus 0 will be shown to be equivalent to the formulae for "top intersections" on the moduli space of smooth curves $\mathscr{M}_g$. For example, three formulae we give there will imply Faber's intersection number conjecture (in: Moduli of Curves and Abelian Varieties, Aspects of Mathematics, vol. E33, Vieweg, Braunschweig, 1999, pp. 109–129) in arbitrary genus with up to three points.

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1. Introduction

If $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are partitions of a positive integer d, the *double Hurwitz number* $H^g_{\alpha,\beta}$ is the number of genus g branched covers of $\mathbb{CP}^1$ with branching corresponding to α and β over 0 and ∞, respectively, and an appropriate number $r = 2g - 2 + m + n$ of other fixed simple branched points (determined by the Riemann–Hurwitz formula). For simple branching, the monodromy of the sheets is a transposition. To simplify the exposition, we assume that the points mapping to 0 and ∞ are *labelled*. Thus the double Hurwitz numbers under this convention are $|\mathrm{Aut}\,\alpha|\,|\mathrm{Aut}\,\beta|$ larger than they would be under the convention in [24].

Let $\mathcal{H}^g_{\alpha,\beta}$ be the Hurwitz scheme parameterizing genus g branched covers of $\mathbb{P}^1$ by smooth curves, with branching over 0 and ∞ given by α and β. Then $H^g_{\alpha,\beta}$ is the degree of the branch morphism to $\mathrm{Sym}^r(\mathbb{P}^1)$ (sending a cover to its branch divisor away from 0 and ∞). Double Hurwitz numbers are naturally top intersections on any compactification of the Hurwitz scheme extending the branch morphism.

Let the universal Picard variety $\mathrm{Pic}_{g,m+n}$ be the moduli space of smooth genus g curves with $m + n$ distinct labelled smooth points $p_1, \ldots, p_m, q_1, \ldots, q_n$, together with a degree 0 line bundle; thus, the points of $\mathrm{Pic}_{g,m+n}$ correspond to ordered triples (smooth genus g curve C, $m + n$ distinct labelled points on C, degree 0 line bundle on C). Then the projection $\mathrm{Pic}_{g,m+n} \to \mathcal{M}_{g,m+n}$ (with fiber the Picard variety of the appropriate curve) has section

$$\sum_{i=1}^{m} \alpha_i p_i - \sum_{j=1}^{n} \beta_j q_j,$$

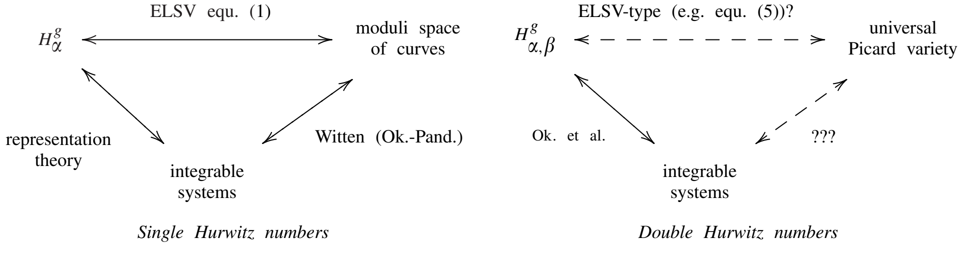
and $\mathcal{H}^g_{\alpha,\beta}$ is a $\mathbb{C}^*$-bundle over the intersection of this section with the 0-section. (The determination of the class of the closure of this intersection in the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,m+n}$ is known as Eliashberg's problem, because of its appearance in symplectic field theory [9,10].) Thus one may speculate that double Hurwitz numbers are naturally top intersections on an appropriate compactification of the universal Picard variety.

Our (long-term) goal is to understand the structure of double Hurwitz numbers, and in particular to determine the possible form of an ELSV-type formula expressing double Hurwitz numbers in terms of intersection theory on some compactified universal Picard variety, presumably related to the one defined by Caporaso [3]. (M. Shapiro has made significant progress in determining what this space might be [55].) An ELSV-type formula would translate all of the structure found here (and earlier, e.g. relations to integrable systems) to the intersection theory of this universal Picard variety.

A second goal is to use the structure of double Hurwitz numbers in genus 0 to understand top intersections on the moduli space of smooth curves, and in particular prove Faber's intersection number conjecture [25].

1.1. Motivation from single Hurwitz numbers: polynomiality, and the ELSV formula

Our methods are extensions of the combinatorial and character-theoretic methods that we have used in the well-developed theory of *single Hurwitz numbers* $H^{\mathbf{g}}_{\alpha}$, where all but possibly one branch point have simple branching. (They are usually called "Hurwitz numbers," but we add the term "single" to distinguish them from double Hurwitz numbers.) Single Hurwitz numbers have surprising connections to geometry, including the moduli space of curves. (For a remarkable recent link to the Hilbert scheme of points on a surface, see for example [42, p. 2]; [59].) Our intent is to draw similar connections in the case of the double Hurwitz numbers. We wish to use the representation-theoretic and combinatorial structure of double Hurwitz numbers to understand the intersection theory of a conjectural universal Picard variety, in analogy with the connection between single Hurwitz numbers and the moduli space of curves, as shown in the following diagram.



Understanding this would give, for example, Toda constraints on the topology of the universal Picard variety.

The history of single Hurwitz numbers is too long to elaborate here (and our bibliography omits many foundational articles), but we wish to draw the reader's attention to ideas leading, in particular, to the ELSV formula ([7,8], see also [26]):

$$\boxed{H^g_\alpha = C(g,\alpha)\int_{\overline{\mathcal{M}}_{g,m}} \frac{1-\lambda_1+\lambda_2-\cdots\pm\lambda_g}{(1-\alpha_1\psi_1)\cdots(1-\alpha_m\psi_m)}}, \tag{1}$$

where

$$C(g,\alpha) = r!\prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \tag{2}$$

is a scaling factor. Here λ_k is a certain codimension k class, and ψ_i is a certain codimension 1 class. (Also $r = 2g-2+d+m$ is the expected number of branch points, as described earlier.) We refer the reader to the original papers for precise definitions, which we will not need. (The original ELSV formula includes a factor of $|\mathrm{Aut}\,\alpha|$ in the denominator, but as stated earlier, we are considering the points over ∞, or equivalently the parts of α, to be labelled.) The right-hand side should be interpreted by expanding the integrand formally, and capping the terms of degree $\dim \overline{\mathcal{M}}_{g,m} = 3g-3+m$ with the fundamental class $[\overline{\mathcal{M}}_{g,m}]$.

The ELSV formula (1) implies that

$$H^g_\alpha = C(g,\alpha)P^g_m(\alpha_1,\ldots,\alpha_m), \tag{3}$$

where P^g_m is a *polynomial whose terms have total degrees between* $2g-3+m$ *and* $3g-3+m = \dim\overline{\mathcal{M}}_{g,m}$. The coefficients of this polynomial are all top intersections on the moduli space of curves involving ψ-classes and up to one λ-class, often written, using Witten's notation, as:

$$\boxed{\langle\tau_{a_1}\ldots\tau_{a_m}\lambda_k\rangle_g := \int_{\overline{\mathcal{M}}_{g,m}} \psi_1^{a_1}\cdots\psi_m^{a_m}\lambda_k = (-1)^k\left[\alpha_1^{a_1}\cdots\alpha_m^{a_m}\right]P^g_m(\alpha_1,\ldots,\alpha_m)} \tag{4}$$

when $\sum a_i + k = 3g-3+m$, and 0 otherwise. (Here we use the notation $[A]B$ for the coefficient of A in B.) This ELSV polynomiality is related to (and implies, by Goulden et al. [24, Theorem 3.2]) an earlier polynomiality conjecture of Goulden and Jackson, describing the form of the generating series for single Hurwitz numbers of genus g [21, Conjecture 1.2], see also [23, Conjecture 1.4]. The conjecture asserts that after a change of variables, the single Hurwitz generating series is "polynomial" (in the sense that its scaled coefficients are polynomials). The conjecture is in fact a genus expansion ansatz for Hurwitz numbers analogous to the ansatz of Itzykson–Zuber [32, (5.32)] (proved in [5,24]). ELSV polynomiality is related to Goulden–Jackson polynomiality by a change of variables arising from Lagrange inversion [24, Theorem 2.5].

Hence, in developing the theory of double Hurwitz numbers, we seek some sort of polynomiality (in this case, piecewise polynomiality) that will tell us something about the moduli space in the background (such as its dimension), as well as a genus expansion ansatz.

1.2. Summary of results

In Section 2, we use ribbon graphs to establish that double Hurwitz numbers (with fixed m and n) are piecewise polynomial of degree up to $4g-3+m+n$ (Piecewise Polynomiality Theorem 2.1), with *no* scaling factor analogous to $C(g,\alpha)$. More precisely, for fixed m and n, we show that $H^g_{(\alpha_1,\ldots,\alpha_m),(\beta_1,\ldots,\beta_n)}$ counts the number of lattice points in certain polytopes, and as the α_i and β_j vary, the facets move. Further, we conjecture that the degree is bounded below by $2g-3+m+n$ (Conjecture 2.2), and verify this conjecture in genus 0, and also for m or $n=1$. We give an example ($(g,m,n)=(0,2,2)$) showing that it is not polynomial in general.

In Section 3, we consider the case $m=1$ ("one-part double Hurwitz numbers"), which corresponds to double Hurwitz numbers with complete branching over 0. One-part double Hurwitz numbers have a particularly tractable structure. In particular, they *are* polynomial: for fixed g, n, $H^g_{(d),(\beta_1,\ldots,\beta_n)}$ is a polynomial in $\beta_1,\ldots,\beta_n$. Theorem 3.1 gives two formulae for these numbers (one in terms of the series $\sinh x/x$ and the other an explicit expression) generalizing formulae of both Shapiro et al. [56, Theorem 6] and Goulden–Jackson [19, Theorem 3.2]. As an application, we prove polynomiality, and in particular show that the resulting polynomials have simple expressions in terms of character theory. Based on this polynomiality, we conjecture an ELSV-type formula for one-part double Hurwitz numbers (Conjecture 3.5):

$$\boxed{H^g_{(d),\beta} = r!d\int_{\overline{\mathrm{Pic}}_{g,n}} \frac{\Lambda_0-\Lambda_2+\cdots\pm\Lambda_{2g}}{(1-\beta_1\psi_1)\cdots(1-\beta_n\psi_n)}.} \tag{5}$$

The space $\overline{\mathrm{Pic}}_{g,n}$ is some as-yet-undetermined compactification of $\mathrm{Pic}_{g,n}$, supporting classes ψ_i and Λ_{2k}, satisfying properties described in Conjecture 3.5. As with the ELSV formula (1), the right-side of (5) should be interpreted by expanding the integrand formally, and capping the terms of dimension $4g-3+n$ with $[\overline{\mathrm{Pic}}_{g,n}]$. The most speculative part of this conjecture is the identification of the $(4g-3+n)$-dimensional moduli space with a compactification of $\mathrm{Pic}_{g,n}$ (see the remarks following Conjecture 3.5).

Motivated by this conjecture, we define a symbol $\langle\langle\cdot\rangle\rangle_g$, the analogue of $\langle\cdot\rangle_g$, by the first equality of

$$\boxed{\langle\langle\tau_{b_1}\cdots\tau_{b_n}\Lambda_{2k}\rangle\rangle_g := (-1)^k\left[\beta_1^{b_1}\cdots\beta_n^{b_n}\right]\left(\frac{H^g_{(d),\beta}}{r!d}\right) = \int_{\overline{\mathrm{Pic}}_{g,n}}\psi_1^{b_1}\cdots\psi_n^{b_n}\Lambda_{2k},} \tag{6}$$

so that Conjecture 3.5 (or (5)) would imply the second equality. (All parts of (6) are zero unless $\sum b_i+2k=4g-3+n$. Also, we point out that the definition of $\langle\langle\cdot\rangle\rangle_g$ is independent of the conjecture.) We show that this symbol satisfies many properties analogous to those proved by Faber and Pandharipande for $\langle\cdot\rangle_g$, including integrals over $\overline{\mathcal{M}}_{g,1}$, and the λ_g-theorem; we generalize these further. We then prove a genus expansion ansatz for $\langle\langle \cdot \rangle\rangle_g$ in the style of Itzykson–Zuber [32, Theorem 3.16]. As consequences, we prove that $\langle\langle \cdot \rangle\rangle_g$ satisfies the string and dilaton equations, and verify the ELSV-type conjecture in genus 0 and 1. A proof of Conjecture 3.5 would translate all of this structure associated with double Hurwitz numbers to the intersection theory of the universal Picard variety.

In Section 4, we give a simple formula for the double Hurwitz generating series in terms of Schur symmetric functions. As an application, we give explicit formulae for double Hurwitz numbers $H^g_{\alpha,\beta}$ for fixed α and β, in terms of linear combinations of gth powers of prescribed integers, extending work of Kuleshov–M. Shapiro [36]. Although this section is placed after Section 3, it can be read independently of Section 3.

In Section 5, we consider *m-part Hurwitz numbers* (those with $m = l(\alpha)$ fixed and β arbitrary). As remarked earlier, polynomiality fails in this case in general, but we still find strong suggestions of geometric structure. We define a (symmetrized) generating series $\mathbf{H}^g_m$ for these numbers, and show that it satisfies a topological recursion (in g, m) (Theorems 5.4, 5.6, and 5.12). The existence of such a recursion is somewhat surprising as, unlike other known recursions in Gromov–Witten theory (involving the geometry of the source curve), it is not a low-genus phenomenon. (The one exception is the Toda recursion of Pandharipande [50] and Okounkov [46], which also deals with double Hurwitz numbers.) We use this recursion to derive closed expressions for $\mathbf{H}^g_m$ for small (g, m), and to conjecture a general form (Conjecture 5.9), in analogy with the original Goulden–Jackson polynomiality conjecture of Goulden–Jackson [21].

1.3. Earlier evidence of structure in double Hurwitz numbers

Our work is motivated by several recent suggestions of strong structure of double Hurwitz numbers. Most strikingly, Okounkov proved that the generating series H for double Hurwitz numbers is a τ-function for the Toda hierarchy of Ueno and Takasaki [4,6], in the course of resolving a conjecture of Pandharipande's on single Hurwitz numbers [50]; see also their joint work Okounkov–Pandharipande [47–49]. Dijkgraaf's earlier description [4] of Hurwitz numbers where the target has genus 1 and all branching is simple, and his unexpected discovery that the corresponding generating series is essentially a quasi-modular form, is also suggestive, as such Hurwitz numbers can be written (by means of a generalized join-cut equation) in terms of double Hurwitz numbers (where $\alpha = \beta$). This quasi-modularity was generalized by Bloch-Okounkov [2].

Signs of structure for fixed g (and fixed number of points) provides a clue to the existence of a connection between double Hurwitz numbers and the moduli of curves (with additional structure), and even suggests the form of the connection, as was the case for single Hurwitz numbers. Evidence for this comes from recent work of Lando–Zvonkine [39], Kuleshov–M. Shapiro [36], and others.

We note that double Hurwitz numbers are relative Gromov–Witten invariants (see for example [40] in the algebraic category, and earlier definitions in the symplectic category [31,41]), and hence are necessarily top intersections on a moduli space (of relative stable maps). Techniques of Okounkov–Pandharipande [47–49] can be used to study double Hurwitz numbers in this guise. A second promising approach, relating more general Hurwitz numbers to intersections on moduli spaces of curves, is due to Shadrin [53] building on work of Ionel [30]. We expect that some of our results are probably obtainable by one of these two approaches. As a notable example, see [54]. However, we were unable to use them to prove any of the conjectures and, in particular, we could prove no ELSV-type formula.

We also alert the reader to other recent work on Hurwitz numbers due to Lando [37] and Zvonkine [62].

1.4. Notation and background

Throughout, the partitions α and β have m and n parts, respectively. We use $l(\alpha)$ for the number of parts of α, and $|\alpha|$ for the sum of the parts of α. If $|\alpha| = d$, we say α is a partition of d, and write $\alpha \vdash d$. For a partition $\alpha = (\alpha_1, \dots)$, let $\operatorname{Aut}\alpha$ be the group of permutations of $\{1, \dots, l(\alpha)\}$ fixing $(\alpha_1, \dots, \alpha_{l(\alpha)})$. Hence, if α has a_i parts equal to i, $i \geqslant 1$, then $|\operatorname{Aut}\alpha| = \prod_{i \geqslant 1} a_i!$. For indeterminates $p_1, \dots$ and $q_1, \dots,$ we write $p_\alpha = \prod_{i \geqslant 1} p_{\alpha_i}$ and $q_\alpha = \prod_{i \geqslant 1} q_{\alpha_i}$. Let $\mathcal{C}_\alpha$ denote the conjugacy class of the symmetric group $\mathfrak{S}_d$ indexed by α, so $|\mathcal{C}_\alpha| = d!/|\operatorname{Aut}\alpha| \prod_i \alpha_i$. We use the notation $[A]B$ for the coefficient of monomial A in a formal power series B.

Genus will in general be denoted by superscript. Let

$$r^g_{\alpha,\beta} := -2 + 2g + m + n. \tag{7}$$

When the context permits, we shall abbreviate this to r.

