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The Generalized Plasma in One Dimension: Evaluation of a Partition Function

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Abstract. Forrester and Jancovici have given sum rules for a two-dimensional generalized plasma with two species of particles interacting through logarithmic potentials with three independent coupling constants. They have also found a specific one-dimensional solvable model which satisfies the analogs of their sum rules. A class of one-dimensional models for which the partition function is evaluable is given as well as a more general result evaluating multidimensional integrals.

1. Introduction

Forrester and Jancovici [1] have given an exactly solved model for a one-dimensional generalized plasma with two species of particles (roman and greek), interacting through logarithmic potentials, with three independent coupling constants. This was motivated by their discovery of sum rules for such a generalized plasma in two dimensions (see also Halperin [2] and Girvin [3]) and the desire to at least verify the one-dimensional analogs of these rules.

The two-dimensional system with roman and greek particles of density ϱ_R and ϱ_G, respectively, and independent coupling constants g_{RR}, g_{RG}, g_{GG} has Hamiltonian

$$\begin{aligned} H = & -g_{RR}\sum_{i>j}\ln r_{ij} - g_{GG}\sum_{\alpha>\beta}\ln r_{\alpha\beta} - g_{RG}\sum_{i,\alpha}\ln r_{i\alpha} \\ & + (g_{RR}\varrho_R + g_{RG}\varrho_G)\sum_i \int \ln|\mathbb{R}_i - \mathbb{R}|d\mathbb{R} \\ & + (g_{GG}\varrho_G + g_{RG}\varrho_R)\sum_\alpha \int \ln|\mathbb{R}_\alpha - \mathbb{R}|dr \\ & - (\tfrac{1}{2}g_{RR}\varrho_R^2 + \tfrac{1}{2}g_{GG}\varrho_G^2 + g_{RG}\varrho_R\varrho_G)\int \ln|\mathbb{R} - \mathbb{R}'|d\mathbb{R}d\mathbb{R}', \end{aligned} \tag{1.1}$$

where the particle-background and background-background interactions have been chosen in a way which compensates the remote particle-particle interactions

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so that we can expect the system to have a well-behaved thermodynamic limit. Forrester and Jancovici's exactly solved model is for the system where all particles lie on a circle of radius R:

$$z_j = \mathrm{Re}^{i\theta_j}, \qquad z_\beta = \mathrm{Re}^{i\theta_\beta},$$

and $g_{RR} = g_{RG} = 2$, $g_{GG} = 4$. We shall consider the more general case of this one-dimensional model,

$$g_{RR} = g_{RG} = 2y, \qquad g_{GG} = 2y+2, \qquad y \in \mathbb{N}.$$

The excess partition function is

$$Z = (2\pi)^{-(a+b)N} R^{-(ya+(y+1)b)N} \prod_{j=1}^{aN} \int_0^{2\pi} d\theta_j \prod_{\alpha=1}^{bN} \int_0^{2\pi} d\theta_\alpha \\ \times \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^{2y} \prod_{\alpha<\beta} |e^{i\theta_\alpha} - e^{i\theta_\beta}|^{2y+2} \prod_{j,\alpha} |e^{i\theta_j} - e^{i\theta_\alpha}|^{2y}. \tag{1.2}$$

We shall show that

$$Z = \frac{(ayN + b(y+1)N)!\, bN!\, R^{-(ya+(y+1)b)N}}{y!^{aN}(y+1)!^{bN}\left(\dfrac{ayN}{y+1}+1\right)_{bN}}, \tag{1.3}$$

where $(y)_n = y(y+1)\dots(y+n-1)$.

2. A General Identity

Since $|e^{i\theta_j} - e^{i\theta_k}|^2 = (1 - e^{i(\theta_j - \theta_k)})(1 - e^{i(\theta_k - \theta_j)})$, the partition function Z as defined in Eq. (1.2) is simply the power of R times the constant term in

$$\prod_{j<k}\left(1-\frac{x_j}{x_k}\right)^y\left(1-\frac{x_k}{x_j}\right)^y \prod_{\alpha<\beta}\left(1-\frac{x_\alpha}{x_\beta}\right)^{y+1}\left(1-\frac{x_\beta}{x_\alpha}\right)^{y+1} \prod_{j,\alpha}\left(1-\frac{x_j}{x_\alpha}\right)^y\left(1-\frac{x_a}{x_j}\right)^y.$$

Equation (1.3) follows from a proposition proved in a more general setting by the authors [4].

Proposition 2.1. *Let $a_1, \dots, a_n$ be positive integers, A be an arbitrary subset of $\{(i,j)\colon 1 \leqq i < j \leqq n\}$, $\mathfrak{S}_A$ be the set of permutations on $\{1, \dots, n\}$ whose inversions are contained in A:*

$$\mathfrak{S}_A = \{\sigma \in \mathfrak{S}_n\colon i<j \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j) \text{ implies } (i,j) \in A\},$$

and let $\chi(T)$ be the characteristic function which is 1 if T is true, 0 otherwise. If [1] denotes the constant term in the succeeding expression, then

$$[1] \prod_{1 \leqq i < j \leqq n}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j - \chi((i,j)\notin A)} \\ = \frac{(a_1+a_2+\dots+a_n)!}{(a_1-1)!(a_2-1)!\dots(a_n-1)!} \sum_{\sigma \in \mathfrak{S}_A} \prod_{l=1}^{n} \frac{1}{(a_{\sigma(1)}+\dots+a_{\sigma(l)})}. \tag{2.1}$$

Equation (1.3) is the special case of this proposition where $n=aN+bN$, $A=\{(i,j)\colon\ 1\leqq i<j\leqq aN \text{ or } aN+1\leqq i<j\leqq aN+bN\}$, $\mathfrak{S}_A\simeq\mathfrak{S}_{aN}\times\mathfrak{S}_{bN}$, and $a_1=\ldots=a_{aN}=y$, $a_{aN+1}=\ldots=a_{aN}+_{bN}=y+1$.

