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A COMBINATORIAL APPLICATION OF MATRIX RICCATI EQUATIONS AND THEIR q-ANALOGUE

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The generating functions for a large class of combinatorial problems involving the enumeration of permutations may be expressed as solutions to matrix Riccati equations. We show that the generating functions for the permutation problem in which the number of inversions is also preserved form a system of matrix Riccati equations in which the differential operator is the Eulerian differential operator. We obtain the classical result of MacMahon concerning permutations.

1. Introduction

The purpose of this paper is to give a new application of matrix Riccati equations involving differential and other operators. The application concerns an area of combinatorial theory which deals with the enumeration of combinatorial structures. The particular structure we consider here is the sequence.

We show that the generating functions $y_t^s(x)$, for $0 \leq s \leq Q-1$ and $0 \leq t \leq P-1$, associated with a large class of permutation enumeration problems, satisfy a system of non-linear ordinary differential equations of the form

$$\mathscr{D}\mathbf{Y}+\mathbf{YA}+\mathbf{YBY}-\mathbf{C}+\mathbf{DY}=\mathbf{0} \tag{1.1}$$

with initial condition $\mathbf{Y}(0)=\mathbf{0}$, where $[\mathbf{Y}]_{s,t}=y_t^s(x)$, and $\mathscr{D}$ denotes $\mathrm{d}/\mathrm{d}x$. In addition, we show that $\mathbf{Y}$ satisfies a system of linear algebraic equations of the form

$$\mathbf{UY}=\mathbf{V} \tag{1.2}$$

in which the elements of $\mathbf{U}$ and $\mathbf{V}$ satisfy the differential equation

$$\mathscr{D}^{\alpha}u=\beta u \tag{1.3}$$

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for some fixed β and fixed positive integer α, and are hence obtained in closed form. Equation (1.2) was previously obtained using a purely algebraic argument by Jackson and Goulden [7] and Gessel [6] attributes a number of expressions of this type to Stanley [12]. A particular case has been given by Carlitz and Scoville [2].

We also show that related permutation and sequence enumeration problems correspond to q-analogues and backward difference analogues of (1.1), (1.2) and (1.3). In these, $\mathscr{D}$ is replaced by the Eulerian differential operator $\mathscr{E}$ (defined in Section 6) and the backward difference operator ∇ (defined in Section 7). For clarity, we refer to (1.1) as a *matrix Riccati $\mathscr{D}$-equation*, and its q-analogue and ∇-analogue as a *matrix Riccati $\mathscr{E}$-equation* and a *matrix Riccati ∇-equation*, respectively.

The interest in permutation and sequence enumeration is explained by the observation that sequences may be used for encoding combinatorial structures. Thus, particular aspects of combinatorial structures may be examined by means of the theory of sequence enumeration. A number of approaches to an algebraic theory of sequence enumeration has been adopted by several authors (e.g. [3, 4, 5, 6, 8, 11]).

It is perhaps unexpected that a class of discrete problems may be transformed into a system of differential equations involving continuous variables. An explanation of this in the present context is given in Section 3.

Let $\{0,1\}^*$ denote the sequence monoid on $\{0,1\}$ with concatenation. If $\boldsymbol{S}, \boldsymbol{T} \subseteq \{0,1\}^*$ then $\boldsymbol{ST}$ denotes $\{\alpha\beta \mid \alpha \in \boldsymbol{S}, \beta \in \boldsymbol{T}\}$, and $\boldsymbol{S}^*$ denotes $\bigcup_{k=0}^{\infty} \boldsymbol{S}^k$, where $\boldsymbol{S}^0 = \{\boldsymbol{\varepsilon}\}$ and $\boldsymbol{\varepsilon}$ is the null sequence. The enumeration problems we consider involve sequences with periodic shape. A sequence $\sigma_1 \cdots \sigma_l$ on $\mathscr{N}_n = \{1, \ldots, n\}$ has *shape* $k_1 \cdots k_{l-1} \in \{0,1\}^*$ if $\sigma_i < \sigma_{i+1}$ when $k_i = 0$ and $\sigma_i \geqslant \sigma_{i+1}$ when $k_i = 1$ (so 1423 has shape 010). We refer to the determination of the generating function for the number of sequences with shape in $\boldsymbol{S} \subseteq \{0,1\}^*$ as the (*enumeration*) *problem* $\boldsymbol{S}$. When $\boldsymbol{S} = \boldsymbol{S}_1 \boldsymbol{W}^* \boldsymbol{S}_2$ for some $\boldsymbol{S}_1$, $\boldsymbol{W}$, $\boldsymbol{S}_2 \subseteq \{0,1\}^*$ we say that the problem $\boldsymbol{S}$ is a *periodic* problem.

Let $c_n(\boldsymbol{S})$, where $\boldsymbol{S} \subseteq \{0,1\}^*$, be the number of permutations in $\mathscr{N}_n$ with shape in $\boldsymbol{S}$ and $c_{n,m}(\boldsymbol{S})$ be the number of these with m inversions, where an *inversion* in σ is a pair (i,j) such that $\sigma_i > \sigma_j$ and $i<j$ (so 1423 has two inversions). Let

$$\left.\begin{aligned} \Phi(\boldsymbol{S}; x) &= \sum_{n \geqslant 1} c_n(\boldsymbol{S}) \frac{x^n}{n!} \\ \Phi_q(\boldsymbol{S}; x) &= \sum_{n \geqslant 1} d_n(\boldsymbol{S}) \frac{x^n}{n!_q} \end{aligned}\right\} \tag{1.4}$$

where

$$d_n(\boldsymbol{S}) = \sum_{m \geqslant 0} c_{n,m}(\boldsymbol{S}) q^m, \qquad n!_q = (1-q)(1-q^2) \cdots (1-q^n),$$

and x and q are indeterminates. Then Φ and Φ_q are the *exponential* and *Eulerian* *generating functions* for the problem $\boldsymbol{S}$. We usually denote $\Phi(\boldsymbol{S}; x)$ by $\Phi(\boldsymbol{S})$ for brevity.