A summary of other globally defined notation is in the table below:

$\langle\cdot\rangle_g$, τ_i	Witten symbol (4)
$H^g_{\alpha,\beta}$, $\tilde{H}^g_{\alpha,\beta}$, H, $\tilde{H}$	double Hurwitz numbers and series, Section 1.4.1
Θ_m, $\mathbf{H}^g_m$, $\mathbf{H}^g_{m,i}$	symmetrization operator, symmetrized genus g m-part Hurwitz function, and its derivatives Sect. 1.4.2
Q, w, w_i, μ, μ_i, Q_i	Lagrange's Implicit Function Theorem 1.3
$P^g_{m,n}$	Piecewise Polynomiality Theorem 2.1
E^α, K_α	character theory (19), (20)
N_i, $c_i = N_i - \delta_{i,1}$, S_{2j}	functions of β, Section 3.1
B_{2k}, ξ_{2k} (and $\xi_{2\lambda}$), v_{2k}, f_{2k}	coefficients of $\frac{x}{e^x-1} + x/2$ (Bernoulli), $\log \frac{\sinh x}{x}$ (Theorem 3.1), $\frac{\sinh(x/2)}{x/2}$, $\frac{x/2}{\sinh(x/2)}$ (Thm. 3.7)
$\langle\langle \cdot \rangle\rangle_g$, $\overline{\operatorname{Pic}}_{g,n}$, Λ_{2k}	ELSV-type Conjecture 3.5
$Q^{(i)}(t)$, $Q^{(\lambda)}(t)$	Section 3.3, Theorem 3.7
s_θ, h_i, p_i	symmetric functions (Schur, complete, power sum), Sections 4, 5.5
$\mathbf{h}^g_m = \Gamma\mathbf{H}^g_m$, $\mathbf{h}^g_{m,i}$	transform of $\mathbf{H}^g_m$, and its partial derivatives, Sect. 5.2

1.4.1. Double Hurwitz numbers

As described earlier, let the *double Hurwitz number* $H^g_{\alpha,\beta}$ denote the number of degree d branched covers of $\mathbb{CP}^1$ by a genus g (connected) Riemann surface, with $r+2$ branch points, of which $r = r^g_{\alpha,\beta}$ are simple, and two (0 and ∞, say) have branching given by α and β, respectively. Then (7) is equivalent to the Riemann–Hurwitz formula. If a cover has automorphism group G, it is counted with multiplicity $1/|G|$. For example, $H^0_{(d),(d)} = 1/d$. The points above 0 and ∞ are taken to be *labelled*.

The *possibly disconnected* double Hurwitz numbers $\tilde{H}^g_{\alpha,\beta}$ are defined in the same way except the covers are not required to be connected.

The double Hurwitz numbers may be characterized in terms of the symmetric group through the monodromy of the sheets around the branch points. This axiomatization is essentially due to Hurwitz [29]; the proof relies on the Riemann existence theorem.

Proposition 1.1 (*Hurwitz axioms*). *For* $\alpha, \beta \vdash d$, $H^g_{\alpha,\beta}$ *is equal to* $|\mathrm{Aut}\,\alpha|\,|\mathrm{Aut}\,\beta|\,/d!$ *times the number of* $(\sigma, \tau_1, \ldots, \tau_r, \gamma)$, *such that*

H1. $\sigma \in \mathcal{C}_\beta$, $\gamma \in \mathcal{C}_\alpha$, $\tau_1, \ldots, \tau_r$ *are transpositions on* $\{1, \ldots, d\}$,
H2. $\tau_r \cdots \tau_1 \sigma = \gamma$,
H3. $r = r^g_{\alpha,\beta}$, *and*
H4. *the group generated by* $\sigma, \tau_1, \ldots, \tau_r$ *acts transitively on* $\{1, \ldots, d\}$.

The number $\tilde{H}^g_{\alpha,\beta}$ *is equal to* $|\mathrm{Aut}\,\alpha|\,|\mathrm{Aut}\,\beta|\,/d!$ *times the number of* $(\sigma, \tau_1, \ldots, \tau_r, \gamma)$ *satisfying* H1–H3.

If $(\sigma, \tau_1, \ldots, \tau_r)$ satisfies H1–H3, we call it an *ordered factorization* of γ, and if it also satisfies H4, we call it a *transitive ordered factorization*.

The *double Hurwitz* (*generating*) *series* H for double Hurwitz numbers is given by

$$H = \sum_{g\geqslant 0, d\geqslant 1} \sum_{\alpha,\beta\vdash d} y^g z^d p_\alpha q_\beta u^{l(\beta)} \frac{H^g_{\alpha,\beta}}{r^g_{\alpha,\beta}!\,|\mathrm{Aut}\,\alpha|\,|\mathrm{Aut}\,\beta|}, \tag{8}$$

and $\tilde{H}$ is the analogous generating series for the possibly disconnected double Hurwitz numbers. Then $\tilde{H} = e^H$, by a general enumerative result (see, e.g., [18, Lemma 3.2.16]). (The earliest reference we know for this result is, appropriately enough, in work of Hurwitz.)

The following result is obtained by using the axiomatization above, and by studying the effect that multiplication by a final transposition has on the cutting and joining of cycles in the cycle decomposition of the product of the remaining factors. The details of the proof are essentially the same as that of Goulden–Jackson [20, Lemma 2.2] and Goulden et al. [23, Lemma 3.1], and are therefore suppressed. A geometric proof involves pinching a loop separating the target $\mathbb{CP}^1$ into two disks, one of which contains only one simple branch point and the branch point corresponding to β.

Lemma 1.2 (*Join-cut equation*).

$$\left(\sum_{i\geqslant 1} p_i \frac{\partial}{\partial p_i} + u\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial y} - 2\right) H$$
$$= \frac{1}{2}\sum_{i,j\geqslant 1}\left(ijp_{i+j}\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial p_j} + (i+j)p_i p_j \frac{\partial H}{\partial p_{i+j}} + ijp_{i+j}y\frac{\partial^2 H}{\partial p_i \partial p_j}\right) \tag{9}$$

with initial conditions $\left[z^i p_i q_i u\right] H = \frac{1}{i}$ *for* $i \geqslant 1$.

Substituting $u\frac{\partial}{\partial u}H = \sum_{i\geqslant 1} q_i \frac{\partial}{\partial q_i} H$ yields the usual, more symmetric version. But the above formulation will be more convenient for our purposes.

1.4.2. The symmetrization operator Θ_m, and the symmetrized double Hurwitz generating series $\mathbf{H}_m^g$

The linear symmetrization operator Θ_m is defined by

$$\Theta_m(p_\alpha) = \sum_{\sigma\in\mathfrak{S}_m} x_{\sigma(1)}^{\alpha_1}\cdots x_{\sigma(m)}^{\alpha_m} \tag{10}$$

if $l(\alpha) = m$, and zero otherwise. (It is *not* a ring homomorphism.) The properties of Θ_m we require appear as Lemmas 4.1–4.3 in [23]. Note that $\Theta_m(p_\alpha)$ has a close relationship with the monomial symmetric function m_α since

$$\Theta_m(p_\alpha) = |\mathrm{Aut}\,\alpha|\, m_\alpha(x_1,\ldots,x_m).$$

We shall study in detail the symmetrization $\sum_{m\geqslant 1, g\geqslant 0} \mathbf{H}_m^g y^g$, of H where

$$\mathbf{H}_m^g(x_1,\ldots,x_m) = \left[y^g\right] \Theta_m(H)|_{z=1}$$
$$= \sum_{d\geqslant 1}\sum_{\substack{\alpha,\beta\vdash d\\ l(\alpha)=m}} m_\alpha(x_1,\ldots,x_m) q_\beta u^{l(\beta)} \frac{H^g_{\alpha,\beta}}{r^g_{\alpha,\beta}!\,|\mathrm{Aut}\,\beta|}, \tag{11}$$

for $m\geqslant 1$, $g\geqslant 0$.

In other words, the redundant variable z is eliminated, and $\mathbf{H}_m^g$ is a generating series containing information about genus g double Hurwitz numbers (where α has m parts).

We use the notation

$$\mathbf{H}^g_{j,i} = x_i \frac{\partial \mathbf{H}^g_j}{\partial x_i}.$$

1.4.3. Lagrange's Implicit Function Theorem

We shall make repeated use of the following form of Lagrange's Implicit Function Theorem (see, e.g., [18, Section 1.2] for a proof) concerning the solution of certain formal functional equations.

Theorem 1.3 (*Lagrange*). *Let $\phi(\lambda)$ be an invertible formal power series in an indeterminate λ. Then the functional equation*

$$v = x\phi(v)$$

has a unique formal power series solution $v = v(x)$. Moreover, if f is a formal power series, then

$$f(v(x)) = f(0) + \sum_{n \geqslant 1} \frac{x^n}{n} \left[\lambda^{n-1}\right] \frac{df(\lambda)}{d\lambda} \phi(\lambda)^n \tag{12}$$

and

$$\frac{f(v(x))}{v} \frac{x\,dv(x)}{dx} = \sum_{n \geqslant 0} x^n \left[\lambda^n\right] f(\lambda)\phi(\lambda)^n. \tag{13}$$

We apply Lagrange's theorem to the functional equation

$$w = xe^{uQ(w)}, \tag{14}$$

where

$$Q(t) = \sum_{j \geqslant 1} q_j t^j,$$

the series in the indeterminates q_j that record the parts of β in the double Hurwitz series (8). The following observations and notation will be used extensively. By differentiating the functional equation with respect to x, and u, we obtain

$$x\frac{\partial w}{\partial x} = w\mu(w), \qquad \frac{\partial w}{\partial u} = wQ(w)\mu(w), \qquad \text{where } \mu(t) = \frac{1}{1 - utQ'(t)}, \tag{15}$$

and we therefore have the operator identity

$$\frac{x\partial}{\partial x} = \mu(w)\frac{w\partial}{\partial w}. \tag{16}$$

We shall use the notation $w_i = w(x_i)$, $\mu_i = \mu(w_i)$, and $Q_i = Q(w_i)$, for $i = 1, \ldots, m$.

2. Piecewise polynomiality

By analogy with the ELSV formula (1), we consider double Hurwitz numbers for fixed g, m, n as functions in the parts of α and β:

$$P^g_{m,n}(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) = H^g_{\alpha,\beta}.$$

Here the domain is the set of $(m+n)$-tuples of positive integers, where the sum of the first m terms equals the sum of the remaining n. In contrast with the single Hurwitz number case, the double Hurwitz numbers have no scaling factor $C(g, \alpha, \beta)$ (see (3)).

Theorem 2.1 (*Piecewise polynomiality*). *For fixed m, n, the double Hurwitz function $H^g_{\alpha,\beta} = P^g_{m,n}$ is piecewise polynomial* (*in the parts of α and β*) *of degrees up to $4g-3+m+n$. The "leading" term of degree $4g-3+m+n$ is non-zero.*

By *non-zero leading term*, we mean that for fixed α and β, $P^g_{m,n}(\alpha_1 t, \ldots, \beta_n t)$ (considered as a function of $t \in \mathbb{Z}^+$) is a polynomial of degree $4g-3+m+n$. In fact, this leading term can be interpreted as the volume of a certain polytope. For example, $P^0_{2,2}(\alpha_1, \alpha_2, \beta_1, \beta_2) = 2\max(\alpha_1, \alpha_2, \beta_1, \beta_2)$, which has degree one. (This can be shown by a straightforward calculation, either directly, or using Section 2.1, or Corollary 4.2. See Corollary 4.2 for a calculation of $P^g_{2,2}$ in general.) In particular, unlike the case of single Hurwitz numbers (see (3)), $P^g_{m,n}$ is not polynomial in general.

We conjecture further:

Conjecture 2.2 (*Strong piecewise polynomiality*). *$P^g_{m,n}$ is piecewise polynomial, with degrees between $2g-3+m+n$ and $4g-3+m+n$ inclusive.*

This conjecture is not clear even in many cases where closed-form formulae for double Hurwitz numbers exist, such as Corollary 4.2. However, as evidence, we prove it when the genus is 0 (Section 2.2), and when m or n is 1 (Corollary 3.2). It may be possible to verify the conjecture by refining the proof of the Piecewise Polynomiality Theorem, but we were unable to do so.

2.1. Proof of the Piecewise Polynomiality Theorem 2.1

We spend the rest of this section proving Theorem 2.1. Our strategy is to interpret double Hurwitz numbers as counting lattice points in certain polytopes. We use a combinatorial interpretation of double Hurwitz numbers that is a straightforward extension of the interpretation of single Hurwitz numbers given in [47, Section 3.1.1] (which is there shown to be equivalent to earlier graph interpretations of Arnol'd [1] and Shapiro et al. [56]). The case $r = 0$ is trivially verified (the double Hurwitz number is $1/d$ if $\alpha = \beta = (d)$ and 0 otherwise), so we assume $r > 0$.

Consider a branched cover of $\mathbb{CP}^1$ by a genus g Riemann surface S, with branching over 0 and ∞ given by α and β, and r other branch points (as in Section 1.4.1). We

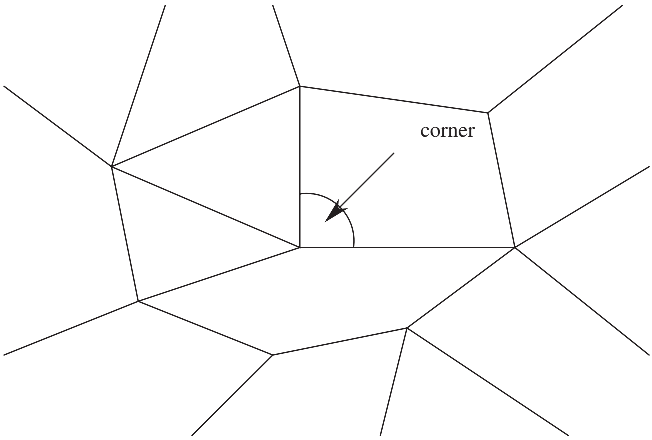


Fig. 1. An example of a corner in a fragment of a ribbon graph.

may assume that the r branch points lie on the equator of the $\mathbb{CP}^1$, say at the r roots of unity. Number the r branch points 1 through r in counterclockwise order around 0.

We construct a ribbon graph on S as follows. The vertices are the m preimages of 0, denoted by $v_1, \ldots, v_m$, where v_i corresponds to α_i. For each of the r branch points on the equator of the target $b_1, \ldots, b_r$, consider the d preimages of the geodesic (or radius) joining b_r to 0. Two of them meet the corresponding ramification point on the source. Together, they form an edge joining two (possibly identical) of the vertices. The resulting graph on the genus g surface has m (labelled) vertices and r (labelled) edges. There are n (labelled) faces, each homotopic to an open disk. The faces correspond to the parts of β: each preimage of ∞ lies in a distinct face. Call such a structure a *labelled* (ribbon) graph. (Euler's formula $m - r + n = 2 - 2g$ is equivalent to the Riemann–Hurwitz formula (7).)

Define a *corner* of this labelled graph to be the data consisting of a vertex, two edges incident to the vertex and adjacent to each other around the vertex, and the face between them (see Fig. 1).

Now place a dot near 0 on the target $\mathbb{CP}^1$, between the geodesics to the branch points r and 1. Place dots on the source surface S at the d preimages of the dot on the target $\mathbb{CP}^1$.

Then the number of dots near vertex v_i is α_i: a small circle around v_i maps to a loop winding α_i times around 0. Moreover, any corner where edge i is counterclockwise of j and $i < j$ must contain a dot. Call such a corner a *descending corner*. The number of dots in face f_j is β_j: move the dot on the target (together with its d preimages) along a line of longitude until it is near the pole ∞ (the d preimages clearly do not cross any edges en route), and repeat the earlier argument.

Thus each cover counted in the double Hurwitz number corresponds to a combinatorial object: a labelled graph (with m vertices, r edges, and n faces, hence genus g), and a non-negative integer (number of dots) associated to each corner, which is positive if the corner is descending, such that the sum of the integers around vertex i is α_i, and the sum of the integers in face j is β_j.

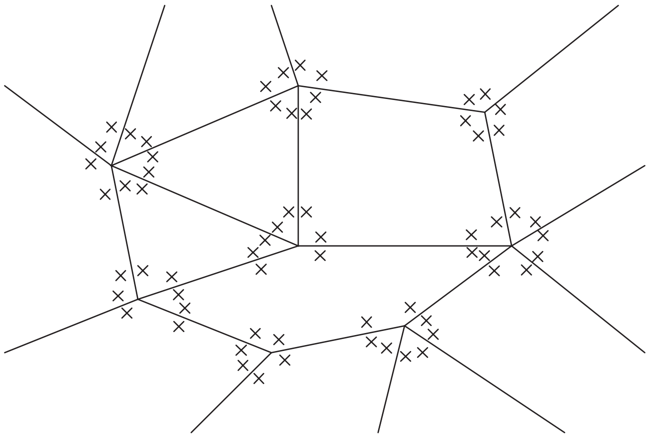


Fig. 2. The number of $\times$'s is twice the number of corners, and four times the number of edges.

It is straightforward to check that the converse is true (using the Riemann existence theorem, see for example [1]): given such a combinatorial structure, one gives the target sphere a complex structure (with branch points at roots of unity), and this induces a complex structure on the source surface.

Hence the double Hurwitz number is a sum over the set of labelled graphs (with m vertices, r edges, and n faces). The contribution of each labelled graph is the number of ways of assigning non-negative numbers to each corner so that each descending corner is assigned a positive integer, and such that the sum of numbers around vertex i is α_i and the sum of the integers in face j is β_j.

For fixed m and n, the contributions to $H^g_{\alpha,\beta}$ is the sum over the same finite set of labelled graphs. Hence to prove the Piecewise Polynomiality Theorem it suffices to consider a single such labelled graph Γ.

This problem corresponds to counting points in a polytope as follows. We have one variable for each corner (which is the corresponding number of dots). The number of corners is easily seen to be twice the number of edges (count $\times$'s in Fig. 2, so there are $2r$ variables $z_1, \ldots, z_{2r}$. We have one linear equation for each vertex (the sum of the variables corresponding to corners incident to vertex i must be α_i) and one for each face (the sum of the variables corresponding to corners incident to face j must be β_j). These equations are dependent since the sum of the m vertex relations is the sum of the n face relations, i.e. $\sum_{i=1}^m \alpha_i = \sum_{j=1}^n \beta_j$.

There are no other dependencies, i.e. the rank of the system is $m+n-1$: suppose otherwise, that one of the equations, for example the equation eq_1 corresponding to vertex i, were a linear combination of the others modulo the sum relation. Pick a face j incident to that vertex. Let z be the variable corresponding to the corner between vertex i and face j. Discard the equation eq_2 corresponding to that face (which is redundant because of the sum relation). Then z appears only in equation eq_1, and hence eq_1 cannot be a linear combination of the other equations.

Thus the contribution to the double Hurwitz number $P^g_{m,n}(\alpha_1, \ldots, \beta_n)$ from this labelled graph Γ is the number of lattice points in a polytope $\mathcal{P}_\Gamma(\alpha_1, \ldots, \beta_n)$ of dimension $2r-(m+n-1)=4g-3+m+n$ in $\mathbb{R}^{2r}$, lying in the linear subspaces defined by

$$\sum_{\substack{\text{corner } k \text{ incident}\\ \text{to vertex } i}} z_k = \alpha_i \qquad \text{and} \qquad \sum_{\substack{\text{corner } k \text{ incident}\\ \text{to face } j}} z_k = \beta_j, \tag{17}$$

bounded by inequalities of the form $z_k \geqslant 0$ or $z_k > 0$ (depending on whether corner k is descending or not). Let $P_\Gamma(\alpha_1, \ldots, \beta_m)$ be this contribution.

