Europ. J. Combinatorics (1981) **2**, 131–135

The Enumeration of Directed Closed Euler Trails and Directed Hamiltonian Circuits by Lagrangian Methods

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1. Introduction

In this paper we demonstrate that the numbers of closed directed Euler trails and directed Hamiltonian circuits in a digraph may be obtained as coefficients in the power series solution of a single system of functional equations. These results, separately, have more elementary proofs, and in fact the first is the well known BEST Theorem [1]. However, it seems of interest that they may both be obtained by application of the Lagrange Theorem to a single power series.

In a previous paper [4], we have demonstrated that the matrix tree theorem [6] and a theorem of Good [3] may similarly be obtained as different coefficients in the power series solution of a particular system of functional equations. In that case, the system of equations was a multivariate generalization of the equation $T = x\,e^T$, for labelled, rooted, abstract trees.

The following notation is used. Let $\mathbf{A}$ be an $n\times n$ matrix whose (i,j)-element is a_{ij}. We write $\mathbf{A}=[a_{ij}]_{n\times n}$ and $a_{ij}=[\mathbf{A}]_{ij}$. We denote the determinant of $\mathbf{A}$ by $|\mathbf{A}|$, $\|a_{ij}\|$ and $\det(\mathbf{A})$, and the permanent of $\mathbf{A}$ by $\mathrm{per}(\mathbf{A})$. The (i,j)-cofactor of $\mathbf{A}$ is denoted by $\mathrm{cof}_{ij}(\mathbf{A})$. If $\mathbf{D}=[d_{ij}]_{n\times n}$, then $\mathbf{D}!$ denotes $\prod_{i,j=1}^{n} d_{ij}!$, and $\mathbf{A}^{\mathbf{D}}$ denotes $\prod_{i,j=1}^{n} a_{ij}^{d_{ij}}$. Let $\alpha,\beta\subseteq\{1,\ldots,n\}=\mathcal{N}_n$. Then $\mathbf{A}[\alpha|\beta]$ denotes the submatrix of $\mathbf{A}$ intercepted between the rows of $\mathbf{A}$ with labels in α and the columns of $\mathbf{A}$ with labels in β, and $\mathbf{A}(\alpha|\beta)$ denotes $\mathbf{A}[\mathcal{N}_n\backslash\alpha|\mathcal{N}_n\backslash\beta]$. If $\mathbf{x}=(x_1,\ldots,x_n)$ and $\mathbf{k}=(k_1,\ldots,k_n)$ then $\mathbf{x}^{\mathbf{k}}$ denotes $\prod_{i=1}^{n} x_i^{k_i}$ and $\mathbf{k}!$ denotes $\prod_{i=1}^{n} k_i!$. If $f(\mathbf{x})$ is a formal power series in the indeterminates $x_1,\ldots,x_n$, then $[\mathbf{x}^{\mathbf{k}}]f(\mathbf{x})$ denotes the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $f(\mathbf{x})$. Finally $\mathbf{1}$ is the vector all of whose entries are 1s.

2. The System of Functional Equations

Let $\Omega_{b,c}^{(\kappa)}(\mathbf{D})$, where $\mathbf{D}=[d_{ij}]_{\kappa\times\kappa}$, be the number of non-empty sequences on $\mathcal{N}_\kappa$ which begin with $b\in\mathcal{N}_\kappa$ and end with $c\in\mathcal{N}_\kappa$, and have d_{ij} occurrences of the substring ij, for $i,j=1,\ldots,\kappa$. If $\mathbf{n}=(n_1,\ldots,n_\kappa)$, where n_i is the number of occurrences of i in such a sequence, then $n_i-\delta_{ic}=\sum_{j=1}^{\kappa} d_{ij}$, $n_i-\delta_{ib}=\sum_{j=1}^{\kappa} d_{ji}$, for $i=1,\ldots,\kappa$, by the obvious combinatorial argument. Let $\mathbf{A}=[a_{ij}]_{\kappa\times\kappa}$, $\mathbf{x}=(x_1,\ldots,x_\kappa)$ where the a_{ij} and $x_1,\ldots,x_\kappa$ are indeterminates, and define the power series