Let $\mathbf{i} = (i_1, \ldots, i_n)$ and $c_{\mathbf{i}}(\boldsymbol{S})$ be the number of sequences over $\mathcal{N}_n$ with shape in $\boldsymbol{S}$ and with i_j occurrences of j, for $j = 1, \ldots, n$. The *ordinary generating function* for $\boldsymbol{S}$ is

$$\Psi_n(\boldsymbol{S}; \mathbf{x}) = \sum_{\mathbf{i}} c_{\mathbf{i}}(\boldsymbol{S}) x_1^{i_1} \cdots x_n^{i_n},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ are indeterminates. We denote $\Psi_n(\boldsymbol{S}; \mathbf{x})$ by $\Psi_n(\boldsymbol{S})$.

Sections 3, 4 and 5 deal with the determination of $\Phi(\boldsymbol{S})$ both for the periodic case and for the arbitrary case. Sections 6 and 7 contain extensions of this material for the determination of $\Phi_q(\boldsymbol{S}; x)$ and $\Psi_n(\boldsymbol{S})$, respectively. Section 8 gives an application of the material of Sections 5, 6 and 7 to the derivation of MacMahon's result [9] and its generalisations.

2. Historical background

In 1881, André [1], in his early investigations into the combinatorial properties of Eulerian numbers, considered the problems $(01)^*$ and $(01)^*0$. He termed permutations with these shapes *alternating*. It may be shown by the method of Section 3 that $u(x) = \Phi((01)^*)$ and $v(x) = \Phi((01)^*0)$ satisfy a system of non-linear differential equations

$$\mathcal{D}u = u^2 + 1,$$
$$\mathcal{D}v = uv + u$$

with initial conditions $u(0) = v(0) = 0$. This system may be solved immediately to give $\Phi((01)^*) = \tan x$ and $1 + \Phi((01)^*0) = \sec x$. Thus, the coefficient of $x^4/4!$ in $\Phi((01)^*0)$ is five and, indeed, there are five alternating permutations on $\mathcal{N}_4$, namely, 1324, 1423, 2314, 2413 and 3412.

This historical evidence of André suggests that other periodic problems may be associated with systems of non-linear differential equations and this, indeed, is the case. For example, it will be seen in Section 3 that the system

$$\begin{aligned} \mathcal{D}u &= vw + 1, \\ \mathcal{D}v &= vz + u, \\ \mathcal{D}w &= wz + u, \\ \mathcal{D}z &= z^2 + v + w \end{aligned} \tag{2.1}$$

with initial conditions $u(0) = v(0) = w(0) = z(0) = 0$ is associated with the permutation problem $\boldsymbol{W} = (0^2 1^2)^*$, and that the functions are identified, combinatorially, by $u(x) = \Phi(\boldsymbol{W})$, $v(x) = \Phi(\boldsymbol{W}0)$, $w(x) = \Phi(1\boldsymbol{W})$ and $z(x) = \Phi(1\boldsymbol{W}0)$.

This system may be rewritten as a matrix Riccati $\mathscr{D}$-equation, a fact which may be seen by setting

$$\mathbf{Y}=\begin{bmatrix} u & v \\ w & z \end{bmatrix},$$

and **A**, **B**, **C**, **D** equal to

$$-\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad -\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

respectively. It may be shown from Proposition 4.1 that

$$\left.\begin{matrix} u(x) \\ z(x) \end{matrix}\right\} = (\tan x \pm \tanh x)(1+\sec x \operatorname{sech} x)^{-1}$$

and

$$v(x)=w(x)=\tan x \tanh x\,(1+\sec x \operatorname{sech} x)^{-1}.$$

This problem has been treated by Carlitz and Scoville [2] by a method which does not exploit the connexion with matrix Riccati equations.

In the next section we consider a combinatorial construction which gives systems of differential equations for periodic permutation problems. Although the equations may be established faster by means of exponential generating functions, the familiar approach which we give here suffices in the present context. The construction is modified in Sections 6 and 7 to incorporate, respectively, permutations with inversions, and sequences.

3. The construction of the systems of equations for periodic permutations problems

We describe the method for constructing the systems of differential equations with reference to the particular case (namely (2.1)) given in Section 2, since generalisation to the arbitrary case is immediate. Let $\boldsymbol{W}=(0^2 1^2)^*$ and consider the permutation problem $\boldsymbol{W}0$. Let σ be an arbitrary permutation on $\mathscr{N}_n$ with shape in $\boldsymbol{W}0$. Clearly n must be greater than one. When the largest element, namely n, in σ is deleted, σ decomposes into a left fragment and a right fragment, in two distinct ways. These are, *either*

(i) the left fragment has length r and shape in $\boldsymbol{W}0$; the right fragment has length $n-1-r$ and shape in $1\boldsymbol{W}0$; *or*

(ii) the left fragment has length $n-1$ and shape $\boldsymbol{W}$; the right fragment is empty.

The contribution of (i) to $c_n(\boldsymbol{W}0)$ is $c_r(\boldsymbol{W}0)c_{n-1-r}(1\boldsymbol{W}0)$. Moreover, the set on which the left fragment is constructed may be chosen in $\binom{n-1}{r}$ distinct ways, since the largest element, n, is excluded from this selection. The set on which the right fragment is constructed is then determined. The contribution of (ii) is treated similarly. Thus

$$c_n(\mathbf{W}0) = \sum_{r=0}^{n-1} \binom{n-1}{r} c_r(\mathbf{W}0) c_{n-r-1}(1\mathbf{W}0) + c_{n-1}(\mathbf{W}) \quad \text{for } n > 1$$

whence

$$\sum_{n=1}^{\infty} \frac{c_n(\mathbf{W}0)}{(n-1)!} x^{n-1} = \Phi(\mathbf{W}0)\Phi(1\mathbf{W}0) + \Phi(\mathbf{W}).$$

Finally, using the (formal) differential coefficient, we may rewrite the above equation in the form

$$\mathcal{D}\Phi(\mathbf{W}0) = \Phi(\mathbf{W}0)\Phi(1\mathbf{W}0) + \Phi(\mathbf{W}) \tag{3.1}$$

or $\mathcal{D}v(x) = v(x)z(x) + u(x)$, in the notation of Section 2. Clearly, any particular case may be treated in a similar way.