We are grateful to A. Vainshtein for pointing out that (17) is a well-known transportation polytope, and the next lemma (in the guise of integrality of the transportation polytope) is a classical fact, see e.g. [34, Corollary 1, p. 266]. We have kept the proof for the sake of completeness.

Lemma 2.3. *The vertices of the polytope $\mathcal{P}_\Gamma(\alpha, \beta)$ are lattice points, i.e., the polytope is integral.*

Proof. Let $p \in \mathbb{R}^{2r}$ be a point of the polytope. We show that if p is not a lattice point, then p lies in the interior of a line segment contained in $\mathcal{P}_\Gamma(\alpha, \beta)$, and hence is not a vertex. Construct an auxiliary graph, where the vertices correspond to corners i of Γ such that $z_i(p) \notin \mathbb{Z}$. The edges come in two colors. Red edges join any two distinct vertices incident to a common vertex, and blue edges join any two distinct vertices incident to a common face. By the first (resp. second) equality in (17), each vertex is incident to a red (resp. blue) edge. Thus we may find a cycle of distinct vertices $v_1 = v_{2w+1}, \ldots, v_{2w}$ such that v_{2i-1} and v_{2i} (resp. v_{2i} and v_{2i+1}) are joined by a red (resp. blue) edge: choose any x_1, and then subsequently choose x_2, x_3, etc. (such that x_i and x_{i+1} is joined by an appropriately coloured edge) until the first repetition: $x_j = x_k$ $(j < k)$. If $k - j$ is even, take $v_i = x_{j+i}$ $(1 \leqslant i \leqslant k - j)$, and if $k - j$ is odd, take $v_i = x_{j+i}$ $(1 \leqslant i < k - j)$. (If v_1 and v_2 are joined by a blue edge rather than a red edge, then cyclically permute the v_i by one.)

Then for $|\varepsilon| < \min(z_{v_i})$, the point $p(\varepsilon)$ given by

$$z_j(\varepsilon) = \begin{cases} z_j & \text{for } j \notin \{v_1, \ldots, v_{2w}\}, \\ z_j + \varepsilon & \text{for } j = v_{\text{even}}, \\ z_j - \varepsilon & \text{for } j = v_{\text{odd}} \end{cases}$$

satisfies (17) and $z_j(t) \geqslant 0$, and hence also lies in the polytope. $\square$

As $\mathcal{P}_\Gamma(\alpha, \beta)$ is an integral polytope, by Ehrhart's theorem [6], for t a positive integer, $P_\Gamma(t\alpha_1, \ldots, t\beta_m)$ is a polynomial in t of degree precisely $4g - 3 + m + n$ (the *Ehrhart polynomial* of the polytope), with leading coefficient equal to the volume of $\mathcal{P}_\Gamma(\alpha, \beta)$. (More correctly, Ehrhart's theorem requires that either that all the boundary points are counted, or that none of them are counted. Our argument is by adding the number of points on various open faces using the latter form of the Ehrhart's theorem, yielding a sum of Ehrhart polynomials, which is also polynomial.)

Finally, we recall the following well-known result [45].

Theorem 2.4. *Consider the polytopes in* $\mathbb{R}^L$ *(with coordinates* $z_1, \ldots, z_L$*) defined by equalities*

$$\sum_{i=1}^{L} \mu_{ij} z_i = \eta_j \qquad (1 \leqslant j \leqslant \sigma) \qquad \textit{and} \qquad \sum_{i=1}^{L} v_{ij} z_i \leqslant \zeta_j \qquad (1 \leqslant j \leqslant \tau)$$

as η_j *and* ζ_j *vary (with* μ_{ij} *and* v_{ij} *fixed). When the polytope is integral (for given* η_j *and* ζ_j*), define* $U(\eta_1, \ldots, \eta_\sigma, \zeta_1, \ldots, \zeta_\tau)$ *to be the number of lattice points contained therein. Then the function U is piecewise polynomial on its domain, of degree equal to the dimension of the polytope.*

Thus as α and β vary, the function $P^g_{m,n}$ is piecewise polynomial, concluding the proof of the Piecewise Polynomiality Theorem 2.1.

2.2. Proof of the Strong Piecewise Polynomiality Conjecture 2.2 in genus 0

It is straightforward to prove the result by induction using the join-cut equation (9). Instead we give a geometric argument. For convenience, define $t\alpha := (t\alpha_1, \ldots, t\alpha_m)$, and $t\beta$ similarly. Consider the commutative diagram

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathcal{H}^0_{\alpha,\beta} \arrow[r, "\mu", "\text{degree } t"'] \arrow[d, "\text{degree } H^0_{\alpha,\beta}"'] & \mathcal{H}^0_{t\alpha,t\beta} \arrow[d, "\text{degree } H^0_{t\alpha,t\beta}"] \\
\mathrm{Sym}^r \mathbb{P}^1 \arrow[r, "\text{degree } t^r"] & \mathrm{Sym}^r \mathbb{P}^1
\end{tikzcd} \qquad (18)$$

where the vertical morphisms are branch morphisms, and the horizontal morphisms are induced by $\mathbb{P}^1 \to \mathbb{P}^1$, $[u; v] \mapsto [u^t; v^t]$ ($\mathcal{H}^0_{\alpha,\beta}$ is the Hurwitz scheme, as in Section 1). The morphism μ is given by $(C, [f; g]) \mapsto (C, [f^t; g^t])$, where $\operatorname{div}(f) = \sum \alpha_i p_i$ and $\operatorname{div}(g) = \sum \beta_j q_j$. It is surjective: the preimage of $(C, [F; G])$ is $(C, [F^{1/t}; G^{1/t}])$ where $F^{1/t}$ and $G^{1/t}$ are any tth roots of F and G. (As $g = 0$, $F^{1/t}$ and $G^{1/t}$ are sections of the same line bundle.) The morphisms have degrees as shown in (18), from which $H^0_{t\alpha,t\beta} = t^{r-1} H^0_{t\alpha,t\beta} = t^{2g-3+m+n} H^0_{\alpha,\beta}$ as desired. $\square$

In general genus, the above argument shows that $H^g_{t\alpha,t\beta} \geqslant t^{2g-3+m+n} H^g_{\alpha,\beta}$; the image of μ is only a subset of the components of $\mathcal{H}^g_{t\alpha,t\beta}$. A vague heuristic suggests that a point of $\mathcal{H}^g_{t\alpha,t\beta}$ has a $1/t^{2g}$ chance of being in the image of μ: $\operatorname{div} F^{1/t} - \operatorname{div} G^{1/t}$ one of the t^{2g} t-torsion points of $\operatorname{Pic} C$, and $(C, [F; G])$ is in the image of μ if $\operatorname{div} F^{1/t} - \operatorname{div} G^{1/t}$ is 0 in $\operatorname{Pic} C$. This suggests that $H^g_{t\alpha,t\beta} \propto t^{4g-3+m+n} H^g_{\alpha,\beta}$, in keeping with Theorem 2.1.

3. One-part double Hurwitz numbers ($\alpha = (d)$): polynomiality, explicit formulae, and a conjectured ELSV-type formula in terms of the Picard variety

We use character theory to completely describe double Hurwitz numbers where α has one part, which leads to a conjectural formula in terms of intersection theory on a moduli space. The particular results that are needed from character theory are to be found in [44].

3.1. One-part double Hurwitz numbers through characters

In the group algebra $\mathbb{C}\mathfrak{S}_d$, let $\mathsf{K}_\alpha := \sum_{\sigma\in\mathcal{C}_\alpha}\sigma$. Then $\{\mathsf{K}_\alpha,\ \alpha\vdash d\}$, is a basis for the centre, and if χ_λ^α is the character of the irreducible representation of $\mathfrak{S}_d$ indexed by $\mathcal{C}_\alpha$, evaluated at any element of $\mathcal{C}_\lambda$, then

$$\mathsf{E}^\alpha = \frac{\chi^\alpha_{(1^d)}}{d!}\sum_{\lambda\vdash d}\chi^\alpha_\lambda \mathsf{K}_\lambda, \qquad \alpha\vdash d, \tag{19}$$

gives a basis of orthogonal idempotents. The inverse relations are

$$\mathsf{K}_\alpha = |\mathcal{C}_\alpha|\sum_{\lambda\vdash d}\frac{\chi^\lambda_\alpha}{\chi^\lambda_{(1^d)}}\mathsf{E}^\lambda, \qquad \alpha\vdash d. \tag{20}$$

The following result gives an expression for the double Hurwitz number $H^g_{(d),\beta}$, using various special properties of characters for the one part partition (d). We consider $\beta\vdash d$ with $l(\beta)=n$, and let the number of parts of β equal to i be given by N_i, $i\geqslant 1$. Thus $\sum_{i\geqslant 1}N_i = n$ and $\sum_{i\geqslant 1} iN_i = d$. We also let $c_1 = N_1 - 1$ and $c_i = N_i$, for $i\geqslant 2$, and

$$S_{2j} = \sum_{i\geqslant 1} i^{2j}c_i = -1 + \sum_{i\geqslant 1} i^{2j}N_i = -1 + \sum \beta_i^{2j},$$

for $j\geqslant 1$ (i.e. S_{2j} is a power sum for the partition, shifted by 1). Let $\xi_{2j} = \left[x^{2j}\right]\log(\sinh x/x)$, and let $\xi_\lambda = \xi_{\lambda_1}\xi_{\lambda_2}\ldots$. For any partition $\lambda = (\lambda_1,\ldots)$, let $2\lambda = (2\lambda_1,\ldots)$.

Theorem 3.1. *Let $r = r^g_{(d),\beta}$. For $g\geqslant 0$ and $\beta\vdash d$,*

$$H^g_{(d),\beta} = r!d^{r-1}\left[t^{2g}\right]\prod_{k\geqslant 1}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k} \tag{21}$$

$$= \frac{r!d^{r-1}}{2^{2g}}\sum_{\lambda\vdash g}\frac{\xi_{2\lambda}S_{2\lambda}}{|\mathrm{Aut}\,\lambda|}. \tag{22}$$

Remarks. (1) Eq. 21 is a generalization of a theorem of Shapiro et al.: [56, Theorem 6] is the case $\beta = (1^d)$. To our knowledge, Shapiro et al. [56] contains the first appearance of the generating series $\sinh t/t$ in connection with branched covers of curves (see [51, Section 3] for some of the subsequent connections, for example through the Gopakumar–Vafa conjecture). As noted there, proofs of equivalent statements appear in [17,33], but Shapiro et al. [56] is the first interpretation in terms of Hurwitz numbers.

(2) Eq. (21) also generalizes Theorem 3.2 of Goulden–Jackson [19]:

$$H^0_{(d),\beta} = r!d^{r-1}.$$

(Note that, in [19, Theorem 3.2], the right-hand side of the condition $t_1+\cdots+t_m = n+1$ should be replaced with $(m-1)n+1$.)

Proof. We use the Hurwitz axioms (Proposition 1.1). The group generated by any element of $\mathcal{C}_{(d)}$ acts transitively on $\{1,\ldots,d\}$. But $H^g_{\alpha,\beta}$ is a class function so, for $\alpha = (d)$ and $r = r^g_{(d),\beta} = n-1+2g$, axiom (H4) gives

$$H^g_{(d),\beta} = \frac{1}{\prod \beta_j}\left[\mathsf{K}_\beta\right]\left(\mathsf{K}_{(2,1^{d-2})}\right)^r \mathsf{K}_{(d)} = \frac{1}{d\prod \beta_j}\sum_{\lambda\vdash d}\eta(\lambda)^r\chi^\lambda_{(d)}\chi^\lambda_\beta$$

from (19) and (20), where

$$\eta(\lambda) = \frac{\left|\mathcal{C}_{(2,1^{d-2})}\right|\chi^\lambda_{(2,1^{d-2})}}{\chi^\lambda_{(1^d)}} = \sum_i\binom{\lambda_i}{2} - \sum_i\binom{\widetilde{\lambda}_i}{2},$$

and $\widetilde{\lambda}$ is the conjugate of λ. But $\chi^{(d-k,1^k)}_{(d)} = (-1)^k,\ k = 0,\ldots,d-1$, and $\chi^\lambda_{(d)} = 0$ for all other λ, and $\left|\mathcal{C}_{(d)}\right| = (d-1)!$. Also

$$\sum_{k=0}^{d-1}\chi_\beta^{(d-k,1^k)}y^k = \prod_{i\geqslant 1}\left(1-(-y)^i\right)^{c_i}, \tag{23}$$

and $\eta((d-k,1^k)) = \binom{d-k}{2} - \binom{k+1}{2} = \binom{d}{2} - dk$. Thus

$$\begin{aligned}
H^g_{(d),\beta} &= \frac{1}{\prod\beta_j}d^{r-1}\sum_{k=0}^{d-1}\left(\frac{d-1}{2}-k\right)^r(-1)^k\chi_\beta^{(d-k,1^k)}\\
&= \frac{1}{\prod\beta_j}d^{r-1}\left[\frac{t^r}{r!}\right]\sum_{k=0}^{d-1}e^{\left(\frac{d-1}{2}-k\right)t}(-1)^k\chi_\beta^{(d-k,1^k)}\\
&= \frac{1}{\prod\beta_j}r!d^{r-1}\left[t^r\right]e^{\frac{d-1}{2}t}\prod_{k\geqslant 1}\left(1-e^{-kt}\right)^{c_k},
\end{aligned}$$

by substituting $y=-e^{-t}$ in (23) above. But $\sum_{h\geqslant 1} hc_h = d-1$ and $\sum_{h\geqslant 1} c_h = n-1$, so we obtain

$$H^g_{(d),\beta} = r!d^{r-1}\left[t^{r-n+1}\right]\prod_{k\geqslant 1}\left(\frac{\sinh(kt/2)}{kt/2}\right)^{c_k}.$$

This yields (21). Applying the logarithm,

$$\prod_{k\geqslant 1}\left(\frac{\sinh(kx)}{kx}\right)^{c_k} = \exp\left(\sum_{k\geqslant 1} c_k \sum_{j\geqslant 1}\xi_{2j} i^{2j} x^{2j}\right) = \exp\left(\sum_{j\geqslant 1}\xi_{2j} S_{2j} x^{2j}\right)$$
$$= \sum_{\lambda}\frac{\xi_{2\lambda} S_{2\lambda}}{|\mathrm{Aut}\,\lambda|} x^{2|\lambda|},$$

where the sum is over all partitions λ. Eq. (22) follows. $\square$

Polynomiality is immediate from (22); hence we have proved the following.

Corollary 3.2. *The double Hurwitz numbers $H^g_{(d),\beta}$ satisfy polynomiality and the Strong Piecewise Polynomiality Conjecture* 2.2.

(The polynomials for $g\leqslant 5$ can be read off from Corollary 3.3.)

Even more striking, the polynomial is divisible by d^{n+2g-2}, and $H^g_{(d),\beta}/r!d^{n+2g-2}$ is a polynomial in the parts of β that is independent of the number of parts. These polynomials may immediately be computed in any desired genus. For example, the next corollary gives the formulae in genus up to 5, in terms of the S_i, which are polynomials in the parts of β.

Corollary 3.3. *For $g\leqslant 5$, explicit expressions for $H^g_{(d),\beta}$ are given by*

$$H^0_{(d),\beta} = (n-1)!d^{n-2},$$

$$H^1_{(d),\beta} = \frac{(n+1)!}{24} d^n S_2,$$

$$H^2_{(d),\beta} = \frac{(n+3)!d^{n+2}}{5760}\left(5S_2^2 - 2S_4\right),$$

$$H^3_{(d),\beta} = \frac{(n+5)!d^{n+4}}{2^{10}\cdot 3^4\cdot 5\cdot 7}\left(16S_6 - 42S_2S_4 + 35S_2^3\right),$$

$$H^4_{(d),\beta} = \frac{(n+7)!d^{n+6}}{2^8}\left(-\frac{S_8}{37800} + \frac{S_2S_6}{17010} + \frac{S_4^2}{64800} - \frac{S_2^2S_4}{12960} + \frac{S_2^4}{31104}\right),$$

$$H^5_{(d),\beta} = \frac{(n+9)!d^{n+8}}{2^{10}}\left(\frac{S_{10}}{467775} - \frac{S_2S_8}{226800} - \frac{S_4S_6}{510300} + \frac{S_2^2S_6}{204120} + \frac{S_2S_4^2}{388800} - \frac{S_2^3S_4}{233280} + \frac{S_2^5}{933120}\right).$$

Notice how constants associated to intersection theory on moduli spaces of low genus curves (such as 1/24 and 1/5760 for genus 1 and 2, respectively) make their appearance.

In addition, we note the following attractive formula for the number of branched covers of any genus and degree, with complete branching over two fixed points, and simple branching over $2g$ other fixed distinct points.

Corollary 3.4.

$$H^g_{(d),(d)} = (2g)!d^{2g-2}\left[t^{2g}\right]\frac{\sinh(dt/2)}{\sinh(t/2)} = d^{2g-2}\sum_{k=-\frac{d-1}{2}}^{\frac{d-1}{2}} k^{2g}.$$

Proof. The first equality is (21), and the second comes after straightforward manipulation. □

3.2. From polynomiality to the symbol $\langle\langle\cdot\rangle\rangle_g$, and intersection theory on moduli spaces

Theorem 3.1 strongly suggests the existence of an ELSV-type formula for one-part double Hurwitz numbers, and even suggests the shape of such a formula. In particular, we are in a much better position than we were for single Hurwitz numbers when the ELSV formula (1) was discovered. At that point, polynomiality was conjectured [21, Conjecture 1.2], see also [23, Conjecture 1.4]. Even today, polynomiality has only been proved by means of the ELSV formula; no character-theoretic or combinatorial reason is known.

In the one-part double Hurwitz case we have much more.