$$f_b=f_b(\mathbf{A},\mathbf{x})=\sum_{\mathbf{D}}\Omega_{b,c}^{(\kappa)}(\mathbf{D})\mathbf{A}^{\mathbf{D}}\mathbf{x}^{\mathbf{n}}$$

for $b=1,\ldots,\kappa$, where the summation is over all appropriate $\mathbf{D}$, and $\mathbf{n}$ is determined from $\mathbf{D}$ by the above equations. These power series satisfy the system of functional equations given in the following result.

Lemma 2.1

$$\Omega_{c,c}^{(\kappa)}(\mathbf{D})=[\mathbf{A}^{\mathbf{D}}\mathbf{x}^{\mathbf{n}}]f_c(\mathbf{A},\mathbf{x})$$

where $f_i=x_i\{\delta_{ic}+\sum_{j=1}^{\kappa} a_{ij}f_j\}$ *for* $i=1,\ldots,\kappa$.

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PROOF. The power series f_i is the generating function for non-empty sequences beginning with i and ending with c, in which an occurrence of j in the sequence is marked by the indeterminate x_j, for $j=1,\ldots,\kappa$ and an occurrence of the substring jl is marked by a_{jl}, for $j, l=1,\ldots,\kappa$. Such a sequence can have length one only when $i=c$, and in this case consists of a single symbol c. Thus sequences of length one contribute $\delta_{ic}x_i$ to the generating function f_i. Any other such sequence has second element j, for some $j=1,\ldots,\kappa$. These consist of the element i prefixing a non-empty sequence beginning with j and ending with c. The latter are enumerated by f_j, and in addition we record the initial i by x_i and the initial substring ij by a_{ij}. Thus we have

$$f_i=\delta_{ic}x_i+\sum_{j=1}^{\kappa} x_i a_{ij} f_j, \quad \text{for } i=1,\ldots,\kappa,$$

and the result follows.

The following two results will be used in the solution of this system of functional equations.

THEOREM 2.2 (GOOD [2], TUTTE [7]). *Let $\boldsymbol{\phi}=(\phi_1,\ldots,\phi_\kappa)$ and γ be formal power series in the indeterminates $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_\kappa)$ and with no terms with negative exponents. Suppose that $\boldsymbol{\zeta}=(\zeta_1,\ldots,\zeta_\kappa)$ satisfies $\xi_i=\zeta_i\phi_i(\boldsymbol{\xi})$ for $i=1,\ldots,\kappa$. Then*

$$[\boldsymbol{\zeta}^{\boldsymbol{\nu}}]\gamma(\boldsymbol{\xi}(\boldsymbol{\zeta}))=[\boldsymbol{\xi}^{\boldsymbol{\nu}}]\gamma(\boldsymbol{\xi})\boldsymbol{\phi}^{\boldsymbol{\nu}}(\boldsymbol{\xi})\left\|\delta_{ij}-\frac{\xi_j}{\phi_i(\boldsymbol{\xi})}\frac{\partial\phi_i(\boldsymbol{\xi})}{\partial\xi_j}\right\|.$$

The next corollary is useful in allowing us to avoid the extraction of coefficients from the determinant in Theorem 2.2.

COROLLARY 2.3 (JACKSON AND GOULDEN [5]). *Under the conditions of Theorem 2.2 further suppose that $\phi_i(\boldsymbol{\xi})$ is independent of ξ_j for each $(i,j)\in\mathscr{S}\subseteq\mathscr{N}_\kappa^2$. Then*

$$[\boldsymbol{\zeta}^{\boldsymbol{\nu}}]\boldsymbol{\xi}^{\mathbf{r}}=(\nu_1\cdots\nu_\kappa)^{-1}\sum_{\boldsymbol{\mu}}\|\delta_{ij}\nu_i-\mu_{ij}\|\prod_{i=1}^{\kappa}([\xi_1^{\mu_{i1}}\cdots\xi_\kappa^{\mu_{i\kappa}}]\phi_i^{\nu_i})$$

where the summation is over all non-negative integer $\kappa\times\kappa$ matrices such that

$$\sum_{i=1}^{\kappa}\mu_{ij}=\nu_j-r_j,\ j=1,\ldots,\kappa \quad \text{and} \quad \mu_{ij}=0 \text{ for each } (i,j)\in\mathscr{S}.$$