The following notational conventions are adopted. If $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size, then $[\mathbf{A} : \mathbf{B}_{.s}]_t$ denotes the matrix obtained from $\mathbf{A}$ by replacing column t of $\mathbf{A}$ by column s of $\mathbf{B}$. The Kronecker delta is denoted by δ_j^i. The unit matrix is denoted by $\mathbf{I}$.

4. A special case of the periodic permutation problem

The preliminary system we consider is based on the problem $(0^P 1^Q)^*$ where P and Q are fixed positive integers. Using the method given in Section 3 we may obtain a system of differential equations for $y_t^s = \Phi(1^s(0^P 1^Q)^* 0^t)$. This preliminary system is included here because the method for obtaining its solution may be extended to a more general result given as Theorem 5.2, and because the latter has a more difficult proof whose strategy is an amplification of the former.

Proposition 4.1. *Let* $y_t^s = \Phi(1^s(0^P 1^Q)^* 0^t)$ *and* $[\mathbf{Y}]_{st} = y_t^s$, *where* $\mathbf{Y}$ *is* $Q \times P$. *Then* $\mathbf{Y}$ *satisfies the matrix Riccati* $\mathcal{D}$*-equation*

$$\mathcal{D}\mathbf{Y} + \mathbf{YA} + \mathbf{YBY} - \mathbf{C} + \mathbf{DY} = \mathbf{0}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are $P \times P$, $P \times Q$, $Q \times P$, $Q \times Q$ *respectively, in which the* (s, t)*-elements are* $A_t^s = -\delta_{t-1}^s$, $B_t^s = -\delta_{P-1}^s \delta_t^{Q-1}$, $C_t^s = \delta_0^s \delta_t^0$ *and* $D_t^s = -\delta_t^{s-1}$, *with initial conditions* $\mathbf{Y}(0) = \mathbf{0}$. *Moreover,*

$$y_t^s = |[\mathbf{U} : \mathbf{V}_{.s}]_t| \cdot |\mathbf{U}|^{-1}$$

where

$$[\mathbf{U}]_{st} = U_t^s = \begin{cases} (-1)^{s-t}\phi^{(s-t)} & \text{for } t \leq s \\ (-1)^{Q+s-t}\phi^{(P+Q+s-t)} & \text{for } t > s \end{cases}$$

and

$$[\mathbf{V}]_{st} = V_t^s = (-1)^s \phi^{(s+t+1)}$$

in which

$$\phi^{(r)} = \sum_{k\geq 0} (-1)^{Qk} \frac{x^{(P+Q)k+r}}{((P+Q)k+r)!} \quad \textit{for } 0 \leq r < P+Q.$$

Proof. Consider an arbitrary permutation on $\mathcal{N}_n$ with shape in $(0^P 1^Q)^*$. Then by the construction of Section 3 we have directly, for $0 \leq s \leq Q-1$ and $0 \leq t \leq P-1$,

$$\mathcal{D} y_t^s = \sum_e \sum_f y_f^s \delta_{P-1}^f \delta_e^{Q-1} y_t^e + \sum_e y_t^e \delta_e^{s-1} + \sum_f y_f^s \delta_{t-1}^f + \delta_0^s \delta_t^0.$$

Thus, $\mathcal{D}\mathbf{Y} + \mathbf{YA} + \mathbf{YBY} - \mathbf{C} + \mathbf{DY} = \mathbf{0}$. This may be solved by a standard procedure (see e.g. [10]). Let $\mathbf{U}$ and $\mathbf{V}$ be respectively $Q \times Q$ and $Q \times P$ matrices such that $\mathcal{D}\mathbf{U} = \mathbf{U}(\mathbf{YB} + \mathbf{D})$ and $\mathbf{V} = \mathbf{UY}$ where (cf. (1.2)) $\mathbf{U}$ is non-singular. Accordingly, $\mathbf{U}$ and $\mathbf{V}$ satisfy

$$\mathcal{D}\mathbf{U} - \mathbf{UD} - \mathbf{VB} = \mathbf{0},$$
$$-\mathcal{D}\mathbf{V} + \mathbf{UC} - \mathbf{VA} = \mathbf{0}$$

with initial conditions $\mathbf{V}(0) = \mathbf{0}$ and, for convenience, $\mathbf{U}(0) = \mathbf{I}$. It remains to determine $\mathbf{U}$ and $\mathbf{V}$ as solutions to this system of linear equations. These equations may be rewritten

$$\mathcal{D}U_t^s + U_{t+1}^s + V_{P-1}^s \delta_t^{Q-1} = 0,$$
$$-\mathcal{D}V_t^s + V_0^s \delta_t^0 + V_{t-1}^s \qquad = 0$$

with the convention that $V_{-1}^s = U_Q^s = 0$. Accordingly, we obtain the following basic relationships from which U_t^s and V_t^s may be derived

(i) $\mathcal{D}V_0^s = U_0^s$,

(ii) $\mathcal{D}V_t^s = V_{t-1}^s$ if $t \neq 0$,

(iii) $\mathcal{D}U_t^s = -U_{t+1}^s$ if $t \neq Q-1$,

(iv) $\mathcal{D}U_{Q-1}^s = -V_{P-1}^s$.