(i) We have a non-geometric proof of polynomiality.
(ii) The polynomials $P^g_{1,n}(\beta_1,\beta_2,\ldots) = H^g_{(d),\beta}$ have an excellent description in terms of generating series.
(iii) The polynomials are well-behaved as n increases. (More precisely, as described before Corollary 3.3, the polynomial $H^g_{(d),\beta}$ is divisible by d^{2g-2+n}, and the quotient $H^g_{(d),\beta}/r!d^{n+2g-2}$ is independent of n.)
(iv) Finally, the polynomials may be seen, for non-geometric reasons, to satisfy the string and dilaton equation. (This will be made precise in Proposition 3.10.)

Hence we make the following geometric conjecture. (Formula (24) is identical to (5) in the Introduction.) The conjecture should be understood as: "There exists a moduli space $\overline{\mathrm{Pic}}_{g,n}$ with the following properties…". We emphasize that a proof of this conjecture would be useful not to compute double Hurwitz numbers, but to understand the intersection theory of the universal Picard variety.

Conjecture 3.5 (*ELSV-type formula for one-part double Hurwitz numbers*). *For each* $g \geqslant 0$, $n \geqslant 1$, $(g, n) \neq (0, 1), (0, 2)$,

$$H^g_{(d),\beta} = r^g_{(d),\beta}!d \int_{\overline{\mathrm{Pic}}_{g,n}} \frac{\Lambda_0 - \Lambda_2 + \cdots \pm \Lambda_{2g}}{(1-\beta_1\psi_1)\cdots(1-\beta_n\psi_n)}, \tag{24}$$

where $\overline{\mathrm{Pic}}_{g,n}$, ψ_i, *and* Λ_{2k} *satisfy the following properties.*

- The space $\overline{\mathrm{Pic}}$, and its fundamental class. *There is a moduli space* $\overline{\mathrm{Pic}}_{g,n}$, *with a* (*possibly virtual*) *fundamental class* $[\overline{\mathrm{Pic}}_{g,n}]$ *of dimension* $4g-3+n$, *and an open subset isomorphic to the Picard variety* $\mathrm{Pic}_{g,n}$ *of the universal curve over* $\mathcal{M}_{g,n}$ (*where the two fundamental classes agree*).
- Morphisms from $\overline{\mathrm{Pic}}$. *There is a forgetful morphism* $\pi : \overline{\mathrm{Pic}}_{g,n+1} \to \overline{\mathrm{Pic}}_{g,n}$ (*flat, of relative dimension* 1), *with n sections* σ_i *giving Cartier divisors* $\Delta_{i,n+1}$ $(1 \leqslant i \leqslant n)$. *Both morphisms behave well with respect to the fundamental class*: $[\overline{\mathrm{Pic}}_{g,n+1}] = \pi^*[\overline{\mathrm{Pic}}_{g,n}]$, *and* $\Delta_{i,n+1} \cap \overline{\mathrm{Pic}}_{g,n+1} \cong \overline{\mathrm{Pic}}_{g,n}$ (*with isomorphisms given by* π *and* σ_i), *inducing* $\Delta_{i,n+1} \cap [\overline{\mathrm{Pic}}_{g,n+1}] \cong [\overline{\mathrm{Pic}}_{g,n}]$.
- ψ-classes on $\overline{\mathrm{Pic}}$. *There are n line bundles, which over* $\mathcal{M}_{g,n}$ *correspond to the cotangent spaces of the n points on the curve* (*i.e. over* $\mathcal{M}_{g,n}$ *they are the pullbacks of the* "*usual*" ψ-*classes on* $\mathcal{M}_{g,n}$). *Denote their first Chern classes by* $\psi_1, \ldots, \psi_n$. *They satisfy* $\psi_i = \pi^*\psi_i + \Delta_{i,n+1}$ $(i \leqslant n)$ *on* $\overline{\mathrm{Pic}}_{n+1}$ (*the latter* ψ_i *is on* $\overline{\mathrm{Pic}}_n$), *and* $\psi_i \cdot \Delta_{i,n+1} = 0$.
- Λ-classes. *There are Chow* (*or cohomology*) *classes* Λ_{2k} $(k = 0, 1, \ldots, g)$ *of codimension* $2k$ *on* $\overline{\mathrm{Pic}}_{g,n}$, *which are pulled back from* $\overline{\mathrm{Pic}}_{g,1}$ (*if* $g > 0$) *or* $\overline{\mathrm{Pic}}_{0,3}$; $\Lambda_0 = 1$. *The* Λ-*classes are the Chern classes of a rank* $2g$ *vector bundle isomorphic to its dual.*

The suggestion that the Λ-classes are the Chern classes of a self-dual vector bundle is due to J. Bryan. One might expect that the Λ_{2k} are tautological, given the philosophy that "geometrically natural classes tend to be tautological" (see e.g. [58]).

Remarks. (1) Our motivation for this conjecture included (a) the form of the ELSV-formula (1), (b) the Piecewise Polynomiality Theorem 2.1, and (c) the remaining results of this section. (In particular, the string and dilaton equations, Proposition 3.10, motivated the conditions on the ψ-classes.)

(2) As pointed out in the Introduction, the most speculative part of this conjecture is the identification of the $(4g-3+n)$-dimensional moduli space with a compactification of $\mathrm{Pic}_{g,n}$; the evidence suggests a space of this dimension with a morphism to $\overline{\mathcal{M}}_{g,n}$. We suggest this space because double Hurwitz numbers should be

top intersections on some compactified universal Picard variety, as described in the Introduction.

(3) There are certainly other formulae for double Hurwitz numbers not of this form, for example those involving integrals on the space of stable relative maps. However, to our knowledge, none of these formulae explains polynomiality of one-part double Hurwitz numbers, or the strong features of these polynomials.

(4) A satisfactory proof would connect the geometry of one-part double Hurwitz numbers with (24).

(5) See Conjecture 3.13 relating Λ_{2g} to λ_g.

In analogy with Witten's notation (4), we define $\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g$ by

$$\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g = (-1)^k \left[\beta_1^{b_1} \cdots \beta_n^{b_n}\right] \left(\frac{P^g_{1,n}(\beta_1, \ldots, \beta_n)}{r^g_{(d),\beta}! d} \right) \tag{25}$$

if $(g, n) \neq (0, 1)$, $(0, 2)$, $\sum b_i + 2k = 4g - 3 + n$ and the b_i are non-negative integers, and $\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g := 0$ otherwise. (This is identical to the first equality of (6).) This definition makes sense by Corollary 3.2. Note that the symbol is symmetric in the b_i. Conjecture 3.5 then implies that

$$\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g = \int_{\overline{\mathrm{Pic}}_{g,n}} \psi_1^{b_1} \cdots \psi_n^{b_n} \Lambda_{2k}. \tag{26}$$

3.3. Generating series for $\langle\langle \cdot \rangle\rangle_g$, and the string and dilaton equations

This symbol has some remarkable properties which suggest geometric meaning, in analogy with Witten's symbol $\langle \cdot \rangle_g$. We determine two expressions for a particular generating series for this symbol, and then derive string and dilaton equations, and prove Conjecture 3.5 in genus 0 and 1.

Define $Q^{(i)}(x) = \sum_{j \geqslant 1} q_j j^i x^j$ for $i \geqslant 0$, so $Q^{(0)}(x) = Q(x)$, defined just after (14), and

$$Q^{(i)}(x) = \left(x \frac{d}{dx} \right)^i Q(x), \qquad i \geqslant 0.$$

Of course, we also have

$$x \frac{d}{dx} Q^{(i)}(x) = Q^{(i+1)}(x), \quad i \geqslant 0. \tag{27}$$

The first expression for the generating series follows directly from definition (25) of the symbol $\langle\langle \cdot \rangle\rangle_g$.

Theorem 3.6. *For* $g \geqslant 0$,

$$x \frac{d}{dx} \sum_{n \geqslant 1} \frac{1}{n!} \sum_{k=0}^{g} (-1)^k \sum_{b_1,\ldots,b_n \geqslant 0} \langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g \prod_{i=1}^{n} Q^{(b_i)}(x) = \mathbf{H}_1^g(x)\big|_{u=1}.$$

Proof. From (25), we have

$$\begin{aligned} LHS &= x \frac{d}{dx} \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\beta_1,\ldots,\beta_n \geqslant 0} q_{\beta_1} \cdots q_{\beta_n} x^{\beta_1 + \cdots + \beta_n} \sum_{k=0}^{g} (-1)^k \\ &\quad \times \sum_{b_1,\ldots,b_n \geqslant 0} \beta_1^{b_1} \cdots \beta_n^{b_n} \langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g \\ &= \sum_{d \geqslant 1} d x^d \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\beta_1 + \cdots + \beta_n = d} q_{\beta_1} \cdots q_{\beta_n} \sum_{k=0}^{g} (-1)^k \\ &\quad \times \sum_{b_1,\ldots,b_n \geqslant 0} \beta_1^{b_1} \cdots \beta_n^{b_n} \langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g \\ &= \sum_{d \geqslant 1} d x^d \sum_{n \geqslant 1} \sum_{\substack{\beta \vdash d \\ l(\beta) = n}} \frac{q_\beta}{|\mathrm{Aut}\,\beta|} \sum_{k=0}^{g} (-1)^k \sum_{b_1,\ldots,b_n \geqslant 0} \beta_1^{b_1} \cdots \beta_n^{b_n} \langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g \\ &= RHS, \end{aligned}$$

giving the result. □

The second expression for this generating series follows from Theorem 3.1. To state this result requires some more notation. Define v_{2j}, f_{2j} by

$$\frac{\sinh(x/2)}{x/2} = \sum_{j \geqslant 0} v_{2j} x^{2j}, \qquad \frac{x/2}{\sinh(x/2)} = \sum_{j \geqslant 0} f_{2j} x^{2j}.$$

Then we have (see for example [27, Section 1.41 and 9.6]),

$$v_{2j} = \frac{1}{2^{2j}(2j+1)!}, \qquad f_{2j} = \frac{1 - 2^{2j-1}}{2^{2j-1}(2j)!} B_{2j}, \quad j \geqslant 0,$$

where B_{2j} is a Bernoulli number ($B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$). As Bernoulli numbers alternate in sign after B_2, note that f_{2j} has sign $(-1)^j$, $j \geqslant 0$. For a partition $\beta = (\beta_1, \ldots)$, let $Q^{(\beta)}(x) = \prod_{i \geqslant 1} Q^{(\beta_i)}(x)$.

Theorem 3.7. *For $g \geqslant 0$,*

$$\mathbf{H}_1^g(x)\big|_{u=1} = \sum_{k=0}^{g} f_{2k} \sum_{\substack{\theta \vdash_0 g-k \\ l(\theta) \geqslant 1}} \frac{v_{2\theta}}{|\mathrm{Aut}\,\theta|} \left(x \frac{d}{dx} \right)^{2g-2+l(\theta)} Q^{(2\theta)}(x),$$

where $\theta \vdash_0 g-k$ means that θ is a partition of $g-k$, with 0-parts allowed.

Proof. From (21), we have

$$\begin{aligned}
\left[x^d\right] \mathbf{H}_1^g(x)\big|_{u=1} &= \sum_{\beta \vdash d} \frac{H^g_{(d),\beta}}{r!\,|\mathrm{Aut}\,\beta|} q_\beta \\
&= d^{2g-2} \left[x^d t^{2g}\right] \frac{t/2}{\sinh(t/2)} \exp\left(d \sum_{j \geqslant 1} q_j x^j \frac{\sinh(jt/2)}{jt/2} \right) \\
&= d^{2g-2} \sum_{k=0}^{g} f_{2k} \left[x^d t^{2g-2k}\right] \exp\left(d \sum_{i \geqslant 0} v_{2i} t^{2i} Q^{(2i)}(x) \right) \\
&= d^{2g-2} \sum_{k=0}^{g} f_{2k} \left[x^d\right] \sum_{\theta \vdash_0 g-k} d^{l(\theta)} \frac{v_{2\theta}}{|\mathrm{Aut}\,\theta|} Q^{(2\theta)}(x),
\end{aligned}$$

and the result follows. □

By comparing the two generating series expressions given in Theorems 3.6 and 3.7, we obtain an explicit expression for $\langle\langle \cdot \rangle\rangle_g$, in the following result.

Corollary 3.8. *For all $b_1, \dots, b_n$, k, g, with $b = (b_1, \dots, b_n)$,*

$$\begin{aligned}
\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g = |\mathrm{Aut}\,b|\, f_{2k} (-1)^k \sum_{\substack{\theta \vdash_0 g-k \\ l(\theta)=n}} \frac{v_{2\theta}}{|\mathrm{Aut}\,\theta|} \left[Q^{(b_1)} \cdots Q^{(b_n)} \right] \\
\times \left(x \frac{d}{dx} \right)^{2g-3+n} Q^{(2\theta)}(x).
\end{aligned}$$

Proof. Compare Theorems 3.6 and 3.7. Now $\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g$ is symmetric in the b_i's, so each of the $n!/|\mathrm{Aut}\,b|$ distinct reorderings of b contributes equally to the coefficient of the monomial $Q^{(b_1)} \cdots Q^{(b_n)}$. The result follows from (27). □

From Corollary 3.8 and applying (27), we can obtain a great deal of information about values of $\langle\langle \cdot \rangle\rangle_g$. For example, we immediately have the following non-negativity result. (We believe the analogous result for the Witten symbol $\langle \tau_{a_1} \cdots \tau_{a_n} \lambda_k \rangle_g \geqslant 0$ is known but difficult.)

Corollary 3.9 (*Non-negativity*). *For all* $b_1, \ldots, b_n$, k, g, *we have* $\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g \geqslant 0$.

We can also prove that the symbol $\langle\langle \cdot \rangle\rangle_g$ satisfies the string and dilaton equations, in the following result.

Proposition 3.10 (*String and dilaton equations*). (a) (string equation) *The following equation holds, except when* $g = k = 1, n = 0$:

$$\langle\langle \tau_0 \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g = \sum_{i=1}^{n} \langle\langle \tau_{b_1} \cdots \tau_{b_{i-1}} \tau_{b_i - 1} \tau_{b_{i+1}} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g .$$

In the exceptional case, we have $\langle\langle \tau_0 \Lambda_2 \rangle\rangle_1 = 1/24$.

(b) (dilaton equation) *The following equation holds*:

$$\langle\langle \tau_1 \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g = (2g - 2 + n) \langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g .$$

Note that, by the usual proofs of the string and dilaton equation (see for example [43, Section 1] or [28, Chapter 25]), this proposition would be implied by Conjecture 3.5.

Proof. For formal power series in the variables $Q^{(i)} := Q^{(i)}(x)$, $i \geqslant 0$, we define the partial differential operators

$$\Delta_{-1} = \sum_{i \geqslant 0} Q^{(i+1)} \frac{\partial}{\partial Q^{(i)}}, \quad \Delta_0 = \sum_{i \geqslant 0} Q^{(i)} \frac{\partial}{\partial Q^{(i)}}. \tag{28}$$

Note that we have the operator identity $\Delta_{-1} = x \frac{d}{dx}$, as well as

$$\frac{\partial}{\partial Q^{(i)}} \Delta_{-1} = \Delta_{-1} \frac{\partial}{\partial Q^{(i)}} + \frac{\partial}{\partial Q^{(i-1)}} \tag{29}$$

and

$$Q^{(i)} \Delta_{-1} = \Delta_{-1} Q^{(i)} - Q^{(i+1)}. \tag{30}$$

Now, multiplying (29) on the left by $Q^{(i)}$, applying (30), and then summing over $i \geqslant 0$, we obtain

$$\Delta_0 \Delta_{-1} = \Delta_{-1} \Delta_0 . \tag{31}$$

Also, applying (29) repeatedly with $i = 0, 1$, we obtain

$$\frac{\partial}{\partial Q^{(0)}}\Delta_{-1}^{m} = \Delta_{-1}^{m}\frac{\partial}{\partial Q^{(0)}}, \qquad \frac{\partial}{\partial Q^{(1)}}\Delta_{-1}^{m} = \Delta_{-1}^{m}\frac{\partial}{\partial Q^{(1)}} + m\Delta_{-1}^{m-1}\frac{\partial}{\partial Q^{(0)}}. \tag{32}$$

Now let $x\frac{d}{dx}\Psi_g(x) = \mathbf{H}_1^g(x)\big|_{u=1}$. Then, from Theorem 3.7 and (32), we obtain

$$\left(\frac{\partial}{\partial Q^{(0)}} - \Delta_{-1}\right)\left(\Psi_g + \delta_{g,1}\frac{Q^{(0)}}{24}\right) = 0,$$

and thus the string equation holds with the given exceptional value, from Corollary 3.8.

Also, from Theorem 3.7 and (31), (32), we obtain

$$\left(\frac{\partial}{\partial Q^{(1)}} - \Delta_0 - (2g-2)\right)\Psi_g = 0,$$

and thus the dilaton equation holds, from Corollary 3.8. □

3.3.1. Virasoro constraints?

In the case of the moduli space of curves, the string and dilaton equations are essentially the first two Virasoro constraints (see for example [28, Section 25.2]). It is natural then to ask whether there is a full set of Virasoro constraints. Even in the case of single Hurwitz numbers, this is not known. However, in the single Hurwitz number case, the highest-degree terms (of the polynomial defined in (3)) are polynomials with coefficients of the form $\langle\tau_{a_1}\cdots\tau_{a_m}\rangle_g$ (i.e. with no λ-class), which *do* satisfy Virasoro constraints, by Witten's conjecture (Kontsevich's theorem) [35,60]. (Indeed, this idea led to Okounkov and Pandharipande's proof of Witten's conjecture [47].) Thus one may ask a weaker question: are there Virasoro constraints on the asymptotics of one-part double Hurwitz numbers, i.e. on $\langle\langle\tau_{b_1}\cdots\tau_{b_n}\rangle\rangle_g := \langle\langle\tau_{b_1}\cdots\tau_{b_n}\Lambda_0\rangle\rangle_g$? Given Conjecture 3.5, this is the analogue of Witten's conjecture on the compactified Picard variety.

We have not yet been able to produce a set of Virasoro constraints, but our partial results suggest additional hidden structure, so we report them here without proof.

