Preparing for Euclid 2016

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Euclid Contest Details

- Written Tuesday 12 April 2016 in North and South America
- $2\frac{1}{2}$ hours long
- Short answer and full solution problems
- 100 total marks
- Average normally between 45 and 50
- Written in schools

Euclid Contest Details

- Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.
- While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

Euclid Coverpage

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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Euclid Contest

Tuesday, April 12, 2016
(in North America and South America)

Wednesday, April 13, 2016
(outside of North America and South America)

UNIVERSITY OF
WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

Question formats

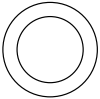
1. (a) What is value of $\dfrac{10^2 - 9^2}{10 + 9}$?

   (b) If $\dfrac{x+1}{x+4} = 4$, what is the value of $3x + 8$?

   (c) If $f(x) = 2x - 1$, determine the value of $(f(3))^2 + 2(f(3)) + 1$.

2. (a) If $\sqrt{a} + \sqrt{a} = 20$, what is the value of a?

   (b) Two circles have the same centre. The radius of the smaller circle is 1. The area of the region between the circles is equal to the area of the smaller circle. What is the radius of the larger circle?

   (c) There were 30 students in Dr. Brown's class. The average mark of the students in the class was 80. After two students dropped the class, the average mark of the remaining students was 82. Determine the average mark of the two students who dropped the class.

Why write the Euclid?

- For the challenge
- To help earn a scholarship to Waterloo Math
- To help earn an acceptance to Waterloo Math
  (You cannot hurt your chance of being accepted by writing)
- To help prepare yourself for first-year studies
- To help earn a spot in the "advanced section" courses in first year
- To see how you stack up against others across the country and around the world
- To have fun

How do I prepare for the Euclid?

- Attend March Break Open House!
- Work through past contests available through CEMC website
- Work through the Euclid eWorkshop on CEMC website
- Work through CEMC Grade 12 Courseware on CEMC website
- Work with friends at your school
- Try problems before you read the solutions
- "Brush up" on standard topics: geometry, trigonometry, exponents and logarithms, systems of equations, sequences, number theory (see Euclid website for more complete list)

Tips for writing the Euclid

- Prepare in advance! (see previous slide)
- Set your expectations
- Start with problems that you know that you can do
- Write better solutions to fewer problems rather than worse solutions to more problems
- Read all of the problems (even the hard ones!)
- Try at least the early parts of all of the problems, including the last few problems

Format of today's session

- We won't work through the entire package, though we will touch on questions throughout the package.
- I will suggest a problem, have you look at it briefly and then I will show a solution
- I will not give you enough time to solve the problems entirely. Often, the key with Euclid problems is knowing how to start. Think about this in the time that I give you.
- Try to understand the "high level strategy" of the solutions that I show you. Don't worry as much about the "technical details".
- There could be lots of ways of doing any one problem – this is one of the great things about doing math in a group.

Format of today's session

- If you get the problem quickly or already know how to do it, try a later problem, and be respectful of those who are trying to do the problem or who are listening to the solution.
- We will finish with a summary of mathematical facts to remember from today's session.
- We will distribute packages as we begin to solve problems. Please put up your hand if you would like one. (Students get priority!)

Problem 1(d)

The lines $ax + y = 30$ and $x + ay = k$ intersect at the point $P(6, 12)$. Determine the value of k.

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Solution

Since $P(6, 12)$ is on both lines, then the point satisfies both equations.

From the first equation, $6a + 12 = 30$ or $6a = 18$ or $a = 3$.

From the second equation, $x + 3y = k$, and so $k = 6 + 3(12) = 42$.

Tip #1

If a point lies on a line, its coordinates satisfy the equation of the line. In other words, substitute!

Problem 2(c)

John and Mary wrote the Euclid Contest. Two times John's score was 60 more than Mary's score. Two times Mary's score was 90 more than John's score. Determine the average of their two scores.