We consider U_t^s first. By iterating (iii) we have $\mathcal{D}^Q U_0^s = (-1)^{Q-1} \mathcal{D} U_{Q-1}^s = (-1)^Q V_{P-1}^s$ from (iv). By iterating (ii) we have $\mathcal{D}^{P-1} V_{P-1}^s = V_0^s$ so, eliminating V_{P-1}^s between the two expressions we obtain $\mathcal{D}^{P+Q} U_0^s = (-1)^Q \mathcal{D} V_0^s = (-1)^Q U_0^s$ from (i). Thus U_0^s, and therefore U_t^s, satisfy the differential equation (cf. (1.3))

$$\mathcal{D}^{P+Q} w = (-1)^Q w.$$

A similar argument shows that V_t^s satisfies the same differential equation. To derive the expression for U_t^s we consider first the element U_s^s of $\mathbf{U}$. Let

$\omega_1, \ldots, \omega_{P+Q}$ satisfy $z^{P+Q} = (-1)^Q$. Thus

$$U_s^s = \sum_{j=1}^{P+Q} a_s(j) e^{\omega_j x}$$

for some set of constants $a_s(1), \ldots, a_s(P+Q)$. But $\mathbf{U}(0) = \mathbf{I}$ so $U_s^s(0) = 1$. In addition, from (iii), we have $\mathscr{D}^r U_s^s|_{x=0} = (-1)^r U_{s+r}^s|_{x=0} = 0$ if $0 < r < P+Q$. Thus $\sum_{j=1}^{P+Q} \omega_j^r a_t(j) = \delta_0^r$ for $0 < r < P+Q$. But $\sum_{j=1}^{P+Q} \omega_j^r = (P+Q)\delta_0^r$ so we may set $a_t(j) = (P+Q)^{-1}$, whence

$$U_s^s = (P+Q)^{-1} \sum_{j=1}^{P+Q} e^{\omega_j x}.$$

It follows that $U_s^s = \phi^{(0)}$. But $\mathscr{D}\phi^{(0)} = (-1)^Q \phi^{(P+Q-1)}$ and $\mathscr{D}\phi^{(r)} = \phi^{(r-1)}$, $r > 0$. Thus, from (iii) we have, for $t \leq s$, $U_t^s = (-1)^{s-t} \phi^{(s-t)}$, and for $t > s$ we have $U_t^s = (-1)^{Q+s-t} \phi^{(P+Q+s-t)}$. This completes the determination of $\mathbf{U}$. For $\mathbf{V}$ we note that, from (ii), $\mathscr{D} V_0^s = \mathscr{D}^{t+1} V_t^s$ so, from (i), $\mathscr{D}^{t+1} V_t^s = U_0^s = (-1)^s \phi^{(s)}$ whence $V_t^s = (-1)^s \phi^{(s+t+1)}$. This completes the evaluation of $\mathbf{V}$. Finally, $\mathbf{Y} = \mathbf{U}^{-1}\mathbf{V}$ so, by Cramer's Rule, we have

$$Y_t^s = |[\mathbf{U} : \mathbf{V}_{.s}]_t| \cdot |\mathbf{U}|^{-1}$$

which completes the proof. □

When $P = Q = 2$, we note that this lemma may be specialised to give the generating functions for the permutation problem $(0^2 1^2)^*$ exhibited in Section 2.

5. The general periodic permutation problem

We now consider permutations with arbitrary periodic shape

$$\Omega_{nN}^{mM} = 1^M (0^{p_{m+1}} 1^{q_{m+1}} \cdots 0^{p_K} 1^{q_K})(0^{p_1} 1^{q_1} \cdots 0^{p_K} 1^{q_K})^* (0^{p_1} 1^{q_1} \cdots 0^{p_{n-1}} 1^{q_{n-1}}) 0^N.$$

This problem entails the additional examination of permutations with shape

$$\omega_{nN}^{mM} = 1^M (0^{p_{m+1}} 1^{q_{m+1}} \cdots 0^{p_{n-1}} 1^{q_{n-1}}) 0^N.$$

The theorem which is derived demonstrates that the exponential generating function for this problem may be obtained as the solution to a pair of matrix Riccati $\mathscr{D}$-equations, and that these may be solved explicitly. One of these equations deals with periodic permutations whose shape is in the prescribed set Ω_{nN}^{mM}, while the other deals with certain permutations of fixed shape ω_{nN}^{mM}. The latter equation does not, of course, appear in Proposition 4.1 since in that case the construction of Section 3 does not involve such sequences. The equation, and its analogues, are used in Section 8.

To simplify the statement of Theorem 5.2, a number of special constants is needed. These constants, together with an indexing convention for matrices, are given in Remark 5.1. The proof of Theorem 5.2 closely follows that of Proposition 4.1 in strategy, although the details are more intricate. This connexion between the two proofs has been exploited to the extent that technical details to support an assertion in the proof of Theorem 5.2 have been omitted when these details may be provided by the reader after consulting the proof of Proposition 4.1 on the corresponding point. The following notation is used throughout the remaining portion of the paper.

Remark 5.1. Let $p_1, \ldots, p_K$, $q_1, \ldots, q_K$ be fixed positive integers and M, N be fixed non-negative integers. Let $P = \sum_{i=1}^{K} p_i$ and $Q = \sum_{i=1}^{K} q_i$. Let **A**, **B**, **C**, **D**, **E** be $P \times P$, $P \times Q$, $Q \times P$, $Q \times Q$, $Q \times P$ matrices, respectively, defined by $[\mathbf{A}]_{ij} = A_{nN}^{mM}$, and similarly for **B**, **C**, **D**, **E** with the indexing convention that

$$i = \begin{cases} M + \sum\limits_{k<m} p_k & \text{if the matrix has } P \text{ rows,} \\ M + \sum\limits_{k<m} q_k & \text{if the matrix has } Q \text{ rows,} \end{cases}$$

$$j = \begin{cases} N + \sum\limits_{k<n} p_k & \text{if the matrix has } P \text{ columns,} \\ N + \sum\limits_{k<n} q_k & \text{if the matrix has } Q \text{ columns.} \end{cases}$$

Also

$$A_{nN}^{mM} = -\delta_n^m \delta_{N-1}^{M-1}, \qquad B_{nN}^{mM} = -\delta_n^m \delta_{p_m-1}^{M} \delta_N^{q_n-1},$$

$$C_{nN}^{mM} = (\delta_k^m \delta_n^1 + \delta_{n-1}^m) \delta_0^M \delta_N^0, \qquad D_{nN}^{mM} = -\delta_n^m \delta_N^{M-1}, \qquad E_{nN}^{mM} = \delta_{n-1}^m \delta_0^M \delta_N^0.$$