For formal power series in the variables $Q^{(i)}$, we define the partial differential operators Δ_{-1}, Δ_0 (as in (28)),

$$\Delta_0' = \sum_{i\geqslant 0} iQ^{(i)}\frac{\partial}{\partial Q^{(i)}}, \quad \Delta_1' = \sum_{i\geqslant 1} iQ^{(i-1)}\frac{\partial}{\partial Q^{(i)}}, \quad \Delta_1'' = \sum_{i\geqslant 1} i^2Q^{(i-1)}\frac{\partial}{\partial Q^{(i)}}.$$

Now let

$$\Psi = \sum_{g\geqslant 0}\Psi_g\big|_{k=0},$$

where Ψ_g is as above. (Note that the value of g is recoverable from the partition condition on the monomials.) Then the string equation translates to an annihilator A_{-1} for Ψ (up to initial conditions), where

$$A_{-1} = \frac{\partial}{\partial Q^{(0)}} - \Delta_{-1}.$$

The dilaton equation translates to an annihilator A_0 for Ψ, where

$$A_0 = \frac{\partial}{\partial Q^{(1)}} - \frac{1}{2}\left(\Delta_0 + \Delta_0' - 1\right).$$

It is an easy computation that

$$\left[A_{-1}, A_0\right] = \tfrac{1}{2} A_{-1}.$$

Now Itzykson–Zuber [32, p. 5689] suggests letting $B_{-1} = -\frac{1}{2}A_{-1}$ and $B_0 = -2A_0$, so that

$$\left[B_{-1}, B_0\right] = -B_{-1}.$$

We then sought a candidate B_1 (analogous to Witten's L_1) involving a term of the form $\frac{\partial}{\partial Q^{(2)}}$. There are many such annihilators, and the simplest we found was

$$A_1 = 4\frac{\partial}{\partial Q^{(2)}} + \frac{1}{3}\frac{\partial^3}{\partial Q^{(0)3}} + 2\left(\Delta_1'' + \Delta_1'\right)\frac{\partial}{\partial Q^{(1)}} - 12\frac{\partial}{\partial Q^{(2)}}\frac{\partial}{\partial Q^{(1)}}.$$

However, we have been unable to find a candidate B_1 satisfying the desired Virasoro commutation relations with B_0 and B_{-1}.

3.3.2. Verifying Conjecture 3.5 in low genus

Proposition 3.11. *Conjecture* 3.5 *is true in genus* 0, *taking* $\overline{\mathrm{Pic}}_{0,n} = \overline{\mathcal{M}}_{0,n}$, *and in genus* 1, *taking* $\overline{\mathrm{Pic}}_{1,n} = \overline{\mathcal{M}}_{1,n+1}$ *and* $\Lambda_2 = \pi^*[pt]/24$, *where pt is the class of a point on* $\overline{\mathrm{Pic}}_{1,1}$, *and* π *is the morphism* $\overline{\mathrm{Pic}}_{1,n} \to \overline{\mathrm{Pic}}_{1,1}$.

We have two proofs, neither of which is fully satisfactory (in the sense of Remark 4 after Conjecture 3.5). First, the geometric arguments of Vakil [57] apply with essentially no change; this argument is omitted for the sake of brevity. The following second proof is purely algebraic.

Proof. For genus 0, if $n \geqslant 3$, then

$$r^0_{(d),\beta}!d \int_{\overline{\mathcal{M}}_{0,n}} \frac{1}{(1-\beta_1\psi_1)\cdots(1-\beta_n\psi_n)} = r^0_{(d),\beta}!d(\beta_1+\cdots+\beta_n)^{n-3}$$

(using the string equation; see for example Hori et al. [28, Exercise 25.2.8]), so we are done by Corollary 3.3.

For genus 1, we will prove (26). We need only prove the base cases $\langle\langle\tau_2\Lambda_0\rangle\rangle_1 = \frac{1}{24}$ and $\langle\langle\tau_0\Lambda_2\rangle\rangle_1 = \frac{1}{24}$ (obtained by unwinding Corollary 3.3), as the rest follow by the string and dilaton equation. The first is

$$\int_{\overline{\mathcal{M}}_{1,2}} \psi_1^2 = \frac{1}{24},$$

which is well-known (e.g. [28, Exercise 25.2.9]; combinatorialists may prefer to extract it from the ELSV formula (1)), and the second is immediate from the definition of Λ_2. $\square$

3.4. *Explicit formulae for* $\langle\langle\cdot\rangle\rangle_g$

We can also determine explicit formulae for many instances of this symbol, just as such formulae have been given for Witten's symbol $\langle\cdot\rangle_g$, most notably by Faber and Pandharipande. In particular:

Integrals over $\overline{\mathcal{M}}_{g,1}$. It is a straightforward consequence of Witten's conjecture that

$$\langle\tau_{3g-2}\rangle_g = \frac{1}{24^g g!} \tag{33}$$

(see for example just before (4) in [11]). Also, [13, Eq. (5)]:

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2}\lambda_g = \frac{2^{2g-1}-1}{2^{2g-1}(2g)!}\left|B_{2g}\right|. \tag{34}$$

Generalizing both of these statements is [13, Theorem 2]:

$$1+\sum_{g\geqslant 1}\sum_{i=0}^{g} t^{2g}k^i \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2+i}\lambda_{g-i} = \left(\frac{t/2}{\sin(t/2)}\right)^{k+1}. \tag{35}$$

The "λ_g-theorem". The main theorem of [14] is

$$\langle\tau_{b_1}\cdots\tau_{b_n}\lambda_g\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{b_1}\cdots\psi_n^{b_n}\lambda_g = \binom{2g-3+n}{b_1,\ldots,b_n}\mathbf{b}_g. \tag{36}$$

(This was first conjectured in [15, Eq. (16)].) More precisely, it was shown to be a consequence of the Virasoro conjecture for constant maps to $\mathbb{CP}^1$.) The constant $\mathbf{b}_g$ can be evaluated using (34) by taking $(b_1, b_2, b_3, \ldots) = (3g-2, 0, 0, \ldots)$.

We now deduce analogues and generalizations of these results for double Hurwitz numbers. A proof of Conjecture 3.5 would thus give these results important geometric meaning.

In analogy with the λ_g-theorem (36), we have the following result, which follows immediately from Theorem 3.1 (in the same way as did Corollary 3.3).

Proposition 3.12.

$$\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g = \left[\beta_1^{b_1} \cdots \beta_n^{b_n}\right] \left(\mathbf{c}_g d^{r-2} + \textit{higher terms in } \beta' s\right) \tag{37}$$

($d = \sum \beta_j$), where $\mathbf{c}_g$ depends only on g. As $b_1 + \cdots + b_n = 2g-3+n = r-2$, we have

$$\langle\langle \tau_{b_1} \cdots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g = \binom{2g-3+n}{b_1, \ldots, b_n} \mathbf{c}_g$$

for some constant $\mathbf{c}_g$.

(By "higher terms in β's" we mean terms of homogeneous degree greater than $r-2 = 2g-3+n$.) We note that (37) is analogous to the version [14, Eq. (18)] of the λ_g-theorem used in the proof of Faber and Pandharipande.

In analogy with (33), we have

$$\langle\langle \tau_{4g-2} \rangle\rangle_g = \frac{1}{2^{2g}(2g+1)!}. \tag{38}$$

In analogy with (34), we have

$$\langle\langle \tau_{2g-2} \Lambda_{2g} \rangle\rangle_g = \frac{(-1)^g(1-2^{2g-1})}{2^{2g-1}(2g)!} B_{2g} = \frac{2^{2g-1}-1}{2^{2g-1}(2g)!} \left|B_{2g}\right|. \tag{39}$$

Thus we have evaluated $\mathbf{c}_g$ in the previous Proposition. Remarkably, it is the same constant appearing in Faber and Pandharipande's expression (34), leading us to speculate the following.

Conjecture 3.13. *There is a structure morphism $\pi : \overline{\mathrm{Pic}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, and $\pi_* \Lambda_{2g} = \lambda_g$.*

Generalizing (38) and (39), we have the following result, in analogy with (but not identical to) (35).

Proposition 3.14. *For* $g \geqslant 1$, *and* $k = 0, \ldots, g$,

$$\langle\langle \tau_{b_1} \Lambda_{2k} \rangle\rangle_g = (-1)^k f_{2k} v_{2g-2k} = \frac{(-1)^k f_{2k}}{2^{b_1-2g+2}(b_1 - 2g + 3)!}$$

for $b_1 + 2k = 4g - 2$. *Equivalently*,

$$1 + \sum_{g \geqslant 1} t^{2g} \sum_{k=0}^{g} x^{2k} \langle\langle \tau_{b_1} \Lambda_{2k} \rangle\rangle_g = \frac{x \sinh(t/2)}{\sin(xt/2)}.$$

Proof. This follows immediately from Corollary 3.8, since the only choices of θ in the summation are partitions with a single part. $\square$

This result can be extended to expressions for terms with more τ's. For example, part (a) of the following proposition gives a closed-form expression for any term involving two τ's. There are also formulae for any mixture of τ_2's and τ_3's (where the number of τ_3's is held fixed); the first three examples are parts (b)–(d) below. We know of no analogue for $\langle \cdot \rangle_g$.

Proposition 3.15.

(a) *For* $k = 0, \ldots, g$, *and* $g \geqslant 2$,

$$\langle\langle \tau_{b_1} \tau_{b_2} \Lambda_{2k} \rangle\rangle_g = \frac{(-1)^k f_{2k}}{2^{2g-2k+1}(2g - 2k + 2)!} \sum_{i>0 \text{ odd}} \binom{2g - 2k + 2}{i} \times \left(\binom{2g - 1}{b_1 + 1 - i} + \binom{2g - 1}{b_2 + 1 - i} \right)$$

for $b_1 + b_2 = 4g - 2k - 1$.

(b) *For* $k = 0, \ldots, g$, *and* $g \geqslant 1$, *except* $(k, g) = (1, 1)$,

$$\langle\langle \tau_2^{4g-3-2k} \Lambda_{2k} \rangle\rangle_g = \frac{(-1)^k f_{2k}}{24^{g-k}(g - k)!}(6g - 7 - 2k)!!$$

where $(2m - 1)!! = (2m - 1)(2m - 3) \cdots (3)(1)$ *for* m *a positive integer*, *and* $(-1)!! = 1$.

(c) *For* $k = 0, \ldots, g$, *and* $g \geqslant 2$, *except* $(k, g) = (2, 2)$,

$$\langle\langle \tau_2^{4g-5-2k} \tau_3 \Lambda_{2k} \rangle\rangle_g = \frac{(-1)^k f_{2k}}{24^{g-k}(g - k)!}(6g - 7 - 2k)!! \frac{6g - 4 - 4k}{3}.$$

(d) *For* $k = 0, \ldots, g$, *and* $g \geqslant 2$, *except* $(k, g) = (1, 2), (2, 2), (3, 3)$,

$$\begin{aligned}
&\langle\langle \tau_2^{4g-7-2k} \tau_3^2 \Lambda_{2k} \rangle\rangle_g \\
&\quad = \frac{(-1)^k f_{2k}}{24^{g-k}(g-k)!}(6g-9-2k)!! \\
&\qquad \times \frac{(3g-4-k)((6g-4-4k)(6g-7-4k)-(6g-2-6k))}{9}.
\end{aligned}$$

Proof. These results all follow from Corollary 3.8 in a routine way, using Leibniz's Rule. For part (a), the only choices of θ in the summation are partitions with two parts. For parts (b)–(d), all parts of θ must be 0's or 1's only. $\square$

3.5. *A genus expansion ansatz for* $\langle\langle \cdot \rangle\rangle_g$ *in the style of Itzykson and Zuber*

We next prove an analogue of the genus expansion ansatz of Itzykson and Zuber for intersection numbers on the moduli space of curves [32, (5.32)]. The Itzykson–Zuber ansatz was proved by Eguchi et al. [15] and later by Goulden et al. [24, Theorem 3.1]; the latter proof (and generalization) is similar in approach to the argument in this paper.

Theorem 3.16 (*Genus expansion ansatz*). *For* $g \geqslant 0$,

$$\mathbf{H}_1^g(x)\big|_{u=1} = \sum_{k=0}^{g} f_{2k} \sum_{\theta \vdash g-k} \frac{v_{2\theta}}{|\mathrm{Aut}\,\theta|} \left(x\frac{d}{dx}\right)^{2g-2+l(\theta)} \left(\frac{Q^{(2\theta)}(w)}{1-Q^{(1)}(w)} - \delta_{\theta\emptyset}\right). \tag{40}$$

Remarks. (1) Unlike the Itzykson–Zuber ansatz, this result has explicitly computable coefficients.

(2) w and x are related by (14).

Proof. From Theorem 3.7, we obtain

$$\left[x^d\right] \mathbf{H}_1^g(x)\big|_{u=1} = \sum_{k=0}^{g} f_{2k} \sum_{\theta \vdash g-k} \frac{v_{2\theta}}{|\mathrm{Aut}\,\theta|} d^{2g-2+l(\theta)} \left[x^d\right] Q^{(2\theta)}(x) \sum_{m \geqslant 0} \frac{d^m Q^{(0)}(x)^m}{m!},$$

where m is the number of 0's in the partition with 0-parts allowed. The result follows from Lagrange's Implicit Function Theorem 1.3(13) (with $u = 1$ in (15) and (16)). $\square$

To obtain explicit results from Theorem 3.16, we modify (27), using (15) and (16), to obtain

$$x\frac{d}{dx} Q^{(i)}(w) = \frac{Q^{(i+1)}(w)}{1-Q^{(1)}(w)}, \quad i \geqslant 0.$$

For example, with $g = 0$ in Theorem 3.16, and $i = 0$ above, we obtain

$$x \frac{d}{dx} \mathbf{H}_1^0(x)\Big|_{u=1} = Q^{(0)}(w) = Q(w). \tag{41}$$

With $g = 1$ and $i = 1$, we obtain

$$\mathbf{H}_1^1(x)\Big|_{u=1} = \tfrac{1}{24}\left(Q^{(3)}(w)\mu(w) + Q^{(2)}(w)^2\mu(w)^2 - \mu(w) + 1\right), \tag{42}$$

since, with $u = 1$, we have $\mu(w) = 1/(1 - Q^{(1)}(w))$.

Remarks. (1) In genus 0, there is a direct connection between the generating series for single Hurwitz numbers (with a partition β) and one-part double Hurwitz numbers. More precisely, these generating series are identical, under $\frac{j^j}{j!} p_j \leftrightarrow q_j$ and $s \leftrightarrow w$. Here s is the solution to the functional equation

$$s = x\, e^{\phi_0(s)},$$

and $\phi_i(x) = \sum_{j \geqslant 1} \frac{j^{j+i}}{j!} p_j x^j$, as described in [24, Section 2.3], so, for example, $\phi_0(x) \leftrightarrow Q^{(0)}(x) = Q(x)$. This is a purely formal statement that the formula for single Hurwitz numbers and that for one-part double Hurwitz numbers are "essentially" the same in genus 0. We do not know if there is any geometric or combinatorial reason for this coincidence.

(2) More generally, in arbitrary genus, there is also such a connection. In this case, the direct analogue of Theorem 3.6 holds for the single Hurwitz number series, under $\phi_i(x) \leftrightarrow Q^{(i)}(x)$ and $\langle \cdot \rangle_g \leftrightarrow \langle\langle \cdot \rangle\rangle_g$, as described in [24, Section 2.4]. However, there is no analogue of Theorem 3.7 that we know for the single Hurwitz number series. From this point of view, the Itzykson–Zuber ansatz for the single Hurwitz number series is the analogue of the form given by Theorem 3.16 under $s \leftrightarrow w$.

(3) We note that the substitution for x by a series in w specified by the functional equation (14) is the key technical device used in Section 5, as considered in (47). However, the approach in Section 5 is completely different from that of the present section, so the appearance of w again suggests that it is significant, and that a geometric or combinatorial explanation for this would be enlightening.

Caution: The results of this section, especially the genus expansion ansatz and the string and dilaton equations, seem to lead inescapably to Conjecture 3.5, but this is not quite the case. The simple structure of the polynomials $P^g_{1,n}$ allows other possible statements as well. For example, the correct statement might be

$$H^g_{(d),\beta} = r^g_{(d),\beta}! \int_{\overline{\mathrm{Pic}}'_{g,n+1}} \frac{\Lambda'_0 - \Lambda'_2 + \cdots \pm \Lambda'_{2g}}{(1 - \beta_1\psi'_1)\cdots(1 - \beta_n\psi'_n)},$$

where the space $\overline{\mathrm{Pic}}'_{g,n+1}$ and classes $\psi'_1, \ldots, \psi'_n$, $\Lambda'_0, \ldots, \Lambda'_{2g}$ satisfy the itemized hypotheses of Conjecture 3.5. (Note that there is no "d" in the numerator, as there is in Conjecture 3.5.) We use primes to indicate that these objects need not be the same as in Conjecture 3.5.

The $(n+1)$th point should correspond to α (the point mapping to 0 in the target $\mathbb{CP}^1$). $\overline{\mathrm{Pic}}'_{g,n+1}$ should admit an action of $\mathfrak{S}_n$ (permuting the points corresponding to β), but not necessarily $\mathfrak{S}_{n+1}$. (This insight comes from M. Shapiro, who has suggested that the correct moduli space of curves for the double Hurwitz problem in general should have two "colors" of points, one corresponding to α, and one corresponding to β.) The string and dilaton equations are again satisfied.

4. A symmetric function description of the Hurwitz generating series

In this section, we use character theory again, to give a good description of the double Hurwitz generating series H. This gives algebraic, rather than geometric, insight into Hurwitz numbers, and thereby give a means of producing explicit formulae, for example extending results of Kuleshov–M. Shapiro [36].

For the purposes of this section, we regard each of the indeterminates p_k and q_k as power sum symmetric functions in two sets of indeterminates, one for p_k and the other for q_k. This may be done since the power sum symmetric functions in an infinite set of indeterminates are algebraically independent. The following result gives such an expression, stated in terms of symmetric functions. Let $s_\lambda(p_1, p_2, \ldots)$ be the Schur symmetric function, written as a polynomial in the power sum symmetric functions $p_1, p_2, \ldots$. This is the generating series for the irreducible $\mathfrak{S}_d$-characters $\left(\chi^\alpha_\lambda \colon \lambda \vdash d\right)$ with respect to the power sum symmetric functions. For $\alpha \vdash d$, the expression, and its inverse, is

$$s_\alpha = \frac{1}{d!} \sum_{\lambda \vdash d} |\mathcal{C}_\lambda|\, \chi^\alpha_\lambda p_\lambda, \qquad p_\alpha = \sum_{\lambda \vdash d} \chi^\lambda_\alpha s_\lambda. \tag{43}$$

From the expression for H that we give next, we shall determine how $H^g_{\alpha,\beta}$ depends on g for fixed α and β.