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John and Mary wrote the Euclid Contest. Two times John's score was 60 more than Mary's score. Two times Mary's score was 90 more than John's score. Determine the average of their two scores.

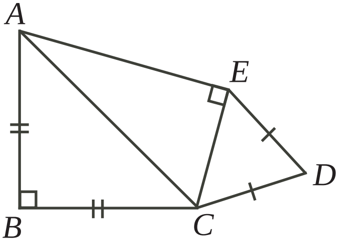
Solution
Suppose that John's score was x and Mary's score was y.
Then $2x = y + 60$ and $2y = x + 90$.
Adding these two equations, we obtain $2x + 2y = x + y + 150$ or $x + y = 150$ or $\dfrac{x+y}{2} = \dfrac{150}{2} = 75$.
Therefore, the average of their scores was 75.

Tip #2

Make sure to answer the question being asked. Sometimes, there might be a more direct way to do this.

Problem 3(b)

In the diagram, $AB = BC = 2\sqrt{2}$ and $CD = DE$. Also, $\angle CDE = 60^\circ$ and $\angle EAB = 75^\circ$. Determine the perimeter of figure $ABCDE$.

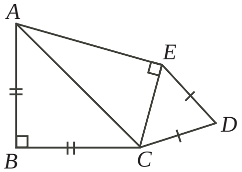


Problem 3(b)

Solution

Since $\triangle ABC$ is isosceles and right-angled, then $\angle BAC = 45^\circ$.

Since $\angle EAB = 75^\circ$ and $\angle BAC = 45^\circ$, then $\angle CAE = \angle EAB - \angle BAC = 30^\circ$.



Also, $AC = \sqrt{2}AB = \sqrt{2}(2\sqrt{2}) = 4$.

Since $\triangle AEC$ is right-angled and has a 30° angle, then $\triangle AEC$ is a 30°-60°-90° triangle.

Thus, $EC = \frac{1}{2}AC = 2$ (since EC is opposite the 30° angle) and $AE = \frac{\sqrt{3}}{2}AC = 2\sqrt{3}$ (since AE is opposite the 60° angle).

In $\triangle CDE$, $ED = DC$ and $\angle EDC = 60^\circ$, so $\triangle CDE$ is equilateral.

Therefore, $ED = CD = EC = 2$.

Overall, the perimeter of $ABCDE$ is

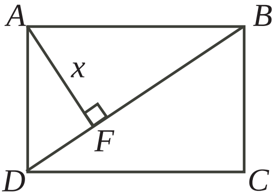
$$AB+BC+CD+DE+EA = 2\sqrt{2}+2\sqrt{2}+2+2+2\sqrt{3} = 4+4\sqrt{2}+2\sqrt{3}$$

Tip #3

In problems involving angles, chase the angles. If side lengths are also involved, look for special triangles.

Problem 5(b)

In rectangle $ABCD$, F is on diagonal BD so that AF is perpendicular to BD. Also, $BC = 30$, $CD = 40$ and $AF = x$. Determine the value of x.



Problem 5(b)

In rectangle $ABCD$, F is on diagonal BD so that AF is perpendicular to BD. Also, $BC = 30$, $CD = 40$ and $AF = x$. Determine the value of x.

Solution

Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.

By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then $BD = \sqrt{30^2 + 40^2} = \sqrt{2500} = 50$.

We calculate the area of $\triangle ADB$ is two different ways.

First, using AB as base and AD as height, we obtain an area of $\frac{1}{2}(40)(30) = 600$.

Next, using DB as base and AF as height, we obtain an area of $\frac{1}{2}(50)x = 25x$.

We must have $25x = 600$ and so $x = \frac{600}{25} = 24$.

Tip #4

Many geometry problems have several different solutions. Calculating an area in two different ways can sometimes provide a "simple" solution.

Problem 9(c)

The numbers a, b, c, in that order, form a three term arithmetic sequence and $a + b + c = 60$. The numbers $a - 2$, b, $c + 3$, in that order, form a three term geometric sequence. Determine all possible values of a, b and c.