The following constants are used:

$$a = N + \sum_{k=n+1}^{K} q_k, \qquad b = \sum_{k=n}^{K} q_k, \qquad s = N + \sum_{k=n+1}^{K} (p_k + q_k),$$

$$t = -(N+1) + \sum_{k=n}^{K} (p_k + q_k), \qquad r = M + \sum_{k=m+1}^{K} q_k$$

and

$$l = M + \sum_{k=m+1}^{K} (p_k + q_k).$$

Finally, the following set of matrices is used

$$U_{nN}^{mM} = (-1)^{a+r} \Xi_{l-s}, \qquad S_{nN}^{mM} = (-1)^{a+r} \xi_{l-s},$$

$$V_{nN}^{mM} = (-1)^{b+r} \Xi_{l-k}, \qquad T_{nN}^{mM} = (-1)^{b+r} \xi_{l-t}$$

where $\Xi_h = \sum_{k \geq 0} (-1)^{Qk} \xi_{(P+Q)k+h}$. The definition of ξ_h depends on the problem considered. □

Theorem 5.2. *Let* $\mathbf{Y}$ *and* $\boldsymbol{\mu}$ *be* $Q\times P$ *matrices given (see Remark* 5.1) *by* $[\mathbf{Y}]_{ij}=\Phi(\Omega_{nN}^{mM})$ *and* $[\boldsymbol{\mu}]_{ij}=\Phi(\omega_{nN}^{mM})$. *Then* $\boldsymbol{\mu}$ *and* $\mathbf{X}=\mathbf{Y}+\boldsymbol{\mu}$ *satisfy*

$$\mathscr{D}\boldsymbol{\mu}+\boldsymbol{\mu}\mathbf{A}+\boldsymbol{\mu}\mathbf{B}\boldsymbol{\mu}-\mathbf{E}+\mathbf{D}\boldsymbol{\mu}=\mathbf{0},$$

$$\mathscr{D}\mathbf{X}+\mathbf{X}\mathbf{A}+\mathbf{X}\mathbf{B}\mathbf{X}-\mathbf{C}+\mathbf{D}\mathbf{X}=\mathbf{0}$$

with initial conditions $\boldsymbol{\mu}(0)=\mathbf{Y}(0)=\mathbf{0}$. *Moreover*

$$\Phi(\omega_{nN}^{mM})=|[\mathbf{S}:\mathbf{T}_{.i}]_j|$$

and

$$\Phi(\Omega_{nN}^{mM})=|[\mathbf{U}:\mathbf{V}_{.i}]_j|\cdot|\mathbf{U}|^{-1}-\Phi(\omega_{nN}^{mM})$$

where

$$\xi_h=\begin{cases}\dfrac{x^h}{h!} & \text{if } h\geqslant 0\\ 0 & \text{if } h<0\end{cases}\quad \text{in Remark 5.1.}$$

Proof. The matrix Riccati $\mathscr{D}$-equations for $\mathbf{X}$ and $\boldsymbol{\mu}$ follow directly from the construction of Section 3. We linearise the equation for $\mathbf{X}$ by the method reported in Reid [10], and we follow the strategy employed in the proof of Proposition 4.1.

Let $\mathbf{U}$ be a $Q\times Q$ non-singular matrix, and $\mathbf{V}$ be a $Q\times P$ matrix such that $\mathscr{D}\mathbf{U}=\mathbf{U}(\mathbf{X}\mathbf{B}+\mathbf{D})$ and $\mathbf{V}=\mathbf{U}\mathbf{X}$. Then $\mathbf{U}$ and $\mathbf{V}$ satisfy the equations

$$\mathscr{D}\mathbf{U}=\mathbf{U}\mathbf{D}+\mathbf{V}\mathbf{B}\quad\text{and}\quad\mathscr{D}\mathbf{V}=\mathbf{U}\mathbf{C}-\mathbf{V}\mathbf{A}.$$

Clearly, $\mathbf{V}(0)=\mathbf{0}$, and, for definiteness $\mathbf{U}(0)=\mathbf{I}$. By repeated application of the above equation we have

$$U_{nN}^{mM}=(-1)^a\mathscr{D}^sU_{K0}^{mM}\quad\text{and}\quad V_{nN}^{mM}=(-1)^b\mathscr{D}^tU_{K0}^{mM}.$$

It follows from these, after routine calculation, that

$$\mathscr{D}^{P+Q}U_{nN}^{nN}=(-1)^QU_{nN}^{nN}.$$

Now U_{mM}^{mM} is a diagonal element of $\mathbf{U}$ so $U_{mM}^{mM}=\Xi_0$, satisfying the initial condition $\mathbf{U}(0)=\mathbf{I}$. But putting $n=m$ and $N=M$, we have $U_{mM}^{mM}=(-1)^r\mathscr{D}^lU_{K0}^{mM}$ so $U_{K0}^{mM}=(-1)^r\Xi_l$. It follows that $U_{nN}^{mM}=(-1)^{a+r}\Xi_{l-s}$, since $\Xi_{-i}=(-1)^Q\Xi_{P+Q-i}$, and consequently that $V_{nN}^{mM}=(-1)^{b+r}\Xi_{l-t}$. This completes the evaluation of $\mathbf{U}$ and $\mathbf{V}$, where $\mathbf{X}=\mathbf{U}^{-1}\mathbf{V}$.

The linearisation of the matrix Riccati $\mathscr{D}$-equation for $\boldsymbol{\mu}$ is similar. Let $\mathbf{S}$ be a $Q\times Q$ non-singular matrix and let $\mathbf{T}$ be a $Q\times P$ matrix such that $\mathscr{D}\mathbf{S}=\mathbf{S}(\boldsymbol{\mu}\mathbf{B}+\mathbf{D})$ and $\mathbf{T}=\mathbf{S}\boldsymbol{\mu}$. Then $\mathbf{S}$ and $\mathbf{T}$ satisfy the equations

$$\mathscr{D}\mathbf{S}=\mathbf{S}\mathbf{D}+\mathbf{T}\mathbf{B}\quad\text{and}\quad\mathscr{D}\mathbf{T}=\mathbf{S}\mathbf{E}-\mathbf{T}\mathbf{A}.$$