3. CLOSED DIRECTED EULER TRAILS

We first obtain an expression for the number of closed directed Euler trails of a digraph. This result is the BEST Theorem [1].

THEOREM 3.1. *Let e be the number of closed directed Euler trails in a digraph on vertex set $\{1,\ldots,\kappa\}$, with adjacency matrix $\mathbf{D}=[d_{ij}]_{\kappa\times\kappa}$ and in-degree (i) = out-degree $(i)=k_i$ for $i=1,\ldots,\kappa$. Then*

$$e=(\mathbf{k}-\mathbf{1})!\,\mathrm{cof}_{cc}[\delta_{ij}k_i-d_{ij}]_{\kappa\times\kappa}$$

where $\mathbf{k}=(k_1,\ldots,k_\kappa)$, for any $c=1,\ldots,\kappa$.

PROOF. If, in Lemma 2.1, we consider $f_1,\ldots,f_\kappa$ as power series in $x_1,\ldots,x_\kappa$, with coefficients which are polynomials in the indeterminates a_{ij}, then

$$\Omega_{c,c}^{(\kappa)}(\mathbf{D})=[\mathbf{A}^{\mathbf{D}}]([\mathbf{x}^{\mathbf{n}}]f_c),$$

where $f_i = x_i(\delta_{ic} + \sum_{j=1}^{\kappa} a_{ij} f_j)$, for $i = 1, \ldots, \kappa$. Thus we can apply Corollary 2.3 to determine $[\mathbf{x}^{\mathbf{n}}]f_c$, by making the following identifications:

$$\xi_i = f_i, \quad \zeta_i = x_i, \quad \nu_i = n_i, \quad r_i = \delta_{ic}, \quad \phi_i = \delta_{ic} + \sum_{j=1}^{\kappa} a_{ij} f_j,$$

for $i = 1, \ldots, \kappa$, and $\mathcal{S} = \varnothing$. We thus obtain

$$[\mathbf{x}^{\mathbf{n}}]f_c = (n_1 \cdots n_\kappa)^{-1} \sum_{\boldsymbol{\mu}} \|\delta_{ij} n_i - \mu_{ij}\| \prod_{i=1}^{\kappa} [f_1^{\mu_{i1}} \cdots f_k^{\mu_{i\kappa}}] \phi_i^{n_i}$$

where the summation is over all $\boldsymbol{\mu}$ with the column-sum restrictions $\sum_{i=1}^{\kappa} \mu_{ij} = n_j - \delta_{jc}$ for $j = 1, \ldots, \kappa$. But

$$[f_1^{\mu_{i1}} \cdots f_\kappa^{\mu_{i\kappa}}] \phi_i^{n_i} = n_i! \left(\prod_{j=1}^{\kappa} a_{ij}^{\mu_{ij}} \middle/ \mu_{ij}! \right) \frac{\delta_{ic}^{n_i - \sum_{j=1}^{\kappa} \mu_{ij}}}{\left(n_i - \sum_{j=1}^{\kappa} \mu_{ij} \right)!},$$

so we have a non-zero contribution only when we have the row restrictions $\sum_{j=1}^{\kappa} \mu_{ij} = n_i$ for $i \neq c$. The column restrictions on $\boldsymbol{\mu}$ mean that $\sum_{j=1}^{\kappa} \sum_{i=1}^{\kappa} \mu_{ij} = n_1 + \cdots + n_\kappa - 1$, and thus we deduce the final row restriction $\sum_{j=1}^{\kappa} \mu_{cj} = n_c - 1$, for a non-zero contribution to the coefficient. Accordingly,