Problem 9(c)

The numbers a, b, c, in that order, form a three term arithmetic sequence and $a + b + c = 60$. The numbers $a - 2$, b, $c + 3$, in that order, form a three term geometric sequence. Determine all possible values of a, b and c.

Solution

Since a, b, c form an arithmetic sequence, then we can write $a = b - d$ and $c = b + d$ for some real number d.
Since $a + b + c = 60$, then $(b - d) + b + (b + d) = 60$ or $b = 20$.
Therefore, we can write a, b, c as $20 - d, 20, 20 + d$.
Thus, $a - 2 = 20 - d - 2 = 18 - d$ and $c + 3 = 23 + d$, so we can write $a - 2, b, c + 3$ as $18 - d, 20, 23 + d$.

Problem 9(c)

Solution (cont'd)

Since these three numbers form a geometric sequence, then

$$\begin{aligned} \frac{20}{18-d} &= \frac{23+d}{20} \\ 20^2 &= (23+d)(18-d) \\ 400 &= -d^2-5d+414 \\ d^2+5d-14 &= 0 \\ (d+7)(d-2) &= 0 \end{aligned}$$

Therefore, $d=-7$ or $d=2$.

If $d=-7$, then $a=27$, $b=20$ and $c=13$.

If $d=2$, then $a=18$, $b=20$ and $c=22$.

Tip #5

Remember definitions of mathematical concepts such as arithmetic sequences, geometric sequences, etc.

Problem 10(b)

Determine all values of x for which $12^{2x+1} = (2^{3x+7})(3^{3x-4})$.

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Determine all values of x for which $12^{2x+1} = (2^{3x+7})(3^{3x-4})$.

Solution

Since $12 = 2^2 3^1$, the equation $12^{2x+1} = (2^{3x+7})(3^{3x-4})$ becomes $2^{4x+2}3^{2x+1} = 2^{3x+7}3^{3x-4}$.

Rearranging, $2^{x-5} = 3^{x-5}$ or $\frac{2^{x-5}}{3^{x-5}} = 1$ or $\left(\frac{2}{3}\right)^{x-5} = 1$.

Therefore, $x - 5 = 0$, and so $x = 5$.

Tip #6

Review exponent rules. Don't assume that exponents are integers. Combine terms withe same base. Use prime factorizations to create common bases.

Problem 10(d)

Solve the equation $\log_{27}(12 \cdot 3^{2x} - 27 \cdot 3^{x}) = x$.

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Solve the equation $\log_{27}(12 \cdot 3^{2x} - 27 \cdot 3^x) = x$.

Solution

Using logarithm and exponent laws,

$$\begin{aligned}
\log_{27}(12 \cdot 3^{2x} - 27 \cdot 3^x) &= x \\
12 \cdot 3^{2x} - 27 \cdot 3^x &= 27^x \\
12(3^x)^2 - 27(3^x) &= (3^x)^3 \\
0 &= u^3 - 12u^2 + 27u \qquad \text{(where } u = 3^x\text{)} \\
0 &= u(u^2 - 12u + 27) \\
0 &= u(u-3)(u-9)
\end{aligned}$$

and so $u = 3^x = 0$ or $u = 3^x = 3$ or $u = 3^x = 9$.

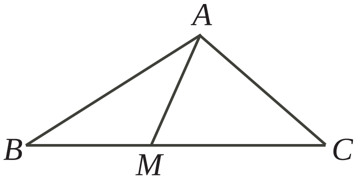
The first is not possible, so $x = 1$ or $x = 2$.

Tip #7

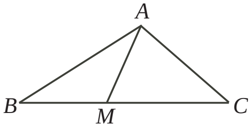
Review logarithm rules. Don't make up new ones.

Problem 11(b)

In $\triangle ABC$, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC.