Clearly $\mathbf{T}(0)=\mathbf{0}$ and, for definiteness, we may set $\mathbf{S}(0)=\mathbf{I}$. We note that this case differs from the case for $\mathbf{X}$ only in the condition that $\mathscr{D}T_{10}^{mM}=0$. Iterating these equations as before we have

$$S_{nN}^{mM} = (-1)^a \mathscr{D}^s S_{K0}^{mM} \quad \text{and} \quad T_{nN}^{mM} = (-1)^b \mathscr{D}^T S_{K0}^{mM}.$$

Again, it follows from these after routine calculation that

$$\mathscr{D}^{P+Q} S_{nN}^{mM} = 0$$

so S_{nN}^{mM} is a polynomial in x. The initial condition is satisfied by $S_{mM}^{mM} = 1$. Setting $n = m$ and $N = M$ we have $S_{mM}^{mM} = (-1)^r \mathscr{D}^l S_{K0}^{mM}$ so $S_{K0}^{mM} = (-1)^r \xi_l$, since the constants of integration are zero, to satisfy the initial conditions. It follows that $S_{nN}^{mM} = (-1)^{a+r} \xi_{l-s}$ and $T_{nN}^{mM} = (-1)^{b+r} \xi_{l-t}$, so the condition $\mathscr{D} T_{10}^{mM} = 0$ is satisfied. This completes the evaluation of $\mathbf{S}$ and $\mathbf{T}$ where $\boldsymbol{\mu} = \mathbf{S}^{-1}\mathbf{T}$. The result follows by Cramer's Rule. □

6. The periodic permutation problem for inversions

The purpose of this section is to derive $\Phi_q(\boldsymbol{W}; x)$ where $\boldsymbol{W}$ is an arbitrary set of periodic shapes. Our reason for doing this is that it provides us with a means for obtaining q-analogues of Theorem 5.2. We may proceed as for $\Phi(\boldsymbol{W})$, although a slight adjustment to the argument of Section 3 is necessary. The following observations are required (see, for example [8]).

(i) Let σ be a permutation on $\mathcal{N}_n$. If n occurs in position i of σ then n contributes $n-i$ to the number of inversions in σ. (Thus 1432 has 2 + number of inversions in 132 = 2 + 1 + number of inversions in 12 = 3.)

(ii) Let $I(\alpha, \beta)$ be the number of inversions (i, j) in any permutation $\sigma_1 \cdots \sigma_n = \sigma = \sigma'\sigma''$ on $\mathcal{N}_n$ such that σ' is a sequence over $\alpha \subseteq \mathcal{N}_n$, $|\alpha| = r$, σ'' is a sequence over $\beta = \mathcal{N}_n - \alpha$ and $\sigma_i \in \alpha$, $\sigma_j \in \beta$. Then

$$\sum_{\substack{\alpha \subseteq \mathcal{N}_n \\ |\alpha| = k}} q^{I(\alpha, \beta)} = \binom{n}{k}_q = n!_q / k!_q (n-k)!_q,$$

the Gaussian coefficient.

The argument of Section 3 may now be modified quite easily. By considering the contributions from cases (i) and (ii) in Section 3, and by using observations (i) and (ii) above, we have, in the notation of Section 1, with $\boldsymbol{W} = (0^2 1^2)^*$,

$$d_n(\boldsymbol{W}0) = \sum_{r=0}^{n-1} \binom{n-1}{r}_q d_r(\boldsymbol{W}0)\, d_{n-r-1}(1\boldsymbol{W}0) q^{n-r-1} + d_{n-1}(\boldsymbol{W}).$$

Thus

$$\sum_{n \geq 1} d_n(\boldsymbol{W}0) \frac{x^{n-1}}{(n-1)!_q} = \Phi_q(\boldsymbol{W}0; x) \Phi_q(1\boldsymbol{W}0; qx) + \Phi_q(\boldsymbol{W}; x).$$

For a formal power series $f(x)$ let $\mathscr{E}f(x)=x^{-1}\{f(x)-f(qx)\}$, so, in particular,

$$\mathscr{E}\frac{x^n}{n!_q}=\frac{x^{n-1}}{(n-1)!_q}.$$

$\mathscr{E}$ is called the *Eulerian differential operator* for formal power series. Accordingly, the above equation may be written

$$\mathscr{E}\Phi_q(\boldsymbol{W}0;x)=\Phi_q(\boldsymbol{W}0;x)\Phi_q(1\boldsymbol{W}0;qx)+\Phi_q(\boldsymbol{W};x).$$

This is a q-analogue of (3.1). Let $u_q(x)=\Phi_q(\boldsymbol{W};x)$, $v_q(x)=\Phi_q(\boldsymbol{W}0;x)$, $w_q(x)=\Phi_q(1\boldsymbol{W};x)$ and $z_q(x)=\Phi_q(1\boldsymbol{W}0;x)$. Then these generating functions satisfy the following system of non-linear Eulerian differential equations

$$\begin{aligned}
&\mathscr{E}u_q(x)=v_q(x)w_q(qx)+1,\\
&\mathscr{E}v_q(x)=v_q(x)z_q(qx)+u_q(x),\\
&\mathscr{E}w_q(x)=w_q(x)z_q(qx)+u_q(qx),\\
&\mathscr{E}z_q(x)=z_q(x)z_q(qx)+v_q(qx)+w_q(x)
\end{aligned}$$

with initial conditions $u_q(0)=v_q(0)=w_q(0)=z_q(0)=0$. This is a q-analogue of (2.1). We note that these equations may be rewritten in the following matrix form

$$\mathscr{E}\mathbf{Y}(x)+\mathbf{Y}(x)\mathbf{A}^{(1)}(xq)+\mathbf{Y}(x)\mathbf{B}^{(1)}\mathbf{Y}(qx)-\mathbf{C}^{(1)}+\mathbf{D}^{(1)}(x)\mathbf{Y}(qx)=\mathbf{0}.$$

Accordingly, we call this a matrix Riccati $\mathscr{E}$-equation. Its solution may be obtained as follows. Let $\mathbf{U}$ be a $Q\times Q$ non-singular matrix such that $\mathbf{V}=\mathbf{U}\mathbf{Y}$. Then $\mathbf{U},\mathbf{V}$ satisfy

$$\begin{aligned}
&\mathscr{E}\mathbf{U}(x)=\mathbf{V}(x)\mathbf{B}^{(1)}(x)+\mathbf{U}(x)\mathbf{D}^{(1)}(x),\\
&\mathscr{E}\mathbf{V}(x)=\mathbf{U}(x)\mathbf{C}^{(1)}(x)-\mathbf{V}(x)\mathbf{A}^{(1)}(x)
\end{aligned}$$

since $\mathscr{E}(\mathbf{F}(x)\mathbf{G}(x))=(\mathscr{E}\mathbf{F}(x))\mathbf{G}(qx)+\mathbf{F}(x)\mathscr{E}\mathbf{G}(x)$. Accordingly, Theorem 5.2 (and, of course, Proposition 4.1) may be generalised to the Eulerian case.

