From Pythagoras to Einstein: The Geometry of Space and Time

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“There is geometry in the humming of the strings, there is music in the spacing of the spheres.”

attributed to Pythagoras (570 - 495 BC)

“The laws of nature are but the mathematical thoughts of God.”

Euclid (323 - 283 BC)

“The Cosmos is all that is or ever was or ever will be. Our feeblest contemplations of the Cosmos stir us - there is a tingling in the spine, a catch in the voice, a faint sensation, as if a distant memory, of falling from a height. We know we are approaching the greatest of mysteries.”

Carl Sagan (1934 - 1996)
Euclidean Geometry

Pythagoras (570 - 495 BC) and Euclid (323 - 283 BC)
The Rule of Pythagoras

Rule of Pythagoras: $ds^2 = dx^2 + dy^2$. 
Using the rule of Pythagoras twice, we get
\[ ds^2 = h^2 + dz^2 = dx^2 + dy^2 + dz^2. \]
The Rule of Pythagoras in 4 Dimensions

\[ \vec{v} = (dx, dy, dz, dt) \]

Similarly, following the pattern, in 4 dimensions we get

\[ ds^2 = dx^2 + dy^2 + dz^2 + dt^2. \]

There is nothing different about any of these 4 dimensions, unlike space and time which are clearly different. How can we modify the geometry to make 1 dimension different from the other 3?
Special Relativity results from studying the geometry of the Minkowski spacetime, where the lengths of vectors are defined by

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2.$$ 

Notice that there is clearly something different about one of the dimensions now. But are we allowed to do this? Sure, why not!
Riemannian Geometry

Carl Friedrich Gauss (1777 - 1855) and Bernhard Riemann (1826 - 1866)
Spheres are Intrinsically 2 Dimensional

Gauss and Riemann realized that the geometry of a sphere or any other surface may be described by looking at a collection of flat maps of the surface, called an atlas. For example, if you want to drive to the Grand Canyon, you might use an atlas of the US to plan your trip. However, maps usually distort distances somewhat: Greenland and Antarctica are not as big as they appear in Mercator projections of the Earth, as seen on the right.
Since maps typically distort distances, the shortest distance between two points is not necessarily a straight line on the map. The true length of a curve may be computed by integrating the lengths of the velocity vectors to the curve. Gauss’s key insight was realizing that all of the geometry intrinsic to the surface was determined by knowing the length of every vector.
The Geometry of the Unit Sphere

This modified “Rule of Pythagoras” captures the geometry of the unit sphere $S^2$. What other geometries might there be?
Other Geometries

Well-known examples of other geometries include:

\[ ds^2 = dx^2 + dy^2 \]  (Flat 2D Euclidean space)
\[ ds^2 = dx^2 + \sin^2(x)dy^2 \]  (The sphere of radius 1)
\[ ds^2 = R^2 dx^2 + R^2 \sin^2(x)dy^2 \]  (The sphere of radius R)
\[ ds^2 = dx^2 + \sinh^2(x)dy^2 \]  (Hyperbolic space)
\[ ds^2 = (dx^2 + dy^2)/y^2 \]  (also Hyperbolic space)

What about:

\[ ds^2 = 4dx^2 + 9dy^2 = (2dx)^2 + (3dy)^2 = dX^2 + dY^2 \]
\[ ds^2 = a \, dx^2 + b \, dy^2 + c \, dz^2 + d \, dt^2 \]  (depends ... )

If \( a, b, c, d > 0 \), then we have 4D Euclidean space again. But if \( a, b, c > 0 \) and \( d < 0 \) then we get the Minkowski spacetime:

\[ ds^2 = 4dx^2 + 9dy^2 + 16dz^2 - 25dt^2 = (2dx)^2 + (3dy)^2 + (4dz)^2 - (5dt)^2 = dX^2 + dY^2 + dZ^2 - dT^2 \]
The Geometry of the Minkowski Spacetime

The Minkowski spacetime has 3 types of vectors: spacelike, timelike, and null. The geometry of the null cone naturally divides timelike vectors into future and past components.
The Geometry of the Minkowski Spacetime

Principle: Geometric quantities correspond to physical observables. For example, the length of your *world line*, which always goes in future timelike directions, equals the time you experience.
The 1+1 Dimensional Minkowski Spacetime

future timelike vectors \((ds^2 < 0)\)

null vectors \((ds^2 = 0)\)

spacelike vectors \((ds^2 > 0)\)

past timelike vectors \((ds^2 < 0)\)

Let’s remove the \(y\) and \(z\) directions for now. The null cone is now really a null “X.”
The 1+1 Dimensional Minkowski Spacetime

- Future timelike vectors ($ds^2 < 0$)
- Past timelike vectors ($ds^2 < 0$)
- Null vectors ($ds^2 = 0$)
- Spacelike vectors ($ds^2 > 0$)

What do “rotations” (which fix the origin and $ds^2$ for all vectors) look like in this new geometry?