Theorem 4.1. *Let*

$$Z = 1 + \sum_{d \geqslant 1} z^d \sum_{\lambda \vdash d} e^{\eta(\lambda)t} s_\lambda(p_1 t^{-1}, p_2 t^{-1}, \ldots) s_\lambda(q_1 u\, t^{-1}, q_2 u\, t^{-1}, \ldots).$$

Then $H|_{y=t^2} = t^2 \log Z$ *and* $\tilde{H}\Big|_{y=t^2} = t^2 Z$.

Proof. Following the method of proof of Theorem 3.1, we have

$$\frac{\tilde{H}^{(g)}_{\alpha,\beta}}{|\mathrm{Aut}\,\alpha|\,|\mathrm{Aut}\,\beta|} = \frac{|\mathcal{C}_\beta|}{d!} \left[\mathsf{K}_\beta\right] \left(\mathsf{K}_{(2,1^{d-2})}\right)^r \mathsf{K}_\alpha = \frac{|\mathcal{C}_\alpha| \cdot |\mathcal{C}_\beta|}{d!^2} \sum_{\lambda \vdash d} \eta(\lambda)^r \chi^\lambda_\alpha \chi^\lambda_\beta. \tag{44}$$

Now multiply by $p_\alpha q_\beta u^{l(\beta)} z^d t^r / r!$, and sum over $\alpha, \beta \vdash d$, $d \geqslant 0$, and $r \geqslant 0$ (this number is 0 unless r has the same parity as $l(\alpha) + l(\beta)$), using (43), to obtain

$$\sum_{r\geqslant 0}\sum_{d\geqslant 0} z^d \sum_{\alpha,\beta\vdash d} p_\alpha q_\beta u^{l(\beta)} t^r \frac{\tilde{H}^{(g)}_{\alpha,\beta}}{|\mathrm{Aut}\,\alpha|\,|\mathrm{Aut}\,\beta|\,r!}$$

$$= 1 + \sum_{d\geqslant 1} z^d \sum_{\lambda\vdash d} e^{\eta(\lambda)t} s_\lambda(p_1,\ldots) s_\lambda(q_1 u,\ldots).$$

This series is an exponential generating series in both z, marking sheets, and t, marking transposition factors (we have divided by both $r!$ and $d!$, the latter in the Hurwitz axioms Proposition 1.1). To transform the exponent of t from number of transposition factors to genus, we apply the substitutions $p_i \mapsto p_i t^{-1}, q_i \mapsto q_i t^{-1}$, $i \geqslant 1$, to obtain $p_\alpha q_\beta t^r \mapsto p_\alpha q_\beta t^{r-l(\alpha)-l(\beta)} = p_\alpha q_\beta t^{2g-2}$, from (7), and the result now follows. (Note that Z is clearly an even series in t since $\eta(\widetilde{\lambda}) = -\eta(\lambda)$ and $s_{\tilde{\lambda}}(p) = s_\lambda(-p)$.) $\square$

4.1. *Expressions for $H^g_{\alpha,\beta}$ for varying g and fixed α, β*

Theorem 4.1 may be used to obtain $H^g_{\alpha,\beta}$ for fixed α, β. The expressions are linear combinations of certain powers of non-negative integers. In particular, the results of Kuleshov–M. Shapiro [36] for $d = 3$, 4 and 5 can be obtained and extended, using Maple to carry out the routine manipulation of series.

As an example, we give an explicit expression for $H^g_{(\alpha_1,\alpha_2),(\beta_1,\beta_2)}$, in the case that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are distinct.

Corollary 4.2. *Let $\alpha = (\alpha_1, \alpha_2) \vdash d$ and $\beta = (\beta_1, \beta_2) \vdash d$ where $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, $\alpha_1 < \beta_1$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ are distinct. Then*

$$H^g_{\alpha,\beta} = \frac{2}{\alpha_1\alpha_2\beta_1\beta_2} \sum_{i=1}^{\alpha_1} \left(\left(\binom{d+1}{2} - di \right)^{2g+2} - \left(\binom{d+1}{2} - di - \alpha_2\beta_1 \right)^{2g+2} \right).$$

Proof. Since $\alpha_1, \alpha_2, \beta_1, \beta_2$ are distinct, we have

$$H^g_{\alpha,\beta} = \tilde{H}^{(g)}_{\alpha,\beta} = \frac{\left|\mathcal{C}_\beta\right| \cdot |\mathcal{C}_\alpha|}{d!^2} \sum_{\lambda\vdash d} \eta(\lambda)^r \chi^\lambda_\alpha \chi^\lambda_\beta,$$

from (44). But here we have $r = 2g+2$ is even, and $\left|\mathcal{C}_\beta\right| \cdot |\mathcal{C}_\alpha| / d!^2 = 1/\alpha_1\alpha_2\beta_1\beta_2$. Moreover, $\eta(\widetilde{\lambda}) = -\eta(\lambda)$, and $\chi^{\widetilde{\lambda}}_\alpha \chi^{\widetilde{\lambda}}_\beta = \chi^\lambda_\alpha \chi^\lambda_\beta$. Finally, from (43), $\chi^\lambda_\alpha = 0$ exactly when $[p_\alpha] s_\lambda = 0$, so we have

$$H^g_{\alpha,\beta} = \frac{2}{\alpha_1\alpha_2\beta_1\beta_2} \sum_{\lambda\in\mathcal{P}_{\alpha,\beta}} \eta(\lambda)^{2g+2} \chi^\lambda_\alpha \chi^\lambda_\beta,$$

where $\mathcal{P}_{\alpha,\beta} = \{\lambda \colon [p_\alpha]s_\lambda \neq 0, [p_\beta]s_\lambda \neq 0, \eta(\lambda) > 0\}$. Now, we can give an explicit description of $\mathcal{P}_{\alpha,\beta}$, using the Murnaghan–Nakayama formula for the irreducible characters of the symmetric group (see, e.g., [44]). The details can be routinely verified. First, $\left|\mathcal{P}_{\alpha,\beta}\right| = 2\alpha_1$, so we let $\mathcal{P}_{\alpha,\beta} = \left\{\lambda^{(1)}, \ldots, \lambda^{(2\alpha_1)}\right\}$, where $\lambda^{(1)} \succ \ldots \succ \lambda^{(2\alpha_1)}$. (Here $\succ$ denotes reverse lexicographic order on partitions, so $(3) \succ (2,1) \succ (1^3)$.) Then, for $i = 1, \ldots, \alpha_1$, we have $\lambda^{(i)} = \left(d - i + 1, 1^{i-1}\right)$ (which is independent of β), so

$$\eta\left(\lambda^{(i)}\right) = \binom{d+1}{2} - di, \qquad \chi_\alpha^{\lambda^{(i)}} \chi_\beta^{\lambda^{(i)}} = 1, \quad i = 1, \ldots, \alpha_1.$$

Also, for $i = 1, \ldots, \alpha_1$, we have $\lambda^{(\alpha_1+i)} = \left(d + 1 - \beta_1 - i, \alpha_1 + 2 - i, 2^{i-1}, 1^{\beta_1 - \alpha_1 - 1}\right)$, so

$$\eta\left(\lambda^{(\alpha_1+i)}\right) = \binom{d+1}{2} - di - \alpha_2\beta_1, \qquad \chi_\alpha^{\lambda^{(\alpha_1+i)}} \chi_\beta^{\lambda^{(\alpha_1+i)}} = -1, \quad i = 1, \ldots, \alpha_1.$$

The result follows immediately. $\square$

For example, $\mathcal{P}_{(3,8),(4,7)} = \left((11), (10,1), (9,1^2), (7,4), (6,3,2), (5,2^3)\right)_\succ$ and

$$H^g_{(3,8),(4,7)} = \frac{2}{3 \cdot 8 \cdot 4 \cdot 7}\left(55^{2g+2} + 44^{2g+2} + 33^{2g+2} - 23^{2g+2} - 12^{2g+2} - 1^{2g+2}\right).$$

Similar expressions may be obtained when α and β have three parts. For example,

$$\begin{aligned} H^g_{(1,2,6),(1,3,5)} = \tfrac{1}{180}\Big(&2^{2g+4} - 6^{2g+4} + 10^{2g+4} + 12^{2g+4} - 18^{2g+4} - 20^{2g+4} \\ &-28^{2g+4} + 36^{2g+4}\Big). \end{aligned}$$

The sum is over $\mathcal{P}_{\alpha,\beta}$, but contributions from some partitions of this set are exactly canceled as a consequence of "identities" between parts of α and parts of β (for example, $1 + 2 = 3$ and $6 = 1 + 5$, where the left and right-hand sides, respectively, refer to α and β). Furthermore, other terms are introduced as a consequence of the same identities.

As an example with more parts, with $d = 8$, we obtain

$$\begin{aligned} H^g_{(2,2,4),(1,2,2,3)} = \tfrac{1}{48}\Big(&3 \cdot 2^{2g+5} + \tfrac{9}{2} \cdot 4^{2g+5} + 3 \cdot 6^{2g+5} - 10^{2g+5} - 14^{2g+5} - 16^{2g+5} \\ &+\tfrac{1}{2} \cdot 28^{2g+5}\Big). \end{aligned}$$

5. m-Part double Hurwitz numbers ($m = l(\alpha)$ fixed): topological recursions and explicit formulae

We next consider more generally the case where α has a fixed number m of parts, and β is arbitrary. (One of our results, Corollary 5.5, will lead to a proof of Faber's intersection number conjecture [12] in arbitrary genus with up to 3 points [25], and we hope to extend this to prove the conjecture in general.) The behaviour is qualitatively different from that of the $m = 1$ case, which was considered in Section 3, as might be expected by the failure of polynomiality. This will require us to utilize more sophisticated algebraic tools.

We prove a topological recursion relation consistent with a description of double Hurwitz numbers in terms of the moduli space of curves. This relation is obvious neither from the currently understood geometry of double Hurwitz numbers nor from the combinatorial interpretation in terms of the join-cut equation. For expository reasons, we will give three versions of this topological recursion: a genus 0 recursion (Theorem 5.4), a "cleaner" version of the genus 0 recursion involving rational rather than transcendental functions (Theorem 5.6), and a version in arbitrary genus (Theorem 5.12).

The topological recursion will enable us to find closed-form expressions for double Hurwitz numbers for small g and m, and in principle for larger g and m. (For a much simpler example of topological recursions implying closed-form expressions for single Hurwitz numbers, see [57].) We conjecture the form of a closed-form expression for $g = 0$ and arbitrary m (Conjecture 5.9).

The reader will notice that except for the cases $(g, m) = (0, 1)$ and $(0, 2)$ (when there is no Deligne–Mumford moduli stack $\overline{\mathcal{M}}_{g,m}$), the explicit expressions that we obtain for $\mathbf{H}_m^g$ are all rational functions in the intrinsic variable u. Moreover, the denominator has explicit linear factors. The topological recursions that we obtain for $\mathbf{H}_m^g$ are integrals over u, and the integrand is quadratic in lower order terms. We conjecture that $\mathbf{H}_m^g$ is, except for the two initial cases, always a rational function in u, with specified linear factors in the denominator. To prove this by induction, we would need to obtain a rational integrand by the induction hypothesis, and then prove (to avoid a logarithm in the integrated form) that the inverse linear terms in the partial fraction expansion of the integrand disappear. We have been unable to prove this in general, since it seems to require a stronger induction hypothesis.

5.1. *The symmetrized join-cut equation at genus* 0

We apply the symmetrization operator Θ_m (defined in (10)) to the join-cut equation (9) to obtain partial differential equations for the symmetrized series $\mathbf{H}_m^0(x_1, \ldots, x_m)$. As a preliminary, we begin with $\mathbf{H}_1^0$. The more general results will be an extension of this idea.

Lemma 5.1.

$$\mathbf{H}_{1,1}^0(x_1) = uQ_1.$$

(Recall that $\mathbf{H}^g_{j,i} = x_i \frac{\partial \mathbf{H}^g_j}{\partial x_i}$.) Although Lemma 5.1 has already been proved in the previous section, in (41), we give a second proof to illustrate the methodology that will be used throughout this section.

Proof. By applying Θ_1 to the join-cut equation (9) and setting $y = 0$, it follows immediately that $\mathbf{H}^0_1$ satisfies the partial differential equation

$$\left(1 + u\frac{\partial}{\partial u} + 0 - 2\right)\mathbf{H}^0_1 = \left(u\frac{\partial}{\partial u} - 1\right)\mathbf{H}^0_1 = \frac{1}{2}\left(\left(\mathbf{H}^0{}_{1,1}\right)^2 + 0 + 0\right) = \frac{1}{2}\left(\mathbf{H}^0{}_{1,1}\right)^2$$

with initial condition $[u]\,\mathbf{H}^0_{1,1} = Q(x_1)$. Apply $\frac{x_1}{u^2}\frac{\partial}{\partial x_1}$ to the above equation and let $G = \frac{1}{u}\mathbf{H}^0_{1,1}$, to obtain

$$\frac{\partial G}{\partial u} = G x_1 \frac{\partial G}{\partial x_1}. \tag{45}$$

In terms of G, the initial condition becomes $\left[u^0\right] G = Q(x_1)$. But, applying $\frac{\partial}{\partial u}$ to the functional equation (14), we obtain

$$\frac{\partial w}{\partial u} = w\mu(w)Q(w), \tag{46}$$

and comparing with (15), we check that $G(x_1) = Q_1$ is the unique solution to (45). $\square$

To state an equation for $\mathbf{H}^0_m$ for $m \geqslant 2$, we need some additional notation. For $\alpha = \{\alpha_1, \ldots, \alpha_j\} \subseteq \{1, \ldots, m\}$, let $x_\alpha = x_{\alpha_1}, \ldots, x_{\alpha_j}$. Let $\Omega_{m,i}$ be the set of unordered pairs $\{\alpha, \zeta\}$ such that $\alpha, \zeta \subseteq \{1, \ldots, m\}$ with $\alpha \cup \zeta = \{1, \ldots, m\}$ and $\alpha \cap \zeta = \{i\}$. Let $\overline{\{l\}} = \{1, \ldots, m\} \setminus \{l\}$.

Theorem 5.2 (*Symmetrized join-cut equation in genus* 0). *For* $m \geqslant 2$, $\mathbf{H}^0_m$ *satisfies the equation*

$$\left(u\frac{\partial}{\partial u} + m - 2 - \sum_{i=1}^{m} u Q_i x_i \frac{\partial}{\partial x_i}\right)\mathbf{H}^0_m = \sum_{i=1}^{m} \sum_{\substack{\{\alpha,\zeta\}\in\Omega_{m,i} \\ l(\alpha), l(\zeta) \geqslant 2}} \mathbf{H}^0_{|\alpha|,i}(x_\alpha)\mathbf{H}^0_{|\zeta|,i}(x_\zeta) + \sum_{\substack{1\leqslant k,l\leqslant m \\ k\neq l}} \frac{x_l \mathbf{H}^0_{m-1,k}(x_{\overline{\{l\}}})}{x_k - x_l}.$$

The two parts of the right-hand side of the above equation correspond to the first two parts of the right-hand side of the join-cut equation (Lemma 1.2); the third part of join-cut does not arise in genus 0.

Proof. By applying Θ_m to (9) for fixed $m \geqslant 2$ and setting $y = 0$, we find that $\mathbf{H}_m^0(x_1, \ldots, x_m)$ satisfies

$$\left(u\frac{\partial}{\partial u} + m - 2\right)\mathbf{H}_m^0 = \sum_{i=1}^{m} \sum_{\{\alpha,\zeta\}\in\Omega_{m,i}} \mathbf{H}^0_{|\alpha|,i}(x_\alpha)\mathbf{H}^0_{|\zeta|,i}(x_\zeta) + \sum_{\substack{1\leqslant k,l\leqslant m\\ k\neq l}} \frac{x_l \mathbf{H}^0_{m-1,k}(x_{\overline{\{l\}}})}{x_k - x_l}.$$

Moving the contribution of $\{\alpha, \zeta\} \in \Omega_{m,i}$ where $l(\alpha) = 1$ or $l(\zeta) = 1$ on the right-hand side of this equation to the left-hand side, we obtain

$$\left(u\frac{\partial}{\partial u} + m - 2 - \sum_{i=1}^{m} \mathbf{H}^0_{1,i}(x_i)x_i\frac{\partial}{\partial x_i}\right)\mathbf{H}_m^0 = \sum_{i=1}^{m} \sum_{\substack{\{\alpha,\zeta\}\in\Omega_{m,i}\\ l(\alpha),l(\zeta)\geqslant 2}} \mathbf{H}^0_{|\alpha|,i}(x_\alpha)\mathbf{H}^0_{|\zeta|,i}(x_\zeta) + \sum_{\substack{1\leqslant k,l\leqslant m\\ k\neq l}} \frac{x_l \mathbf{H}^0_{m-1,k}(x_{\overline{\{l\}}})}{x_k - x_l},$$

and the result follows from Lemma 5.1. □

A key observation is the following. The right-hand side of the equation in Theorem 5.2 involves the series $\mathbf{H}_j^0$ for $j < m$ only, so if we can invert the partial differential operator that is applied to $\mathbf{H}_m^0$ on the left-hand side, then we have a recursive solution for $\mathbf{H}_m^0$, $m \geqslant 2$.

5.2. *A transformation of variables and the recursive solution to the symmetrized join-cut equation*

We now find a solution to the partial differential equation for $\mathbf{H}_m^0$ that is given in Theorem 5.2. The key is to change variables in $\mathbf{H}_m^0$, for $m \geqslant 1$, from $x_1, \ldots, x_m$ to $w_1, \ldots, w_m$, using (14), to obtain

$$\mathbf{h}_m^0(u, w_1, \ldots, w_m) := \mathbf{H}_m^0(w_1 e^{-uQ_1}, \ldots, w_m e^{-uQ_m}), \qquad m \geqslant 2. \tag{47}$$

We denote this transformation by Γ, so

$$\Gamma\mathbf{H}_m^0(x_1, \ldots, x_m) = \mathbf{h}_m^0(u, w_1, \ldots, w_m).$$

We regard $\mathbf{h}_m^0$ as an element of the ring of formal power series in $u, w_1, \ldots, w_m$, with coefficients that are polynomials in $q_1, q_2, \ldots$. It is straightforward to invert this, and recover $\mathbf{H}_m^0$ from $\mathbf{h}_m^0$ in (47) by Lagrange inversion, as specified in Theorem 1.3. For this ring, let D_u be the first partial derivative, with respect to u, for the purposes of which $w_1, \ldots, w_m$ are regarded as algebraically independent variables, with no dependence on u. Henceforth, we use $\mathbf{h}_m^0$ and $\mathbf{H}_m^0$ interchangeably.