Theorem 6.1. *Let* $\mathbf{Y}$ *and* $\boldsymbol{\mu}$ *be* $Q\times P$ *matrices given (see Remark* 5.1) *by* $[\mathbf{Y}]_{ij}=\Phi_q(\Omega_{nN}^{mM})$ *and* $[\boldsymbol{\mu}]_{ij}=\Phi_q(\omega_{nN}^{mM})$. *Then* $\boldsymbol{\mu}$ *and* $\mathbf{X}=\mathbf{Y}+\boldsymbol{\mu}$ *satisfy*

$$\begin{aligned}
&\mathscr{E}\boldsymbol{\mu}(x)+\boldsymbol{\mu}(x)\mathbf{A}+\boldsymbol{\mu}(x)\mathbf{B}\boldsymbol{\mu}(qx)-\mathbf{E}+\mathbf{D}\boldsymbol{\mu}(qx)=\mathbf{0},\\
&\mathscr{E}\mathbf{X}(x)+\mathbf{X}(x)\mathbf{A}+\mathbf{X}(x)\mathbf{B}\mathbf{X}(qx)-\mathbf{C}+\mathbf{D}\mathbf{X}(qx)=\mathbf{0}
\end{aligned}$$

with initial conditions $\boldsymbol{\mu}(0)=\mathbf{Y}(0)=\mathbf{0}$. *Moreover,*

$$\Phi_q(\omega_{nN}^{mM})=|[\mathbf{S}:\mathbf{T}_{.i}]_j|$$

and

$$\Phi_q(\Omega_{nN}^{mM})=|[\mathbf{U}:\mathbf{V}_{.i}]_j|\cdot|\mathbf{U}|^{-1}-\Phi_q(\omega_{nN}^{mM})$$

where

$$\xi_h = \begin{cases} \dfrac{x^h}{h!_q} & \text{if } h \geq 0 \\ 0 & \text{if } h < 0 \end{cases} \quad \text{in Remark 5.1.}$$

Proof. The matrix Riccati $\mathscr{E}$-equations for $\mathbf{X}$ and $\boldsymbol{\mu}$ follow directly by means of the construction indicated at the beginning of this section. To linearise the equation for $\mathbf{X}$ we follow the proof of Theorem 5.2, with $\mathscr{D}$ replaced by $\mathscr{E}$. The result follows immediately. □

7. The periodic sequence problem

The combinatorial problems which we have considered in the earlier sections have been concerned with permutations, but the problems may equally well be solved for sequences in general. The purpose of this section is to derive $\Psi_n(\boldsymbol{S})$ when $\boldsymbol{S}$ is periodic. We may use the argument of Section 3 but, as was the case for its adjustment to account for inversions in Section 6, so do we now adjust it to account for sequences.

Now $\nabla\Psi_n(\boldsymbol{S}) = \Psi_n(\boldsymbol{S}) - \Psi_{n-1}(\boldsymbol{S})$ is the ordinary generating function for all sequences over $\mathcal{N}_n$ with shape in $\boldsymbol{S}$ and with at least one occurrence of the element n. We now look for the first occurrence of n from the left in each sequence and we apply the argument of Section 3, having observed that left fragments of sequences obtained during this process are, of course, sequences over $\mathcal{N}_{n-1}$. Thus, putting $\boldsymbol{W} = (0^2 1^2)^*$, we have

$$\nabla\Psi_n(\boldsymbol{W}0) = x_n\{\Psi_{n-1}(\boldsymbol{W}0)\Psi_n(1\boldsymbol{W}0) + \Psi_{n-1}(\boldsymbol{W})\}.$$

This is a ∇-analogue of (3.1). Let $u_n = \Psi_n(\boldsymbol{W})$, $v_n = \Psi_n(\boldsymbol{W}0)$, $w_n = \Psi_n(1\boldsymbol{W})$ and $z_n = \Psi_n(1\boldsymbol{W}0)$. Then these generating functions satisfy the following system of non-linear ∇-equations

$$\begin{aligned} \nabla u_n &= x_n(v_{n-1}w_n + 1), \\ \nabla v_n &= x_n(v_{n-1}z_n + u_{n-1}), \\ \nabla w_n &= x_n(w_{n-1}z_n + u_n), \\ \nabla z_n &= x_n(z_{n-1}z_n + v_n + w_{n-1}) \end{aligned}$$

with initial conditions $u_n = v_n = w_n = z_n = 0$ at $n = 0$. This is a ∇-analogue of (2.1).

We may rewrite these equations in the form

$$\nabla\mathbf{Y}_n + x_n\{\mathbf{Y}_{n-1}\mathbf{A}_n^{(2)} + \mathbf{Y}_{n-1}\mathbf{B}^{(2)}\mathbf{Y}_n - \mathbf{C}^{(2)} + \mathbf{D}_n^{(2)}\mathbf{Y}_n\} = \mathbf{0},$$

a *matrix Riccati ∇-equation*. The solution may be obtained as follows. Let $\mathbf{U}_n$ be a non-singular matrix such that $\mathbf{V}_n = \mathbf{U}_n\mathbf{Y}_n$. Then $\mathbf{U}_n$ and $\mathbf{V}_n$ satisfy

$$\nabla\mathbf{U}_n = x_n\{\mathbf{V}_{n-1}\mathbf{B}^{(2)} + \mathbf{U}_{n-1}\mathbf{D}_n^{(2)}\},$$
$$\nabla\mathbf{V}_n = x_n\{\mathbf{U}_{n-1}\mathbf{C}^{(2)} - \mathbf{V}_{n-1}\mathbf{A}_n^{(2)}\}$$

since $\nabla(\mathbf{F}_n\mathbf{G}_n) = (\nabla\mathbf{F}_n)\mathbf{G}_n + \mathbf{F}_{n-1}(\nabla\mathbf{G}_n)$. Accordingly, Theorem 5.2 may be generalised to the sequence case as follows.