**Vector Equation:**
$$\vec{v} = (dx, dt)$$

**Minkowski Metric:**
$$ds^2 = dx^2 - dt^2$$

**Distance:**
$$ds = \sqrt{|ds^2|}$$
Rotations (left diagram) and Boosts (right diagram)

Analogous to rotations in Euclidean space, *boosts* are linear transformations of the Minkowski spacetime which fix the origin and preserve the dot product between every pair of vectors.
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A: B has velocity $5/13$.
A: L has velocity 1.

B: A has velocity $-5/13$ (guess).
B: L has velocity ... hmmm.

Suppose A, B, and L all start at the origin and then travel in straight lines as shown. How fast does L appear to be going according to B?
Relative Velocity

A: B has velocity 5/13.  
A: L has velocity 1.

B: A has velocity -5/13.  
B: L has velocity 1.

The trick is to rotate, or more precisely boost, the coordinates to make the answer to the problem clear. In this case, boost the coordinate chart so that B is now going in the (0, 12) direction.
Relative Velocity

A: B has velocity $\frac{5}{13}$.  
A: L has velocity 1.

B: A has velocity $-\frac{5}{13}$.  
B: L has velocity 1.

Both A and B agree that L has speed 1, even though A and B are moving with respect to one another. The geometry of the Minkowski spacetime requires that there be a special speed, namely speed 1, that is observed to be the same by all observers.
Relative Velocity

A: B has velocity $5/13$.
A: L has velocity $1$.

B: A has velocity $-5/13$.
B: L has velocity $1$.

In 1887, the Michelson-Morley experiment determined that the speed of light, $c$, was the same in every reference frame. This result is consistent with the geometry of the Minkowski spacetime if we simply define $c = 1$. Distance and time then have the same units.
The Constancy of the Speed of Light

Boosts, by definition, preserve $ds^2$. Hence, null directions must be transformed into other null directions, which all have speed 1. Thus, if we define $c = 1$, the speed of light will be observed to be the same by all observers.
The Twin “Paradox”

Principle: Geometric quantities correspond to physical observables. For example, the length of your *world line*, which always goes in future timelike directions, equals the time you experience.
The Twin “Paradox” - just the geometry of spacetime

The **longest** distance between 2 points is a straight line! In this example, the twin who stays on Earth experiences 26 years while the twin who goes at velocity \(\frac{5}{13}\), then velocity \(-\frac{5}{13}\), experiences only 24 years. An astronaut going close to the speed of light following the green curve would experience almost no time.
The Minkowski spacetime has, in some sense, 3 different kinds of spheres, depending on whether the distance squared from the origin is positive, negative, or zero, as drawn on the right. For comparison, the unit sphere of Euclidean space is drawn on the left.
The Unit “Spheres” of the Minkowski Spacetime

Just as the unit sphere in Euclidean space has constant curvature, the blue and red hypersurfaces drawn on the right also have constant curvature. Can you guess the geometry of each blue hypersurface?

\[ ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \]

\[ ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \]
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Hyperbolic Space

Hyperbolic space is the spacelike unit sphere of the Minkowski spacetime. Because of the symmetries (both rotations and boosts) of the Minkowski spacetime, every point and every direction of hyperbolic space is the same. Hence, it has constant curvature.
By symmetry, the red curve has constant curvature, just like a circle does in Euclidean space. Physically, this represents constant acceleration.

\[ \alpha(s) = \left( \frac{1}{a} \cosh(as), \frac{1}{a} \sinh(as) \right) \]
\[ \alpha'(s) = (\cosh(as), \sinh(as)) \]
\[ |\alpha'(s)| = 1 \]

Thus, \( s \) is the time experienced by the astronaut.

For small \( s \), we recover the Newtonian analogue:

\[ d = \frac{1}{a} (\cosh(as) - 1) \approx \frac{1}{a} \frac{1}{2} (as)^2 = \frac{1}{2} as^2. \]
By symmetry, the red curve has constant curvature, just like a circle does in Euclidean space. Physically, this represents \textit{constant acceleration}.

