SPECIAL K
Saturday November 3, 2012
10:00 am - 1:00 pm

1: Let $f(x) = x^4 + 2x^3$. Find the equation of a line which is tangent to the curve $y = f(x)$ at two distinct points.

2: Find the area of the region

$$R = \left\{(x, y) \in \mathbf{R}^2 \middle| \, (x^2 + y^2)^2 \leq 4x^2 \text{ and } x(x^2 + y^2) \leq 2\sqrt{3}\, xy \right\}.$$

3: Let x_n be the number of $2 \times n$ matrices with entries in $\{0, 1\}$ which do not contain the 2×2 block $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n}$.

4: Let $k \geq 3$ be an integer. Let $n = \frac{k(k+1)}{2}$. Let $S \subseteq \mathbf{Z}_n$ with $|S| = k$. Show that $S + S \neq \mathbf{Z}_n$. Note that $|S|$ denotes the cardinality of S and $S + S = \{x + y \,|\, x \in S, y \in S\}$.

5: Let $f : \mathbf{R} \to \mathbf{R}$. Suppose that $\lim\limits_{x\to 0} f(x) = f(0) = 0$ and $\lim\limits_{x\to 0} \dfrac{f(2x) - f(x)}{x} = 0$. Show that f is differentiable at 0 with $f'(0) = 0$.

6: Let $\mathbf{Z}^+$ be the set of positive integers. Show that there exists a bijection $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ with the property that $\prod\limits_{k=1}^{n} f(k)$ is an n^{th} power for every $n \in \mathbf{Z}^+$.

BIG E
Saturday November 3, 2012
10:00 am - 1:00 pm

1: Find the volume of the region
$$R = \left\{(x,y,z) \in \mathbf{R}^3 \,\middle|\, (x^2+y^2+z^2)^2 \le 4x^2 \text{ and } x(x^2+y^2) \le xz^2 \right\}.$$

2: Find the number of $3 \times n$ matrices with entries in $\{0,1\}$ which do not contain the 2×2 block $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3: Let $k \ge 3$ be an integer. Let $n = \frac{k(k+1)}{2}$. Let $S \subseteq \mathbf{Z}_n$ with $|S| = k$. Show that $S + S \ne \mathbf{Z}_n$. Note that $|S|$ denotes the cardinality of S and $S + S = \big\{x + y \big| x \in S, y \in S\big\}$.

4: Let $f : \mathbf{R}^2 \to \mathbf{R}$. Suppose that f is continuous and that $\displaystyle\int_0^1 f(a+tu)\,dt = 0$ for every point $a \in \mathbf{R}^2$ and every vector $u \in \mathbf{R}^2$ with $|u| = 1$. Show that f is constant.

5: Let $\mathbf{Z}^+$ be the set of positive integers. Show that there exists a bijection $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ with the property that $\displaystyle\prod_{k=1}^{n} f(k)$ is an n^{th} power for every $n \in \mathbf{Z}^+$.

6: Let A be an $n \times n$ matrix. Let u be an eigenvector of A for the eigenvalue 1. Suppose that all of the entries of A and all of the entries of u are positive. Show that the eigenspace for the eigenvalue 1 is 1-dimensional.