Problem 11(b)



Solution

Using the cosine law in $\triangle ABM$,

$$\begin{aligned} AM^2 &= AB^2 + BM^2 - 2(AB)(BM)\cos(\angle ABM) \\ 3^2 &= 7^2 + 5^2 - 2(7)(5)\cos(\angle ABM) \\ \cos(\angle ABM) &= \frac{3^2 - 7^2 - 5^2}{-2(7)(5)} \\ \cos(\angle ABM) &= \frac{13}{14} \end{aligned}$$

Since $\angle ABC = \angle ABM$, then $\cos(\angle ABC) = \frac{13}{14}$.

Using the cosine law in $\triangle ABC$, we get

$AC^2 = 7^2 + 11^2 - 2(7)(11)\left(\frac{13}{14}\right) = 27$.

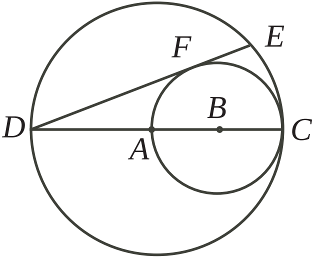
Therefore, $AC = \sqrt{27}$.

Tip #8

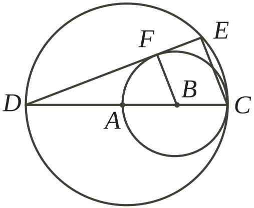
Know when to use the sine law and when to use the cosine law. Know where you are starting and where you are going. Give exact answers!

Problem 12(b)

In the diagram, DC is a diameter of the larger circle centred at A and AC is the diameter of the smaller circle centred at B. If DE is tangent to the smaller circle at F and $DC = 12$, determine the length of DE.



Problem 12(b)



Solution

Join B to F and C to E.

Since DFE is tangent to the circle at F, then $\angle DFB = 90^\circ$.

Since DC is a diameter, then $\angle DEC = 90^\circ$.

Thus, $\triangle DFB$ is similar to $\triangle DEC$ since the two triangles share an angle at D and each is right-angled.

Since the diameter of the larger circle is 12, then its radius is 6.

This means that the diameter of the smaller circle is 6 and so its radius is 3. Thus, $DB = 6 + 3 = 9$.

By the Pythagorean Theorem, $DF = \sqrt{9^2 - 3^2} = \sqrt{72} = 6\sqrt{2}$.

Using similar triangles, $\dfrac{DE}{DC} = \dfrac{DF}{DB}$ and so $DE = \dfrac{12(6\sqrt{2})}{9} = 8\sqrt{2}$.

Tip #9

Join centres to points of tangency. Radii are perpendicular to tangents at points of tangency. The angle subtended by a diameter is always 90°.

2012 Euclid #10

For each positive integer N, an *Eden sequence* from $\{1, 2, 3, \ldots, N\}$ is defined to be a sequence that satisfies the following conditions:

(i) each of its terms is an element of the set of consecutive integers $\{1, 2, 3, \ldots, N\}$,

(ii) the sequence is increasing, and

(iii) the terms in odd numbered positions are odd and the terms in even numbered positions are even.

For example, the four Eden sequences from $\{1, 2, 3\}$ are

$$1 \qquad 3 \qquad 1, 2 \qquad 1, 2, 3$$

(a) Determine the number of Eden sequences from $\{1, 2, 3, 4, 5\}$.

(b) For each positive integer N, define $e(N)$ to be the number of Eden sequences from $\{1, 2, 3, \ldots, N\}$. If $e(17) = 4180$ and $e(20) = 17710$, determine $e(18)$ and $e(19)$.

2012 Euclid #10(a)

Solution

The Eden sequences from $\{1, 2, 3, 4, 5\}$ are

$$1 \quad 3 \quad 5 \quad 1,2 \quad 1,4 \quad 3,4 \quad 1,2,3 \quad 1,2,5 \quad 1,4,5 \quad 3,4,5$$

$$1,2,3,4 \quad 1,2,3,4,5$$

There are 12 such sequences.