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\[ ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \]

For large \( s \), though, the distance traveled grows \textit{exponentially}!

\[ d = \frac{1}{a} (\cosh(as) - 1) \approx \frac{1}{2a} e^{as} \]
So, if you were abducted by aliens after this lecture, how far away from Earth could they take you in your lifetime?
• The star Proxima Centauri is about 4 light years away.
• The Milky Way is about 100,000 light years in diameter.
• The Andromeda galaxy is about 2,500,000 light years away.
• The edge of the observable universe is about 45,000,000,000 light years away.
Alien Abduction

Assuming Special Relativity, nothing can accelerate past the speed of light. So even if the aliens had a space craft that could go close to the speed of light, they could not take you farther than 100 light years from Earth ...
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If the aliens accelerate their spacecraft much faster than \(1g = 9.8 m/s^2\), they might kill you. Let’s assume the aliens want to keep you alive and accelerate their spacecraft at precisely \(1g\), and that they can do this for as long as they like.
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An interesting factoid: $\frac{(1g) \cdot (1\, \text{year})}{g} \approx c = 1$.

Thus, $\frac{1}{g} \approx 1\, \text{year} = 1\, \text{light year}$.
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Thus, $\frac{1}{g} \approx 1 \text{ year} = 1 \text{ light year}.$

Plugging into our constant acceleration formula, we get …
Alien Abduction

\[ d - \text{distance traveled}; \quad s - \text{time experienced by the travelers} \]

\[ \frac{d}{1 \text{ light year}} = \cosh \left( \frac{s}{1 \text{ year}} \right) - 1 \]

Some sample approximate values:

<table>
<thead>
<tr>
<th>( s ) (in years)</th>
<th>( d ) (in light years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
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<tr>
<td>5</td>
<td>75</td>
</tr>
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<td>10</td>
<td>10,000</td>
</tr>
<tr>
<td>15</td>
<td>1,500,000</td>
</tr>
<tr>
<td>20</td>
<td>250,000,000</td>
</tr>
<tr>
<td>25</td>
<td>35,000,000,000</td>
</tr>
<tr>
<td>30</td>
<td>5,000,000,000,000</td>
</tr>
</tbody>
</table>
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Practically anywhere, if you eat right and exercise!
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Practically anywhere, if you eat right and exercise!

25 years to accelerate, 25 years to decelerate, and then you would be at rest, 72 billion light years from Earth!
Meanwhile, 72 billion years have passed on the Earth.
Meanwhile, 72 billion years have passed on the Earth.

In effect, a spaceship which can accelerate at 1g for as long as you like is both a tool for space exploration and a time machine. If such spaceships exist someday, space explorers might head out in various directions and agree to meet back at Earth, 1 million or 1 billion years in the future.
Everything we’ve discussed so far is Special Relativity, based on the flat geometry of the Minkowski spacetime

\[ ds^2 = dx^2 + dy^2 + dz^2 - dt^2, \]

which came from changing a sign in the rule of Pythagoras.

What about other spacetime metrics, like

\[ ds^2 = f(x, y, z, t)(dx^2 + dy^2 + dz^2) - h(x, y, z, t)dt^2, \]

for some functions \( f \) and \( h \)?

What happens when we remove the assumption that the spacetime metric is flat?
General Relativity

Principle: Geometric quantities correspond to physical observables. What does the curvature of a spacetime correspond to? Einstein’s great idea, which he called his happiest thought, can be summed up in 3 words:
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“Matter curves spacetime.”

That is, the curvature of spacetime corresponds to matter density.

Tomorrow we’ll explore how matter curves spacetime in more detail, and explore the question of dark matter in the universe.
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Tomorrow we’ll explore how matter curves spacetime in more detail, and explore the question of dark matter in the universe. For now, let’s review the successes of general relativity.
The Earth goes around the Sun because the mass of the Sun curves spacetime, not because of some mysterious $1/r^2$ force law assumed as an axiom without any explanation as to what the mechanism for gravity might be.
Newtonian physics predicts a precession of $1.5436^\circ$ per century, not $1.5556^\circ$ per century, observed since Verrier in 1859. In 1915, Einstein showed that General Relativity gets it right.
Successes of General Relativity: Black Holes

Artist’s rendition of a black hole. Einstein was surprised when Schwarzschild found an exact solution to the highly nonlinear Einstein vacuum equations in 1915. Einstein spent the rest of his life believing that black holes, while existing in his theory, did not actually exist in nature. The idea seemed too radical at the time.
The supermassive black hole Sagittarius A* (4 million solar masses) at the center of the Milky Way Galaxy. The first black hole ever observed, Cygnus X - 1, was discovered in 1970. Today it is believed that most large galaxies have supermassive black holes at their centers.
General Relativity predicts twice the bending angle for light that Newtonian physics predicts and agrees with observations, as observed by Eddington in 1919, on an island off the west coast of Africa during a solar eclipse.
Successes of General Relativity: Gravitational Lensing