The importance of Γ is shown in its action on the partial differential operator that is applied to $\mathbf{H}_m^0$ on the left-hand side of the symmetrized join-cut equation (Theorem 5.2). We show that, under Γ, the partial differential operator is transformed into a linear differential operator involving only D_u.

Lemma 5.3. *Let k be an integer. Then*

$$u^{k-1}\Gamma\left(u\frac{\partial}{\partial u} + k - \sum_{i=1}^{m} uQ_i x_i \frac{\partial}{\partial x_i}\right) = D_u u^k \Gamma.$$

In short, passing Γ through the differential operator simplifies it. From this point of view, the variable u plays an important role, as the only variable, and accounts for our terming it the intrinsic variable of the system.

Proof. For functions of $u, w_1, \ldots, w_m$, the chain rule gives

$$u\frac{\partial}{\partial u} = uD_u + \sum_{i=1}^{m} u\left(\frac{\partial w_i}{\partial u}\right)\frac{\partial}{\partial w_i} = uD_u + \sum_{i=1}^{m} uw_i Q_i \mu_i \frac{\partial}{\partial w_i} = uD_u + \sum_{i=1}^{m} uQ_i x_i \frac{\partial}{\partial x_i},$$

from (46) and the operator identity (16). Then $\Gamma\left(u\frac{\partial}{\partial u} + k - \sum_{i=1}^{m} uQ_i x_i \frac{\partial}{\partial x_i}\right) = (uD_u + k)\,\Gamma$ and the result follows. $\square$

5.3. A (univariate, rational, integral) topological recursion for $\mathbf{H}_m^0$, and explicit formulae

Lemma 5.3 enables us to solve the partial differential equation for $\mathbf{H}_m^0$, $m \geqslant 2$, recursively. The following result gives an integral expression for $\mathbf{h}_m^0$, in terms of $\mathbf{h}_1^0, \ldots, \mathbf{h}_{m-1}^0$. We use the notation

$$\mathbf{h}_{j,i}^0 = w_i \frac{\partial \mathbf{h}_j^0}{\partial w_i}.$$

(A "cleaner" version, not involving exponentials, will be given later, Theorem 5.6.)

Theorem 5.4 (*Genus 0 topological recursion, transcendental form*). *For $m \geqslant 2$,*

$$\mathbf{h}_m^0 = u^{2-m} \int_0^u \left(\sum_{i=1}^m \sum_{\substack{\{\alpha,\zeta\} \in \Omega_{m,i} \\ l(\alpha), l(\zeta) \geqslant 2}} \mu_i^2 \mathbf{h}_{|\alpha|,i}^0(u, w_\alpha) \mathbf{h}_{|\zeta|,i}^0(u, w_\zeta) \right.$$

$$\left. + \sum_{\substack{1 \leqslant k,l \leqslant m \\ k \neq l}} \frac{w_l e^{-uQ_l} \mu_k \mathbf{h}_{m-1,k}^0(u, w_{\overline{\{l\}}})}{w_k e^{-uQ_k} - w_l e^{-uQ_l}} \right) u^{m-3} du,$$

where the integrand is considered as a power series in $u, w_1, \ldots, w_m$, and the integration is carried out with $w_1, \ldots, w_m$ regarded as constants.

Proof. The result follows by applying Γ to Theorem 5.2 with the aid of Lemma 5.3. $\square$

5.4. Explicit expressions for $\mathbf{H}_m^0$ for $m \leqslant 5$, and a conjectured rational form for $\mathbf{H}_m^0$ in general

We now apply Theorem 5.4 for successive values of $m \geqslant 2$, to obtain explicit expressions for the symmetrized series $\mathbf{H}_m^0$. We begin with $m = 2$ and 3, and include the details in a single result, because the resulting expressions can be treated uniformly, also incorporating $m = 1$. This requires some notation. For $m \geqslant 1$, let $V_m = \prod_{1 \leqslant i < j \leqslant m} (w_i - w_j)$, the value of the Vandermonde determinant $\det \left(w_j^{m-i} \right)_{m \times m}$, and let $\mathbf{A}_m$ be the $m \times m$ matrix with $(1, j)$-entry equal to $\mu_j - 1$, for $j = 1, \ldots, m$, and (i, j)-entry equal to w_j^{m-i+1}, for $i = 2, \ldots, m$, $j = 1, \ldots, m$. Let $\Delta_{m,j}$ be the partial differential operator defined by

$$\Delta_{m,j} = \sum_{i=1}^m w_i^j \mu_i \frac{\partial}{\partial w_i} = \sum_{i=1}^m w_i^{j-1} x_i \frac{\partial}{\partial x_i}, \tag{48}$$

for $m, j \geqslant 1$, where the second equality follows from the operator identity (16).

The following attractive formula will be used in a proof of Faber's intersection number conjecture for up to 3 points [25]. (Although the result is for genus 0, it will give Faber's conjecture in *arbitrary* genus.)

Corollary 5.5. *For $m = 1, 2, 3$,*

$$\Delta_{m,1}^{3-m} \mathbf{h}_m^0 = \frac{\det \mathbf{A}_m}{V_m}.$$

Proof. For $m = 1$, the result follows by differentiating the result of Lemma 5.1 and applying (15).

For the case $m=2$, Theorem 5.4 gives

$$\begin{aligned}
\mathbf{h}_2^0 &= \int_0^u \left(\frac{uQ_1w_2e^{-uQ_2} - uQ_2w_1e^{-uQ_1}}{w_1e^{-uQ_1} - w_2e^{-uQ_2}} - Q_1 - Q_2 \right) \frac{du}{u} \\
&= \int_0^u \left(\frac{Q_1w_1e^{-uQ_1} - Q_2w_2e^{-uQ_2}}{w_1e^{-uQ_1} - w_2e^{-uQ_2}} - Q_1 - Q_2 \right) du \\
&= -\log\left(\frac{w_1e^{-uQ_1} - w_2e^{-uQ_2}}{w_1 - w_2} \right) - uQ_1 - uQ_2 \\
&= \log\left(\frac{w_1 - w_2}{x_1 - x_2} \right) - (uQ_1 + uQ_2)\,,
\end{aligned}$$

which is well-formed as a formal power series in u, w_1, w_2, since the argument for the logarithm has constant term equal to 1. Then from (15) we have

$$\mathbf{H}_{2,1}^0 = E_{1,2} - \frac{x_2}{x_1 - x_2}, \quad \mathbf{H}_{2,2}^0 = E_{2,1} - \frac{x_1}{x_2 - x_1}, \quad \text{where} \quad E_{i,j} = \frac{w_j\mu_i}{w_i - w_j}, \tag{49}$$

for $i \neq j$, and, adding these, we obtain

$$\begin{aligned}
\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} \right)\mathbf{H}_2^0 &= E_{1,2} + E_{2,1} + 1 = \frac{w_2\mu_1}{w_1 - w_2} + \frac{w_1\mu_2}{w_2 - w_1} + 1 \\
&= \frac{w_2}{w_1 - w_2}(\mu_1 - 1) + \frac{w_1}{w_2 - w_1}(\mu_2 - 1)
\end{aligned}$$

so the result follows for $m=2$.

For $m=3$, let $x_{i,j} = x_j/(x_i - x_j)$, for $i \neq j$, and let $\sum_{i,j,k}$ denote summation over all distinct i, j, k with $1 \leqslant i, j, k \leqslant 3$. For the case $m=3$, Theorem 5.4 and (15), (49) give

$$\begin{aligned}
\mathbf{h}_3^0 &= \frac{1}{u}\int_0^u \sum_{i,j,k} \left(\frac{1}{2}(E_{i,j} - x_{i,j})(E_{i,k} - x_{i,k}) + x_{i,j}(E_{i,k} - x_{i,k}) \right) du \\
&= \frac{1}{u}\int_0^u \left(-1 + E_{1,2}E_{1,3} + E_{2,1}E_{2,3} + E_{3,1}E_{3,2} \right) du \\
&= \frac{1}{u}\int_0^u \left(-1 + \sum_{i=1}^{3} \left(\prod_{\substack{1 \leqslant j \leqslant 3 \\ j \neq i}} \frac{w_j}{w_i - w_j} \right) \mu_i^2 \right) du \\
&= \frac{1}{u}\left(-u + \sum_{i=1}^{3} \left(\prod_{\substack{1 \leqslant j \leqslant 3 \\ j \neq i}} \frac{w_j}{w_i - w_j} \right) \frac{\mu_i}{w_iQ_i'} \right)\Bigg|_0^u
\end{aligned}$$

$$= -1 + \sum_{i=1}^{3} \left(\prod_{\substack{1 \leqslant j \leqslant 3 \\ j \neq i}} \frac{w_j}{w_i - w_j} \right) \mu_i = \sum_{i=1}^{3} \left(\prod_{\substack{1 \leqslant j \leqslant 3 \\ j \neq i}} \frac{w_j}{w_i - w_j} \right) (\mu_i - 1),$$

and the result follows for $m = 3$. □

We now apply the case $m = 2$ of the previous result to transform the integrand of Theorem 5.4 to a simpler form.

Theorem 5.6 (*Genus* 0 *topological recursion, rational form*). *For* $m \geqslant 4$,

$$\mathbf{h}_m^0 = u^{2-m} \int_0^u \left(\sum_{i=1}^{m} \sum_{\substack{\{\alpha,\zeta\} \in \Omega_{m,i} \\ l(\alpha), l(\zeta) \geqslant 3}} \mu_i^2 \mathbf{h}^0_{|\alpha|,i}(u, w_\alpha) \mathbf{h}^0_{|\zeta|,i}(u, w_\zeta) \right.$$
$$\left. + \sum_{\substack{1 \leqslant k,l \leqslant m \\ k \neq l}} \frac{\mu_k^2 w_l}{w_k - w_l} \mathbf{h}^0_{m-1,k}(u, w_{\overline{\{l\}}}) \right) u^{m-3} du.$$

Proof. The result follows immediately from Theorem 5.4 and (49), since we are able to cancel the terms with denominator $x_k - x_l = w_k e^{-uQ_k} - w_l e^{-uQ_l}$. □

We now apply Theorem 5.6 in the cases of $m = 4$ and 5 parts. For $m \geqslant 3$, let $\mathbf{B}_m^{(n;k)}$ be the $m \times m$ matrix with $(1, j)$-entry equal to $w_j^n \mu_j$, $(2, j)$-entry equal to $w_j \mu_j$, $(3, j)$-entry equal to w_j^k, for $j = 1, \ldots m$, and (i, j)-entry equal to w_j^{m-i+1}, for $i = 4, \ldots, m, \quad j = 1, \ldots, m$.

Corollary 5.7.

$$\mathbf{h}_4^0 = \Delta_{4,1} \left(\frac{\det \mathbf{A}_4}{V_4} \right) - \frac{\det \mathbf{B}_4^{(2;2)}}{V_4}.$$

The proof is similar in approach to that of Corollary 5.5 and therefore omitted.

Note that the right-hand side of Corollary 5.5 with $m = 4$ appears as the "first approximation" to the series $\mathbf{h}_4^0$ in Corollary 5.7.

For $m = 5$, we have found the expressions in Theorem 5.6 to be intractable by hand, but have used Maple to carry out the integration, and obtained the following result.

Let

$$\mathbf{C}_5 = \begin{pmatrix} w_1^4\mu_1 & w_2^4\mu_2 & w_3^4\mu_3 & w_4^4\mu_4 & w_5^4\mu_5 \\ w_1^2\mu_1 & w_2^2\mu_2 & w_3^2\mu_3 & w_4^2\mu_4 & w_5^2\mu_5 \\ w_1\mu_1 & w_2\mu_2 & w_3\mu_3 & w_4\mu_4 & w_5\mu_5 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 & w_5^2 \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{pmatrix},$$

$$\mathbf{D}_5^{(n;k)} = \begin{pmatrix} w_1^n\mu_1^2 & w_2^n\mu_2^2 & w_3^n\mu_3^2 & w_4^n\mu_4^2 & w_5^n\mu_5^2 \\ w_1^k\mu_1 & w_2^k\mu_2 & w_3^k\mu_3 & w_4^k\mu_4 & w_5^k\mu_5 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 & w_5^3 \\ w_1^2 & w_2^2 & w_3^2 & w_4^2 & w_5^2 \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{pmatrix}.$$

Corollary 5.8.

$$\mathbf{h}_5^0 = \Delta_{5,1}^2\left(\frac{\det\mathbf{A}_5}{V_5}\right) - \Delta_{5,1}\left(2\frac{\det\mathbf{B}_5^{(3;3)}}{V_5} - \frac{\det\mathbf{B}_5^{(2;4)}}{V_5}\right) + \Delta_{5,2}\left(\frac{\det\mathbf{B}_5^{(2;3)}}{V_5}\right)$$
$$+\frac{\det\mathbf{C}_5}{V_5} + \frac{\det\mathbf{D}_5^{(3;1)}}{V_5} - \frac{\det\mathbf{D}_5^{(2;2)}}{V_5}.$$

Again, the right-hand side of Corollary 5.5 with $m = 5$ appears as the "first approximation" to $\mathbf{h}_5^0$ in Corollary 5.8.

The results that we have for $m = 1, \ldots, 5$ have not yet suggested a pattern that can be conjecturally generalised. This is because we have been unable to find a sufficiently uniform presentation for them, although the presentation as a sum of bialternants of very elementary matrices is suggestive.

Still, the forms that we have obtained for $\mathbf{h}_m^0$ when $m \leqslant 5$ suggest a general conjecture, stated below. We refer to this as a rational form in u, because each μ_i is an inverse linear function of u. Note that, for $\mathbf{h}_m^0$ to continue to be rational as m increases, the partial fraction expansion of the recursively formed integrand in Theorem 5.6 must continue to have vanishing coefficients for the terms that are linear in μ_i, $i = 1, \ldots, m$.

Conjecture 5.9. *For $m \geqslant 3$, $\mathbf{h}_m^0$ is a sum of terms of the following type*:

$$\Delta_{m,i_1} \ldots \Delta_{m,i_k} P_{m,i_1,\ldots,i_k},$$

where $0 \leqslant k \leqslant m-3$, $i_1 + \cdots + i_k \leqslant m-3$, and $P_{m,i_1,\ldots,i_k}$ is a homogeneous symmetric polynomial in $\mu_1, \ldots, \mu_m$ of degree $k+1$, with coefficients that are rational functions in $w_1, \ldots, w_m$ with degree of numerator minus degree of denominator equal to $i_1 + \cdots + i_k - k$. Moreover, $P_{m,i_1,\ldots,i_k}$ is a symmetric function of $w_1, \ldots, w_m$, where μ_i is considered as $\mu(w_i)$.

Note that this form specializes to the expressions above for $\mathbf{h}_3^0$, $\mathbf{h}_4^0$, $\mathbf{h}_5^0$, so the conjecture is true for the cases $3 \leqslant m \leqslant 5$.

This conjecture should be seen as the genus 0 double Hurwitz analogue of the polynomiality conjecture [21, Conjecture 1.2], proved in [24, Theorem 3.2]. As with the earlier conjecture, the form of Conjecture 5.9 suggests some geometry. For example, present in the polynomial conjecture was the dimension of the moduli space of n-pointed genus g curves; the n points corresponded to the preimages of ∞. In this case, the analogue is $m-3$, the dimension of the moduli space of m-pointed genus 0 curves; again, the m points should correspond to the preimages of ∞ (i.e. the parts of β). However, we have been unable to make precise the link to geometry.

5.5. Application: explicit formulae

As an application of the explicit formulae for $\mathbf{h}_m^0$ for small m, we now extract the appropriate coefficient to give explicit formulae for the corresponding double Hurwitz numbers. We use some standard results for symmetric functions (see for example [44]), particularly the determinantal identity

$$\frac{\det\left(w_j^{\theta_i+m-i}\right)_{m\times m}}{V_m} = \det\left(h_{\theta_i-i+j}(\mathbf{w})\right)_{m\times m}, \tag{50}$$

for non-negative integers $\theta_1,\dots,\theta_m$, where $h_k(\mathbf{w})$ is the complete symmetric function of total degree k, with generating series $\sum_{k\geqslant 0} h_k(\mathbf{w})t^k = \prod_{j=1}^m (1-w_jt^j)^{-1}$. If $\theta=(\theta_1,\dots,\theta_m)$ is a partition (where $\theta_1\geqslant\cdots\geqslant\theta_m$), then both sides of (50) give expressions for the Schur symmetric function $s_\theta(\mathbf{w})$. In the case that θ is not a partition, we shall still denote either side of (50) by $s_\theta(\mathbf{w})$.

Using multilinearity on the first row of $\det\mathbf{A}_m$, we have

$$\frac{\det\mathbf{A}_m}{V_m} = w_1\cdots w_m \sum_{r\geqslant m} a_r s_{(r-m)}(\mathbf{w}) = w_1\cdots w_m \sum_{r\geqslant m} a_r h_{r-m}(\mathbf{w}), \tag{51}$$

from (50), where

$$\mu(w) = \sum_{i\geqslant 1} a_i w^i, \tag{52}$$

and $\mu(w)$ is defined in (15). We write $\alpha\cup\beta$ for the partition with parts $\alpha_1,\dots,\alpha_m$, $\beta_1,\dots,\beta_n$, suitably reordered.

Proposition 5.10. *For $d\geqslant m\geqslant 1$ and $\alpha,\beta\vdash d$, with $\alpha=(\alpha_1,\dots,\alpha_m)$,*

$$\left[x_1^{\alpha_1}\dots x_m^{\alpha_m}u^{l(\beta)}q_\beta\right]\frac{\det\mathbf{A}_m}{V_m} = \sum \frac{l(\rho)!\prod_{j\geqslant 1}\rho_j}{|\mathrm{Aut}\,\rho|}\prod_{j=1}^m \frac{(\alpha_j-|\gamma_j|)\alpha_j^{l(\gamma_j)-1}}{|\mathrm{Aut}\,\gamma_j|},$$

where the summation is over partitions $\rho,\gamma_1,\dots,\gamma_m$, with $\rho\cup\gamma_1\cup\cdots\cup\gamma_m=\beta$, and $|\gamma_j|<\alpha_j$, $j=1,\dots,m$. (Note that $\gamma_1,\dots,\gamma_m$ can be empty, but ρ cannot.)

