SPECIAL K
Saturday November 7, 2015
10:00 am - 1:00 pm

1: Let $x_0 = -1$, $x_1 = 3$ and $x_n = 2x_{n-1} + x_{n-2}$ for $n \geq 2$. Find the product $x_{n-2}x_{n-1}x_n$ where n is the largest integer with $n \geq 2$ for which x_{n-2}, x_{n-1} and x_n are all prime.

2: Let $f : [0,1] \to [0,1]$ be increasing and convex with $f(0) = 0$ and $f(1) = 1$ (f is *convex* means that for all $0 \leq a < b \leq 1$, the line segment from $(a, f(a))$ to $(b, f(b))$ lies on or above the graph of $y = f(x)$ for $a \leq x \leq b$). Show that $f(x)f^{-1}(x) \leq x^2$ for all $x \in [0,1]$.

3: For a positive integer n, let $\tau(n)$ be the number of divisors of n and let $\sigma(n)$ be the sum of the divisors of n. Show that for all integers $n \geq 2$ we have $\dfrac{\sigma(n)}{\tau(n)} \leq \dfrac{n+1}{2}$ with equality if and only if n is prime.

4: Triangle ABC has a right angle at B. The angle bisector at A meets BC at D and the angle bisector at C meets AB at E. Given that $AD = 9$ and $CE = 8\sqrt{2}$, find AC.

5: Let $f_1(x) = x^2 - 1$ and let $f_{n+1}(x) = f_1\big(f_n(x)\big)$ for $n \geq 1$. For each positive integer n, find the number of distinct real roots of the polynomial $f_n(x)$.

6: Let $\mathbf{N}$ be the set of natural numbers. Let S be a set of subsets of $\mathbf{N}$ and let $n \in \mathbf{N}$. Suppose that for all distinct sets $A, B \in S$, the intersection $A \cap B$ has at most n elements. Show that S is finite or countable.

BIG E
Saturday November 7, 2015
10:00 am - 1:00 pm

1: Let $x_0 = 1$ and $x_1 = 2$, and for $n \geq 1$ let $x_{2n} = x_{2n-1} + 2x_{2n-2}$ and $x_{2n+1} = 2x_{2n} - 3x_{2n-1}$. Find a closed form formula for x_{2n} and x_{2n+1}.

2: Let n be a positive integer. Find the smallest positive integer d such that $d = \det(A)$ for some $n \times n$ matrix whose entries all lie in $\{\pm 1\}$.

3: Let $0 < a_n \in \mathbf{R}$ for all integers $n \geq 1$. Let $b_1 = 1$, and let $b_{n+1} = b_n + \dfrac{a_n}{b_n}$ for all $n \geq 1$. Show that $\sum a_n$ converges if and only if $\{b_n\}$ converges.

4: Let G be a group. Suppose the map $\phi : G \to G$ given by $\phi(x) = x^3$ is an injective group homomorphism. Show that G is abelian.

5: For a positive integer n, let $T(n)$ be the product of the positive divisors of n. Show that for all positive integers n and m, if $T(n) = T(m)$ then $n = m$.

6: Show that the integral $\displaystyle\int_0^\infty \sin(x)\sin(x^2)\, dx$ converges..