After Eddington, Einstein becomes a celebrity, still the only scientist to receive a ticker tape parade in NYC, as he did in 1921. Still, it’s not like he won the Super Bowl or anything ...
Successes of General Relativity: The Big Bang
Big Bang cosmologies predicted by General Relativity were discovered in 1922 by Alexander Friedmann (left, who died in 1925) and independently in 1927 by George Lemaitre (middle), years before Edwin Hubble’s (right) landmark discovery of the expanding universe in 1929.
Successes of General Relativity: The *Accelerating Expansion* of the Universe
Successes of General Relativity: The Accelerating Expansion of the Universe

Saul Perlmutter

Brian P. Schmidt

Adam G. Riess

The Nobel Prize in Physics 2011 was awarded “for the discovery of the accelerating expansion of the Universe through observations of distant supernovae” with one half to Saul Perlmutter and the other half jointly to Brian P. Schmidt and Adam G. Riess.
The Mass of the Universe

70%  Dark Energy
     (the cosmological constant of General Relativity used to explain the observed accelerating expansion of the universe)

25%  Dark Matter

5%   Regular Baryonic Matter
     (Gas, Dust, Planets, Stars, etc., composed of particles described by the Standard Model of Particle Physics and Quantum Field Theory)

Which theory best describes Dark Matter?
There is roughly five times more dark matter in the universe than regular baryonic matter represented by the periodic table.

Also, most of the mass of galaxies is dark matter.

1. What is the nature of dark matter?

2. Does dark matter have something to do with spiral structure in galaxies?
The Puzzle of the Spirals

“Much as the discovery of these strange forms may be calculated to excite our curiosity, and to awaken an intense desire to learn something of the laws which give order to these wonderful systems, as yet, I think, we have no fair ground even for plausible conjecture.”

Lord Rosse (1850)

“A beginning has been made by Jeans and other mathematicians on the dynamical problems involved in the structure of the spirals.”

Curtis (1919)

“Incidentally, if you are looking for a good problem...”

Feynman (1963)
The Puzzle of the Spirals

“The old puzzle of the spiral arms of galaxies continues to taunt theorists. The more they manage to unravel it, the more obstinate seems the remaining dynamics. Right now, this sense of frustration seems greatest in just that part of the subject which advanced most impressively during the past decade - the idea of Lindblad and Lin that the grand bisymmetric spiral patterns, as in M51 and M81, are basically compression waves felt most intensely by the gas in the disks of those galaxies. Recent observations leave little doubt that such spiral “density waves” exist and indeed are fairly common, but no one still seems to know why.

To confound matters, not even the $N$-body experiments conducted on several large computers since the late 1960s have yet yielded any decently long-lived regular spirals.”

Toomre (1977)
Spiral Galaxy M81
Spiral Galaxy M74
Spiral Galaxy NGC1365
Spiral Galaxy NGC4622
Spiral Galaxy M51, the Whirlpool Galaxy
Spiral Galaxies 2MASX J00482185-2507365
Spiral Galaxy NGC3314
Spiral Galaxies ARP274
Elliptical Galaxies

Figure: Elliptical galaxies contain ellipsoidal shaped collections of stars in mostly radial orbits. Two examples are M87 (left) and NGC1132 (right).
Galaxy Cluster MS1054-0321

Ground + X-ray

HST

Distant Galaxy Cluster MS1054-0321
Hubble Space Telescope • Wide Field Planetary Camera 2

PRC98-26 • August 19, 1998 • STScI OPO • M. Donahue (STScI) and NASA
The Bullet Cluster
SPIRALS

Figure: From the Dark Matter Awareness Week presentation. Presentation review at arXiv:1102.1184v1 by Paolo Salucci, Christiane Frigerio Martins, and Andrea Lapi.
What about Dark Matter and Spiral Galaxies?

On April 24, 2014, I gave a talk in Rochester: the three main ideas of my talk were:

**Idea 1**: Natural geometric axioms motivate studying the Einstein-Klein-Gordon equations with a cosmological constant. Is the scalar field of the Klein-Gordon equation dark matter?

**Idea 2**: Wave types of equations, such as the Klein-Gordon equation, naturally form density waves in their matter densities.

**Idea 3**: Density waves in dark matter, through gravity, naturally form density waves in the regular baryonic matter. Does this explain the observed spiral density waves in spiral galaxies?

You can find the slides of the talk at [http://www.math.duke.edu/~bray/Presentations/WingLecture2.pdf](http://www.math.duke.edu/~bray/Presentations/WingLecture2.pdf)
Spiral Galaxy Simulation #1

NGC1300 on the left, simulation on the right.
Spiral Galaxy Simulation #2

NGC4314 on the left, simulation on the right.
Spiral Galaxy Simulation #3

NGC3310 on the left, simulation on the right.
Spiral Galaxy Simulation #4

NGC488 on the left, simulation on the right.