For $A=\{(i,j)\colon 1\leqq i<j\leqq n\}$, one observes that

$$\sum_{\sigma\in\mathfrak{S}_n}\prod_{l=1}^{n}\frac{1}{(a_{\sigma(1)}+\ldots+a_{\sigma(l)})}=\frac{1}{a_1a_2\cdot\ldots\cdot a_n}. \tag{2.2}$$

This can be proved by induction on n. For the inductive step, we observe that the only term involving $\sigma(n)$ is

$$(a_{\sigma(1)}+\ldots+a_{\sigma(n)})^{-1}=(a_1+\ldots+a_n)^{-1},$$

which can be factored out of the summation. The sum is now rewritten as a sum over possible images of n of the sum over all permutations of the remaining $n-1$ elements.

Proposition 2.1 thus implies an identity conjectured by Dyson [5] and proved by Gunson [6] and Wilson [7]:

$$[1]\prod_{1\leqq i<j\leqq n}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j}=\frac{(a_1+a_2+\ldots+a_n)!}{a_1!\,a_2!\ldots a_n!}. \tag{2.3}$$

In fact, the proof of Proposition 2.1 will essentially follow Good's proof [8] of Eq. (2.3).

3. Proof

To prove this proposition, we shall need a lemma. Let T be a tournament (a complete directed graph) on n vertices,

$$T\subseteqq\{(i,j)\colon 1\leqq i\neq j\leqq n\},\qquad (i,j)\in T\Leftrightarrow (j,i)\notin T.$$

We say T is *transitive* if it contains no cycles and thus corresponds to a permutation, σ, of $\{1,\ldots,n\}$, where $\sigma(i)$ is the vertex with in-degree $i-1$.

Lemma 3.1. *Let $a=(a_1,\ldots,a_n)$, T be a tournament on n vertices, then*

$$[1]\prod_{(i,j)\in T}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-1}=\begin{cases}\dfrac{(a_1+\ldots+a_n)!}{(a_1-1)!\ldots(a_n-1)!}\displaystyle\prod_{i=1}^{n}\frac{1}{a_{\sigma(1)}+\ldots+a_{\sigma(i)}}, & \text{if } T \text{ is transitive},\\ 0, & \text{otherwise}.\end{cases} \tag{3.1}$$

Proof of Lemma. By the Lagrange interpolation formula (see Good [8]) we have that

$$1=\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{1-\dfrac{x_i}{x_j}}. \tag{3.2}$$

Let $C(T, a) = [1] \prod_{(i,j)\in T}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-1}$. Multiplying both sides of Eq. (3.2) by $\prod_{(i,j)\in T}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-1}$ and taking constant terms yields

$$C(T, a) = \sum_{i=1}^{n} C(T, a-\delta_i), \tag{3.3}$$

where δ_i is the unit vector in the i^{th} direction. We also have the initial conditions that if $a_k = 0$, then

$$C(T, a) = \begin{cases} 1, & \text{if } k=n=1, \\ C(T\backslash k, a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n), & \text{if vertex } k \text{ has in-degree } 0, \\ 0, & \text{otherwise}, \end{cases} \tag{3.4}$$

where $T\backslash k$ means the tournament obtained by removing vertex k and all incident edges. It is readily verified that the right-hand side of Eq. (3.1) also satisfies this recurrence (3.3) and set of initial conditions (3.4). □

Proof of Proposition. We begin the proof of Eq. (2.1) by observing that

$$\prod_{i<j}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-\chi((i,j)\notin A)} = \prod_{i<j}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-1}\prod_{(i,j)\in A}\left(1-\frac{x_j}{x_i}\right). \tag{3.5}$$

We now consider the formal expansion of

$$\prod_{(i,j)\in A}\left(1-\frac{x_j}{x_i}\right).$$

For each pair $(i, j) \in A$, choosing the first term, 1, will leave the product to the left unchanged. Choosing the second term, $-x_j/x_i$, yields

$$\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-1}\left(-\frac{x_j}{x_i}\right) = \left(1-\frac{x_j}{x_i}\right)^{a_j}\left(1-\frac{x_i}{x_j}\right)^{a_i-1}. \tag{3.6}$$

Thus, choosing the second term has the affect of reversing the order of i and j in the corresponding term of the product. Thus we get that

$$\prod_{i<j}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-\chi((i,j)\notin A)} = \sum_{T}\prod_{(i,j)\in T}\left(1-\frac{x_i}{x_j}\right)^{a_i}\left(1-\frac{x_j}{x_i}\right)^{a_j-1}, \tag{3.7}$$

where the sum is over all tournaments on n vertices such that if $(i, j) \in T$ and $j < i$ then $(j, i) \in A$. The proposition follows by taking the constant term of each side of Eq. (3.7) and using Lemma 3.1.

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