Bernoulli Trials Problems for 2017

1: For all integers $n \geq 2$, we have $n = 3 \bmod 6$ if and only if $n^2 + 2^n$ is prime.

2: For all real numbers x and y with $2 \leq x \leq y$, we have $y^{x+1} \leq x\, y^y$.

3: For every sequence $\{a_n\}_{n\geq 1}$ of real numbers, if $\sum\limits_{n=1}^{\infty} a_n$ converges then $\sum\limits_{n=1}^{\infty} {a_n}^3$ converges.

4: There exist 6 open discs in $\mathbf{R}^2$ such that each disc contains the point $(0,0)$ and no disc contains another disc's centre

5: There exist 99 lines in $\mathbf{R}^2$ such that for all $k, l \in \{1, 2, \cdots, 100\}$, one of the lines passes through the interior of the square with vertices at (k,l), $(k-1,l)$, $(k-1,l-1)$ and $(k,l-1)$.

6: For every positive integer n there exist matrices $A, B \in M_n(\mathbf{R})$ such that

$$\left\{X \in M_n(\mathbf{R}) \middle| AX = XA \text{ and } BX = XB\right\} = \left\{cI \middle| c \in \mathbf{R}\right\}.$$

7: When we rearrange the alternating harmonic series so that each positive term is followed by 4 negative terms, as shown below, the resulting series converges and its sum is zero.

$$\tfrac{1}{1} - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{3} - \tfrac{1}{10} - \tfrac{1}{12} - \tfrac{1}{14} - \tfrac{1}{16} + \tfrac{1}{5} - \tfrac{1}{18} - \cdots = 0.$$

8: There exists a bijection $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + f(n)}$ converges.

9: There exists a bijection $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ such that for every $n \in \mathbf{Z}^+$ we have $n \,\Big|\, \sum\limits_{k=1}^{n} f(k)$.

10: There exist disjoint nonempty homeomorphic sets A and B with $A \cup B = [0,1]$.

11: Define $F : \mathbf{Z}[x] \to [0,1]$ as follows. Given a polynomial $f \in \mathbf{Z}[x]$, let $F(f)$ be the real number x with decimal representation $x = 0.a_1a_2a_3\cdots$ where $a_k \in \{0, 1, 2, \cdots, 9\}$ with $a_k = f(k) \bmod 10$. Then the range of F contains less than 12,536 elements.

12: There exists a 20-element set $S \subseteq \mathbf{Z}_{210}$ such that $S + S = \mathbf{Z}_{210}$.

13: Five regular tetrahedra with unit side length can be arranged in space so that they all share a common edge and are otherwise disjoint.

14: Nine points can be arranged on the unit sphere so that each of the 9 points has exactly 4 equidistant nearest neighbours.

15: There exist infinitely many primes p with the property that for all $a, b \in \mathbf{Z}^+$ with $a < b$ and $\gcd(a,b) = 1$ such that a, b and p are distinct modulo p, the exponent of p in the prime factorization of $b^{p-1} - a^{p-1}$ is odd.