Theorem 7.1. *Let* $\mathbf{Y}_J$ *and* $\boldsymbol{\mu}_J$ *be* $Q\times P$ *matrices given (see Remark* 5.1) *by* $[\mathbf{Y}_J]_{ij} = \Psi_J(\Omega_{nN}^{mM})$ and $[\boldsymbol{\mu}]_{ij} = \Psi_J(\omega_{nN}^{mM})$. *Then* $\boldsymbol{\mu}_J$ *and* $\mathbf{X}_J = \mathbf{Y}_J + \boldsymbol{\mu}_J$ *satisfy*

$$\nabla\boldsymbol{\mu}_J + x_J(\boldsymbol{\mu}_{J-1} + \boldsymbol{\mu}_{J-1}\mathbf{B}\boldsymbol{\mu}_J - \mathbf{E} + \mathbf{D}\boldsymbol{\mu}_J) = \mathbf{0},$$
$$\nabla\mathbf{X}_J + x_J(\mathbf{X}_{J-1} + \mathbf{X}_{J-1}\mathbf{B}\mathbf{X}_J - \mathbf{C} + \mathbf{D}\mathbf{X}_J) = \mathbf{0}$$

with initial conditions $\boldsymbol{\mu}_0 = \mathbf{Y}_0 = \mathbf{0}$. *Moreover,*

$$\Psi_J(\omega_{nN}^{mM}) = |[\mathbf{S}:\mathbf{T}_{.i}]_j|$$

and

$$\Psi_J(\Omega_{nN}^{mM}) = |[\mathbf{U}:\mathbf{V}_{.i}]_j| - \Psi_J(\omega_{nN}^{mM})$$

where

$$\xi_h = \sum_{1\le\sigma_1<\cdots<\sigma_h\le J} x_{\sigma_1}\cdots x_{\sigma_h} \qquad \textit{in Remark } 5.1.$$

Proof. The matrix Riccati ∇-equations for $\mathbf{X}_J$ and $\boldsymbol{\mu}_J$ follow directly from the construction given above. To linearise the equation for $\mathbf{X}$, let $\mathbf{U}_J$ be a $Q\times Q$ non-singular matrix, and $\mathbf{V}_J$ be a $Q\times P$ matrix such that $\mathbf{V}_J = \mathbf{U}_J\mathbf{X}_J$ and $\nabla\mathbf{U}_J = x_J\mathbf{U}_{J-1}\{\mathbf{X}_{J-1}\mathbf{B} + \mathbf{D}\}$. Then $\mathbf{U}_J$ and $\mathbf{V}_J$ satisfy the equations

$$\nabla\mathbf{U}_J = x_J\{\mathbf{U}_{J-1}\mathbf{D} + \mathbf{V}_{J-1}\mathbf{B}\},$$
$$\nabla\mathbf{V}_J = x_J\{\mathbf{U}_{J-1}\mathbf{C} - \mathbf{V}_{J-1}\mathbf{A}\}.$$

Let $\nabla'\mathbf{U}_J = x_{J+1}^{-1}\nabla\mathbf{U}_{J+1}$ and $\nabla'\mathbf{V}_J = x_{J+1}^{-1}\nabla\mathbf{V}_{J+1}$. It follows, after some calculation, that $\nabla'^{P+Q}\mathbf{U}_J = (-1)^Q\mathbf{U}_J$ and $\nabla'^{P+Q}\mathbf{V}_J = (-1)^Q\mathbf{V}_J$. The proof now follows from that of Theorem 5.2 with $\mathscr{D}$ replaced by ∇'. □

8. Permutations and sequences with arbitrary shape

Theorem 5.2 may be used immediately to give a well-known result due to MacMahon [9].

Corollary 8.1. *Let* $\nu_1, \ldots, \nu_l$ *be integers greater than zero and let* $s_i = \sum_{j=1}^{i} \nu_j$ *for*

$i=1,\ldots,l$, $s_0=0$ *and* $s_m=k$. *Let* $\boldsymbol{W}=0^{\nu_1-1}10^{\nu_2-1}1\cdots 0^{\nu_l-1}$. *Then*

$$c_k(\boldsymbol{W})=\left\|\binom{k-s_{i-1}}{s_j-s_{i-1}}\right\|_{l\times l}.$$

Proof. We use Theorem 5.2 to compute $\Phi(\omega_{nN}^{mM})$ where m, M, n, N are chosen so that $\omega_{nN}^{mM}=\boldsymbol{W}$. Thus

$$\Phi(\boldsymbol{W})=\left\|\frac{x^{s_j-s_{i-1}}}{(s_j-s_{i-1})!}\right\|_{l\times l}$$

by routine manipulation. The required number is $[x^k/k!]\Phi(\boldsymbol{W})$ and the result follows. □

A corresponding result may be obtained in a similar way from Theorem 6.1. The result is due to Stanley [11] and states, in the notation of Corollary 8.1, that

$$d_k(\boldsymbol{W})=\left\|\binom{k-s_{i-1}}{s_j-s_{i-1}}_q\right\|_{l\times l}$$

Finally, the result for sequences may be obtained from Theorem 7.1. In the notation of Corollary 8.1 we have

$$c_{\mathbf{i}}(\boldsymbol{W})=[x_1^{i_1}\cdots x_n^{i_n}]\,\|\gamma_{s_j-s_{i-1}}\|_{l\times l}$$

where

$$\gamma_r=\sum_{1\leq\sigma_1<\cdots<\sigma_r\leq n} x_{\sigma_1}\cdots x_{\sigma_r}.$$

This is a result due to Gessel [6].

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