Unit 1

1.1 Consider the following expression:

\[ \text{There exists a real number } x \text{ such that for every } y \in \{-1, 1\} \text{ the equation } x^4 + y^4 = 1 \text{ holds} \]

Questions:

- Can you write this expression symbolically, without using any words or ‘\(\neg\)’ symbol?
- Can you write the negation of this expression symbolically, without using any words or ‘\(\neg\)’ symbol?
- Is this a *mathematical statement*, an *open sentence*, or *neither*?
- If this is a mathematical statement, is it *true* or *false*?
- If this is a true mathematical statement, can you prove it? If this is a false mathematical statement, can you disprove it?
- If this is an open sentence, can you explain what variables it depends on?
- What would happen if we change the order of quantifiers as follows:

\[ \text{For every } y \in \{-1, 1\}, \text{ there exists a real number } x \text{ such that the equation } x^4 + y^4 = 1 \text{ holds} \]

1.2 Consider the following statement:

\[ \text{Every odd integer can be written either in the form } 4k + 1 \text{ or in the form } 4k - 1 \text{ for some integer } k \]

A student attempted to write this expression symbolically as follows:

\[ \forall n \in \mathbb{O}, \forall k \in \mathbb{Z}, (n = 4k + 1) \lor (n = 4k - 1) \]

- Is this symbolic expression correct? If the answer is “yes”, explain why. If the answer is “no”, explain what mistakes were made and how would you fix them.
- Can you find at least two different ways of expressing the given statement symbolically, without using any words or ‘\(\neg\)’ symbol?
Unit 2

2.1 Consider the logical expression \((A \land B) \Rightarrow (B \lor C)\).

Questions:
- What is the truth table for this logical expression?
- What is its negation?
- Is this logical expression logically equivalent to \(A \Rightarrow C\)? If the answer is “yes”, can you prove it by a) using truth tables; and by b) using properties of boolean algebra? If the answer is “no”, can you find \(A, B, C\) where the two statements are different?
- Can you write down some other variation of this expression that is still equivalent to the original one?
- Can you prove that the original expression is not logically equivalent to \((A \lor B) \Rightarrow (B \land C)\)?

2.2 Consider the following statement:

*Andrés can get his favourite candy by doing the dishes and by doing his homework*

Questions:
- Can you write this statement in the form of a logical expression?
- Consider the statement

  *If Andrés did the dishes but still did not receive his favourite candy, then he must have not done his homework*

  Is this statement logically equivalent to the original statement?
- Can you write down some other variation of this expression that is still equivalent to the original one?
- What is the negation of this statement?
- What is the contrapositive of this statement?
- What is the converse of this statement?
Unit 3

3.1 Consider the following true mathematical statement:

\[
\text{For all integers } a \text{ and } b, \text{ if } 8 \mid (a^2 + b^2 - 1) \text{ then } a \text{ is even or } b \text{ is even}
\]

• Can you write this statement symbolically?
• Can you provide examples of \(a\) and \(b\) that make the hypothesis of an implication true? Use this example to demonstrate that the conclusion is also true.
• Can you provide examples of \(a\) and \(b\) that make the hypothesis of an implication false?
• What are possible strategies that you can think of for proving this statement? Discuss their advantages and disadvantages.
• Can you prove this statement?
• Is the converse of this statement true or false? If it is “true”, can you prove it? If it is “false”, can you provide a counter example?
• Can this statement be turned into an if and only if statement?

3.2 Consider the following false statement:

\[
\text{For all positive odd integers } a \text{ and } b, \text{ if } a \neq 1 \text{ then } a \nmid (2b - 4) \text{ or } a \nmid (3b - 9)
\]

• Can you find a counter example which disproves this statement?
• Can you find a condition on \(a\) that can be put in the statement below such that a) the hypothesis can be made true for at least one choice of \(a\); and b) that would make the entire statement true?

\[
\text{For all positive odd integers } a \text{ and } b, \text{ if } a \neq 1 \text{ and } \text{ then } a \nmid (2b - 4) \text{ or } a \nmid (3b - 9)
\]

• Can you find a condition on \(a\) that can be put in the statement below that would make the entire statement true?

\[
\text{For all positive odd integers } a \text{ and } b, a \neq 1 \text{ and } \text{ if and only if } a \nmid (2b - 4) \text{ or } a \nmid (3b - 9)
\]
Unit 4

4.1 Consider the sum
\[ \sum_{n=-3}^{3} (2^{n+3} + 1) \]
• What is this sum equal to?
• How would you change the orders of summation so to make this sum equal to 19?
• Is the sum
\[ \sum_{n=-1}^{8} (2^{n+2} + 1) \]
a result of reindexing the sum \( \sum_{n=-3}^{3} (2^{n+3} + 1) \)? Explain why or why not.

4.2 We say that a positive number \( n \) is *triangular* if there exists an integer \( k \) such that \( n = \frac{k(k+1)}{2} \). The first five triangular numbers are 1, 3, 6, 10, 15. Consider the following true statement:

For every \( n \in \mathbb{N} \), the sum of the first \( n \) triangular numbers is equal to \( \frac{n(n+1)(n+2)}{6} \)
• How would you write this statement symbolically using the summation notation?
• When proving this statement by induction, what method would you choose? The Principle of Mathematical Induction or the Principle of Strong Induction?
• Do you need one base case or many base cases? How would you prove them?
• What statement has to be proved on the inductive step?
• Can you prove this statement?
Unit 5

5.1 An integer \( n \) is called a perfect cube if there exists an integer \( \ell \) such that \( n = \ell^3 \).
   Let \( S \) denote the set of all odd integers that are also perfect cubes.
   
   • Can you write this statement using the set builder notation in three different ways?
   • Can you write it as an intersection of two sets?

5.2 Consider the following three sets:

\[
A = \{n \in \mathbb{Z}: n \text{ is odd}\}, \quad B = \{4k + 3: k \in \mathbb{Z}\}, \quad C = \{m \in \mathbb{Z}: 4 \mid (m - 1)\}
\]

• Can you give examples of 3 elements in each of these sets?
• Can you write the following sets using set builder notation:

\[
\begin{align*}
\overline{A} & \quad \overline{B} & \quad \overline{C} \\
\overline{A} \cup \overline{B} & \quad \overline{A} \cup \overline{C} & \quad \overline{B} \cup \overline{C} \\
\overline{A} \cap \overline{B} & \quad \overline{A} \cap \overline{C} & \quad \overline{B} \cap \overline{C} \\
\overline{A} \cup \overline{B} & \quad \overline{A} \cup \overline{C} & \quad \overline{B} \cup \overline{C} \\
\overline{A} \cap \overline{B} & \quad \overline{A} \cap \overline{C} & \quad \overline{B} \cap \overline{C}
\end{align*}
\]

• What are the relations between \( A \), \( B \) and \( C \)? Is \( A \subseteq B \) or \( B \subseteq A \)? Is \( A \subseteq C \) or \( C \subseteq A \)? Is \( B \subseteq C \) or \( C \subseteq B \)? If one is a subset of the other, can you prove that it is a proper subset or that the two sets are equal?

• What should you do to sets \( B \) and \( C \) so to make them equal to \( A \)?