We present a brief justification of why these are all of the sequences.

2012 Euclid #10(a)

Solution (cont'd)

- An Eden sequence of length 1 consists of a single odd integer. The possible choices are 1 and 3 and 5.
- An Eden sequence of length 2 must be an odd integer followed by a larger even integer. Since the only even integers here are 2 and 4, then the sequences are 1, 2 and 1, 4 and 3, 4.
- An Eden sequence of length 3 starts with an Eden sequence of length 2 and appends a larger odd integer to the end. Starting with 1,2, we form 1,2,3 and 1,2,5. Starting with 1,4, we form 1,4,5. Starting with 3,4, we form 3,4,5.
- An Eden sequence of length 4 starts with an Eden sequence of length 3 and appends a larger even integer. The only possible sequence here is 1,2,3,4.
- An Eden sequence of length 5 from $\{1, 2, 3, 4, 5\}$ must include all 5 elements, so is 1,2,3,4,5.

Tip #10

Take time to read and understand the problem. When lacking a better approach, just do it.

Collected Tips (Mathematical)

1. If a point lies on a line, its coordinates satisfy the equation of the line.
2. In problems involving angles, chase the angles. If side lengths are also involved, look for special triangles.
3. If two lines are perpendicular, think about their slopes.
4. Review logarithm and exponent rules.
5. Use prime factorizations to create common bases.
6. Know when to use the sine law and when to use the cosine law.
7. Give exact answers!
8. Join centres to points of tangency. Radii are perpendicular to tangents at points of tangency. The angle subtended by a diameter is always 90°.

Collected Tips (Strategies)

1. Make sure to answer the question being asked.
2. Remember definitions of mathematical concepts such as arithmetic sequences, geometric sequences, etc.
3. Don't make up new mathematical rules.
4. Try to use the information that you are given carefully and sequentially. It is usually the case that you will need all of the information that you are given.
5. Know where you are starting and where you are going.
6. Take time to read and understand the problem.
7. When lacking a better approach, just do it.

Final Problem

An auditorium has a rectangular array of chairs. Each chair is unpainted or is painted black or gold. There are exactly 14 black chairs in each row and exactly 10 gold chairs in each column. If exactly 3 chairs are unpainted, what are the possible numbers of chairs in the auditorium?

Final Problem

Solution

Suppose that the auditorium has r rows and c columns of chairs.
Then there are rc chairs in total.
Each chair is unpainted or black or gold.
There are exactly $14r$ black chairs.
There are exactly $10c$ gold chairs.
There are exactly 3 unpainted chairs.
Therefore, $rc = 14r + 10c + 3$.
We want to find all pairs of positive integers r and c that satisfy this equation.
We note that $c \geq 14$ and $r \geq 10$.

Final Problem

Solution (cont'd)

Manipulating the equation,

$$
\begin{aligned}
rc &= 14r + 10c + 3 \\
rc - 14r &= 10c + 3 \\
r(c - 14) &= 10c + 3 \\
r &= \frac{10c + 3}{c - 14} \\
r &= \frac{10c - 140 + 143}{c - 14} \\
r &= \frac{10c - 140}{c - 14} + \frac{143}{c - 14} \\
r &= 10 + \frac{143}{c - 14}
\end{aligned}
$$

Final Problem

Solution (cont'd)

Since r is an integer, then $10 + \dfrac{143}{c-14}$ is an integer, so $\dfrac{143}{c-14}$ must be an integer.

Therefore, $c - 14$ is a positive divisor of 143.

Since $143 = 11 \times 13$, then its positive divisors are $1, 11, 13, 143$.

We make a table of possible values:

$c-14$	c	r	rc
1	15	153	2295
11	25	23	575
13	27	21	567
143	157	11	1727

Therefore, the four possible values for rc are $567, 575, 1727, 2295$.

(Can you create a grids of these sizes with the required properties?)

Good luck!