Proof. From (51), we have

$$\frac{\det \mathbf{A}_m}{V_m} = \sum_{r \geqslant m} a_r \sum w_1^{i_1} \cdots w_m^{i_m},$$

where $m \geqslant 1$, and the second summation is over $i_1, \ldots, i_m \geqslant 1$, with $i_1 + \cdots + i_m = r$, and a_r is defined above (52). But, applying Theorem 1.3(12) to (14), we obtain

$$\begin{aligned}\left[x^t\right] w^i &= \frac{1}{t}\left[\lambda^{t-1}\right] i\lambda^{i-1} e^{utQ(\lambda)} = \frac{i}{t}\left[\lambda^{t-i}\right] \sum_{n \geqslant 0} \frac{1}{n!}\left(ut \sum_{j \geqslant 1} q_j \lambda^j\right)^n \\ &= \frac{i}{t} \sum_{\gamma \vdash t-i} \frac{t^{l(\gamma)}}{|\mathrm{Aut}\,\gamma|} u^{l(\gamma)} q_\gamma, \quad i, t \geqslant 1,\end{aligned}$$

and from (15), we have

$$\begin{aligned}a_r &= \left[w^r\right](\mu(w) - 1) = \left[w^r\right] \sum_{i \geqslant 1} \left(u \sum_{j \geqslant 1} j q_j w^j\right)^i \\ &= \sum_{\rho \vdash r} \frac{l(\rho)! \prod_{j \geqslant 1} \rho_j}{|\mathrm{Aut}\,\rho|} u^{l(\rho)} q_\rho, \quad r \geqslant 1.\end{aligned}$$

Combining these results, we obtain

$$\left[x_1^{\alpha_1} \ldots x_m^{\alpha_m}\right] \frac{\det \mathbf{A}_m}{V_m} = \sum \frac{l(\rho)! \prod_{j \geqslant 1} \rho_j}{|\mathrm{Aut}\,\rho|} u^{l(\rho)} q_\rho \prod_{j=1}^{m} \left(\frac{i_j}{\alpha_j} \sum_{\gamma_j \vdash \alpha_j - i_j} \frac{\alpha_j{}^{l(\gamma_j)}}{\left|\mathrm{Aut}\,\gamma_j\right|} u^{l(\gamma_j)} q_{\gamma_j}\right),$$

where the summation is over ρ and $i_1, \ldots, i_m \geqslant 1$, with $\rho \vdash i_1 + \cdots + i_m$. The result follows immediately. $\square$

This result allows us to immediately give formulae for genus 0 double Hurwitz numbers when one of the partitions has two or three parts.

Corollary 5.11. *Suppose $\alpha, \beta \vdash d$, with $\alpha = (\alpha_1, \ldots, \alpha_m)$.*

(1) *If $m = 2$, then*

$$\begin{aligned}H^0_{(\alpha_1,\alpha_2),\beta} = \frac{|\mathrm{Aut}\,\beta|\, r!}{d} \sum &\frac{l(\rho)! \prod_{j \geqslant 1} \rho_j}{|\mathrm{Aut}\,\rho| \left|\mathrm{Aut}\,\gamma_1\right| \left|\mathrm{Aut}\,\gamma_2\right|} \\ &\times (\alpha_1 - \left|\gamma_1\right|)(\alpha_2 - \left|\gamma_2\right|) \alpha_1^{l(\gamma_1)-1} \alpha_2^{l(\gamma_2)-1},\end{aligned}$$

where the summation is over partitions ρ, γ_1, γ_2, with $\rho \cup \gamma_1 \cup \gamma_2 = \beta$, and $\left|\gamma_j\right| < \alpha_j$, $j = 1, 2$.

(2) *If* $m = 3$, *then*

$$H^0_{(\alpha_1,\alpha_2,\alpha_3),\beta} = |\mathrm{Aut}\,\beta|\, r! \sum \frac{l(\rho)! \prod_{j\geqslant 1} \rho_j}{|\mathrm{Aut}\,\rho|} \prod_{j=1}^{3} \frac{(\alpha_j - |\gamma_j|)\alpha_j^{l(\gamma_j)-1}}{|\mathrm{Aut}\,\gamma_j|},$$

where the summation is over partitions $\rho, \gamma_1, \gamma_2, \gamma_3$, *with* $\rho \cup \gamma_1 \cup \gamma_2 \cup \gamma_3 = \beta$, *and* $|\gamma_j| < \alpha_j$, $j = 1, 2, 3$.

Proof. From (11), we obtain

$$H^0_{(\alpha_1,\ldots,\alpha_m),\beta} = |\mathrm{Aut}\,\beta|\, r! \left[x_1^{\alpha_1} \ldots x_m^{\alpha_m} u^{l(\beta)} q_\beta\right] \mathbf{H}^0_m(x_1, \ldots, x_m).$$

Both parts of the result then follow from Proposition 5.10 and Corollary 5.5, using (48) to give the factor of d in the case $m = 2$. $\square$

In a similar, but more complicated way, it is possible to obtain explicit formulae for $H^0_{(\alpha_1,\ldots,\alpha_m),\beta}$ in the cases $m = 4, 5$, using multilinearity to expand the determinants that arise in Corollaries 5.7 and 5.8.

We conclude this subsection with a conjecture for $H^0_{\alpha,\beta}$ where α, β satisfy a particular relation.

Conjecture 5.10. If $\alpha, \beta \vdash d$ and $\beta_1, \ldots, \beta_n > \alpha_2 + \cdots + \alpha_m$, then for $m \geqslant 2$,

$$H^0_{\alpha,\beta} = (m+n-2)!\, d^{m-2} \alpha_1^{n-1}.$$

We include an algebraic proof of this conjecture for the case m=3 using Corollary 5.5 to demonstate the methodology for obtaining explicit results from this lemma. The same methodology may be used to prove the conjecture for m=4 and m=5 by Corollaries 5.7 and 5.8, respectively, (and, of course, routinely for m=2). These cases are left to the reader. Shadrin [52] has very recently proved this conjecture geometrically.

We prove Conjecture 5.10 is true for $m = 3$. From (11) $H^0_{\alpha,\beta} = |\mathrm{Aut}\beta| r! \left[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} u^{l(\beta)} q_\beta\right] H^0_3$, where $H^0_3 \equiv \mathbf{H}^0_3(x_1, x_2, x_3)$, $r = m + n - 2$, from (7), and $H^0_3 = |\mathbf{A}_3|/V_3$ from Corollary 5.5. Expanding the numerator by Laplace's expansion for the set of rows containing μ_i's (there is only one such row in this case), we have

$$H^0_3 = \frac{(\mu_1 - 1)w_2 w_3}{(w_1 - w_2)(w_1 - w_3)} + \frac{(\mu_2 - 1)w_1 w_3}{(w_2 - w_1)(w_2 - w_3)} + \frac{(\mu_3 - 1)w_1 w_2}{(w_3 - w_1)(w_3 - w_2)}.$$

A meaning is attached to $(w_i - w_j)^{-1}$ where $1 \leqslant i < j \leqslant 3$ by imposing the total order $w_3 \prec w_2 \prec w_1$ upon the indeterminates, and then defining this expression to be $w_i^{-1}(1 - w_j/w_i)^{-1}$. This then defines a formal power series ring in w_3 with coefficients that are formal Laurent series in w_1, w_2 (see Xin [61]). We shall work in this ring without further comment and deal with each of the three terms in turn.

From the expression for $[x^t]w^i$ in the proof of Proposition 5.10

$$\left[x_3^{\alpha_3} u^{l(\beta)} q_\beta\right] w_3^i = \frac{i}{\alpha_3} \sum_{\gamma \vdash \alpha_3 - i} \frac{\alpha_3^{l(\gamma)}}{|\mathrm{Aut}\gamma|} [u^{l(\beta)} q_\beta] u^{l(\gamma)} q_\gamma.$$

But $|\gamma| < \alpha_3 < \beta_1, \ldots, \beta_n$, so γ is the null partition, and we may therefore set $q_i = 0$ for all $i \geqslant 1$ in (14), whence $w_3 = x_3$, and $\mu_3 = 1$ from (15). Thus $H^0_{\alpha,\beta} = |\mathrm{Aut}\beta| r! \left[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} u^{l(\beta)} q_\beta\right] f$ where

$$f = (\mu_1 - 1)\frac{w_2}{w_1}\frac{x_3}{w_1}\left(1 - \frac{w_2}{w_1}\right)^{-1}\left(1 - \frac{x_3}{w_1}\right)^{-1} + (\mu_2 - 1)\frac{x_3}{w_2}\left(\frac{w_2}{w_1} - 1\right)^{-1}\left(1 - \frac{x_3}{w_2}\right)^{-1}.$$

Let a and b denote the first and second terms, respectively, on the right hand side of this expression.

Now, $\left[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} u^{l(\beta)} q_\beta\right] b = \left[x_1^{\alpha_1} x_2^{\alpha_2} u^{l(\beta)} q_\beta\right] (\mu_2 - 1)\left(\frac{w_2}{w_1} - 1\right)^{-1} w_2^{-\alpha_3}$. But, by an argument similar to that given above, we have $w_2 = x_2$ and $\mu_2 = 1$, since $\alpha_2 + \alpha_3 < \beta_1, \ldots, \beta_n$. Thus the contribution of b to $H^0_{\alpha,\beta}$ is 0.

Finally, $\left[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} u^{l(\beta)} q_\beta\right] a = \left[x_1^{\alpha_1} u^{l(\beta)} q_\beta\right] (\mu_1 - 1) w_1^{-(\alpha_2+\alpha_3)}$. But $x_1 \partial w_1/\partial x_1 = w_1 \mu_1$ from (15) and $\left[x_1^{\alpha_1}\right] x_1 \partial/\partial x_1 = \alpha_1 [x_1^{\alpha_1}]$, so $\left[x_1^{\alpha_1} u^{l(\beta)} q_\beta\right] (\mu_1 - 1) w_1^{-(\alpha_2+\alpha_3)} = \left[x_1^{\alpha_1} u^{l(\beta)} q_\beta\right] (-d) w_1^{-(\alpha_2+\alpha_3)}/(\alpha_2+\alpha_3)$, and the result follows from the above mentioned expression in the proof of Proposition 5.10.

5.6. Positive genus: A topological recursion for $\mathbf{H}^g_m$ and explicit formulae

In the following result, we apply the symmetrization operator Θ_m to the join-cut equation, to obtain a partial differential equation for $\mathbf{H}^g_m$, for genus $g \geqslant 1$. As in the case of genus 0, the change of variables transforms the partial differential operator applied to $\mathbf{H}^g_m$ into the linear differential operator in the intrinsic variable u. Consequently, we are able to express the transformed series

$$\mathbf{h}^g_m(u, w_1, \ldots, w_m) = \Gamma \mathbf{H}^g_m(x_1, \ldots, x_m)$$

as an integral in u.

Theorem 5.12 (*Topological recursion in positive genus*). (1) *For* $g \geqslant 1$,

$$\mathbf{h}^g_1 = \frac{u^{1-2g}}{2} \int_0^u \left(\sum_{j=1}^{g-1} \mathbf{h}^j_{1,1}(u, w_1) \mathbf{h}^{g-j}_{1,1}(u, w_1) + w_2 \frac{\partial}{\partial w_2} \mathbf{h}^{g-1}_{2,1}(u, w_1, w_2)\bigg|_{w_2 = w_1} \right) \times \mu_1^2 u^{2g-2} du.$$

(2) *For* $m \geqslant 2$ *and* $g \geqslant 1$,

$$\mathbf{h}^g_m = u^{2-m-2g} \int_0^u \left(\sum_{i=1}^{m} \sum_{\substack{\{\alpha,\zeta\} \in \Omega_{m,i} \\ l(\alpha), l(\zeta) \geqslant 3}} \mu_i^2 \left(\mathbf{h}^0_{|\alpha|,i}(u, w_\alpha) \mathbf{h}^g_{|\zeta|,i}(u, w_\zeta) \right.\right.$$

$$+\mathbf{h}^g_{|\alpha|,i}(u,w_\alpha)\mathbf{h}^0_{|\zeta|,i}(u,w_\zeta)\Big) + \sum_{j=1}^{g-1}\sum_{i=1}^{m}\sum_{\{\alpha,\zeta\}\in\Omega_{m,i}} \mu_i^2 \mathbf{h}^j_{|\alpha|,i}(u,w_\alpha)\mathbf{h}^{g-j}_{|\zeta|,i}(u,w_\zeta)$$

$$+\sum_{\substack{1\leqslant k,l\leqslant m\\ k\neq l}} \frac{\mu_k^2 w_l}{w_k-w_l}\mathbf{h}^g_{m-1,k}(u,w_{\overline{\{l\}}})$$

$$+\frac{1}{2}\sum_{i=1}^{m}\mu_i^2\left(w_{m+1}\frac{\partial}{\partial w_{m+1}}\mathbf{h}^{g-1}_{m+1,i}(u,w_1,\ldots,w_{m+1})\right)\Bigg|_{w_{m+1}=w_i}\Bigg) u^{m+2g-3}du.$$

In both parts of this result, the integration is carried out with $w_1,\ldots,w_m$ regarded as constants.

We call this a topological recursion because it expresses $\mathbf{h}^g_m$ in terms of $\mathbf{h}^{g'}_{m'}$, where $g'\leqslant g$ and $m'\leqslant m+1$, and either $g'<g$ or $m'<m$.

Remarks. (1) Note that the case $m=1$ is different.

(2) The exponents of u have geometric meaning; this is no coincidence.

(3) This result specializes to the rough form of the genus 0 topological recursion (Theorem 5.4), by taking $g=0$ and $\mathbf{h}^{-1}=0$, after minor manipulation.

Proof. By applying Θ_1 and $[y^g]$ to (9) for fixed $g\geqslant 1$ we find that $\mathbf{H}^g_1(x_1)$ satisfies

$$\left(u\frac{\partial}{\partial u}+2g-1\right)\mathbf{H}^g_1(x_1)=\frac{1}{2}\sum_{j=0}^{g}\mathbf{H}^j_{1,1}(x_1)\mathbf{H}^{g-j}_{1,1}(x_1)+\frac{1}{2}\,x_2\frac{\partial}{\partial x_2}\mathbf{H}^{g-1}_{2,1}(x_1,x_2)\Big|_{x_2=x_1}.$$

Now move the terms $j=0$ and g in the summation on the right-hand side of this equation to the left-hand side, and change variables by applying the operator identity (16), and part 1 of the result follows from Lemma 5.3.

By applying Θ_m and $[y^g]$ to (9) for fixed $m\geqslant 2$ and $g\geqslant 1$ we find that $\mathbf{H}^g_m(x_1,\ldots,x_m)$ satisfies

$$\begin{aligned}\left(u\frac{\partial}{\partial u}+m+2g-2\right)\mathbf{H}^g_m &= \sum_{j=0}^{g}\sum_{i=1}^{m}\sum_{\{\alpha,\zeta\}\in\Omega_{m,i}}\mathbf{H}^j_{|\alpha|,i}(x_\alpha)\mathbf{H}^{g-j}_{|\zeta|,i}(x_\zeta)\\ &\quad+\sum_{\substack{1\leqslant k,l\leqslant m\\ k\neq l}}\frac{x_l\mathbf{H}^g_{m-1,k}(x_{\overline{\{l\}}})}{x_k-x_l}\\ &\quad+\frac{1}{2}\sum_{i=1}^{m}\left(x_{m+1}\frac{\partial}{\partial x_{m+1}}\mathbf{H}^{g-1}_{m+1,i}(x_1,\ldots,x_{m+1})\right)\Bigg|_{x_{m+1}=x_i}.\end{aligned}$$

Now move the contribution of $\{\alpha,\zeta\}\in\Omega_{m,i}$ when $j=0$ or $j=g$, and $l(\alpha)=1$ or $l(\zeta)=1$, in the first summation on the right-hand side of this equation to the left-hand side, and apply (49) to cancel the terms with denominator $x_k - x_l$ on the right-hand side. Then apply Γ, using the operator identity (16), and part 2 of the result follows from Lemma 5.3. $\square$

The topological recursion may be used to give explicit formulae for $\mathbf{H}_m^g$. The cases $g = 1$ and $m = 1, 2$ are given below. We omit the derivation (which is similar in spirit to that for the genus 0 formulae), and simply report the result.

Corollary 5.13.

$$\mathbf{h}_1^1 = u\mu_1 w_1 \frac{\partial}{\partial w_1} Q_1 + u^2\mu_1^3 \left(\left(w_1 \frac{\partial}{\partial w_1} \right)^2 Q_1 \right)^2 + u\mu_1^2 \left(w_1 \frac{\partial}{\partial w_1} \right)^3 Q_1,$$

$$\begin{aligned} \mathbf{h}_2^1 = &\frac{1}{24} \left(x_1 \frac{\partial}{\partial x_1} \right)^2 \left(\frac{w_2}{w_1 - w_2} w_1 \frac{\partial}{\partial w_1} \log \mu_1 \right) \\ &+ \frac{1}{24} \left(x_2 \frac{\partial}{\partial x_2} \right)^2 \left(\frac{w_1}{w_2 - w_1} w_2 \frac{\partial}{\partial w_2} \log \mu_2 \right) \\ &- \frac{1}{24} \Delta_{2,1} \left(\frac{w_2}{w_1 - w_2} \mu_1 + \frac{w_1}{w_2 - w_1} \mu_2 \right) \\ &+ \frac{1}{48} \Delta_{2,1}^2 \left(w_1 \frac{\partial}{\partial w_1} \frac{w_2}{w_1 - w_2} + w_2 \frac{\partial}{\partial w_2} \frac{w_1}{w_2 - w_1} \right). \end{aligned}$$

A similar equation for $\mathbf{h}_1^1$ has also been derived from the genus expansion ansatz, in (42). Of course, these expression agree, by carrying out the differentiations in Corollary 5.13.

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