Eigenvalue, Quadratic Programming, and Semidefinite Programming Bounds for Vertex Separators

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We consider the problem of partitioning the node set of a graph into k sets of given sizes in order to minimize the cut between all vertices in distinct sets i,j <k. This problem is closely related to the graph partitioning problem and has many applications including to chip design, parallel computing, network partitioning, floor planning and solving linear systems with sparse symmetric matrices. To elaborate on one application, suppose we are given a sparse symmetric matrix of size nxn; we want to find $m_1, m_2, .. m_k$ summing to n such that the given matrix can be block diagonalized into matrices of size m_i via a permutations of its rows and columns. This problem is also related to the clustering problem in solving TSP. We look at known bounds and derive new bounds obtained from various relaxations for this NP-hard problem. This includes: the standard eigenvalue bound, projected eigenvalue bounds using both the adjacency matrix and the Laplacian[1], quadratic programming (QP) bounds based on recent successful QP bounds for the quadratic assignment problems, and semidefinite programming bounds.

We prove the negativity and thus redundancy of the standard eigenvalue bound as well as the guaranteed improvement by using the projected eigenvalue bound. To show that the eigenvalue bound is negative we use the Hoffman-Weilandt theorem and the fact that the trace of one of the matrices is 0. To show that the projected eigenvalue bound is strictly better than the basic eigenvalue bound, we use the Rayleigh Principle as well as properties of the Laplacian. We also conjecture that the projected eigenvalue bound holds with equality for a class of structured graphs and prove this is true when all the m_i are equal.

We prove the theoretical result that the extreme points of the doubly stochastic type n x k matrices are partition matrices. This result has practical uses, for instance it allows us to use an efficient LP formulation to find the closest partition matrix to the solution to our quadratic objective. This method of finding the closest partition matrix provides strong bounds and is inexpensive to evaluate. Furthermore. we develop a new quadratic objective function using the adjacency matrix instead of the Laplacian. Having done thousands of experimental tests our new quadratic objective always gave better results than the old one using the Laplacian. We also explicitly calculate the solution to the linear part of our objective which has a very nice combinatorial explanation: Given a partition of the vertices $v_1, v_2, ...v_n$ the linear piece splits the edges evenly between each of the S_i, i.e., we have a form of expected value. This also allows us to compute the solution to the linear piece of our objective in constant time as opposed to solving an LP. By doing this we were able to get some large scale numerics on random graphs.

We derive the SDP formulation by Lagrangian duality showing it is independent of a relevant parameter d and also show the degeneracy of this Semi-definite Program and create an improved final SDP relaxation through facial reduction techniques. We answer the question posed by Wolkowicz and Zhao in [2] that the two constraints D_o (Y) and D_e (Y) are in fact redundant in the facially reduced SDP. We did this by looking at the Schur complement and noticing that our matrix constraints were Positive Semidefinite and applying a result in [2] We also show how to generate a partition matrix given a feasible solution to our SDP. This we do by showing that the first row of our SDP matrix is actually a partition matrix.

We also derive a QP formulation of the vertex separator from our SDP via a suitable relaxation and prove that it is at least as good, but not always strictly better than the projected eigenvalue bound. Finally we present the projected eigenvalue bound for sparse matrices.

Further work will involve the implementation of our algorithms within a branch and bound method. It may be interesting to see if one could prove that the projected eigenvalue bound using the adjacency matrix is always better than the bound using the Laplacian. In our quadratic formulations of Graph Partitioning we tried using the adjacency matrix and the Laplacian. In fact the diagonal does not matter. It is an interesting question however to choose the best diagonal entries for our objective function. For graph partitioning this was done in[3] for the basic eigenvalue bound using a zero duality gap property in quadratic programming.[3] It is nontrivial and interesting to generalize this to the projected eigenvalue bound or to the vertex separator problem.

[1] F. Rendl, A. Lisser, and M. Piacentini. Bandwidth, vertex separators and eigenvalue optimization. In Discrete Geometry and Optimization, volume 69 of The Fields Institute for Research in Mathematical Sciences, Communications Series, pages 249–263. Springer, 2013.
[2] H. Wolkowicz and Q. Zhao. Semidefinite programming relaxations for the graph partitioning problem. Discrete Appl. Math., 96/97:461–479, 1999. Selected for the special Editors' Choice, Edition 1999.
[3] Kurt Anstreicher and Henry Wolkowicz. On lagrangian relaxation of quadratic matrix constraints. SIAM J. Matrix Anal, 1998.