$$[\mathbf{x}^{\mathbf{n}}] f_c = (\mathbf{n} - \mathbf{1})! \sum_{\boldsymbol{\mu}} \|\delta_{ij} n_i - \mu_{ij}\| \mathbf{A}^{\boldsymbol{\mu}} (\boldsymbol{\mu}!)^{-1}$$

where the summation is over all $\boldsymbol{\mu}$ such that $n_i - \delta_{ic} = \sum_{j=1}^{\kappa} \mu_{ij} = \sum_{j=1}^{\kappa} \mu_{ji}$, for $i = 1, \ldots, \kappa$. Finally, we have

$$\begin{aligned} \Omega_{c,c}^{(\kappa)}(\mathbf{D}) &= [\mathbf{A}^{\mathbf{D}}]([\mathbf{x}^{\mathbf{n}}]f_c) \\ &= [\mathbf{A}^{\mathbf{D}}](\mathbf{n} - \mathbf{1})! \sum_{\boldsymbol{\mu}} \|\delta_{ij} n_i - \mu_{ij}\| \mathbf{A}^{\boldsymbol{\mu}} (\boldsymbol{\mu}!)^{-1} \\ &= (\mathbf{n} - \mathbf{1})! \|\delta_{ij} n_i - d_{ij}\| (\mathbf{D}!)^{-1}, \end{aligned}$$

since $\mathbf{D}$ satisfies exactly the same restrictions as $\boldsymbol{\mu}$.

Now a closed directed Euler trail in a digraph with adjacency matrix $\mathbf{D}$ can be represented as a sequence, by listing the names of the vertices which are traversed in succession, starting at an arbitrary vertex c, and terminating there as well. Such a sequence begins and ends with c, has d_{ij} occurrences of the substring ij, and $n_i = k_i + \delta_{ic}$ occurrences of i, for $i, j = 1, \ldots, \kappa$, since the trail passes through each edge exactly once. Thus there are $\Omega_{c,c}^{(\kappa)}(\mathbf{D})$ such sequences. Moreover, since edges in the trail are distinct, and the sequence representation of the trail can be started at any of the k_c occurrences of vertex c, then

$$\begin{aligned} e &= k_c^{-1} \mathbf{D}! \Omega_{c,c}^{(\kappa)}(\mathbf{D}) \\ &= (\mathbf{k} - \mathbf{1})! \|\delta_{ij}(k_i + \delta_{ic}) - d_{ij}\|. \end{aligned}$$

Using row c to expand the determinant, we obtain

$$\|\delta_{ij}(k_i + \delta_{ic}) - d_{ij}\| = \|\delta_{ij} k_i - d_{ij}\| + \mathrm{cof}_{cc}[\delta_{ij} k_i - d_{ij}]_{\kappa \times \kappa},$$

but $\|\delta_{ij} k_i - d_{ij}\| = 0$ since $\sum_{j=1}^{\kappa} d_{ij} = \sum_{j=1}^{\kappa} d_{ji} = k_i$, for $i = 1, \ldots, \kappa$, and the result follows.

Since from [6], $\mathrm{cof}_{cc}[\delta_{ij} k_i - d_{ij}]_{\kappa \times \kappa}$ is τ_c, the number of out-directed spanning arborescences rooted at vertex c for a digraph with adjacency matrix $\mathbf{D}$, then Theorem 3.1 can be restated as $e = (\mathbf{k} - \mathbf{1})! \tau_c$, a result given in van Aardenne-Ehrenfest and de Bruijn [1].

4. Directed Hamiltonian Circuits

We now consider the enumeration of directed Hamiltonian circuits in a digraph.

THEOREM 4.1. *Let h be the number of directed Hamiltonian circuits in a digraph on κ vertices with adjacency matrix* **D**. *Then*

$$h=\sum_{\alpha}(-1)^{|\alpha|}(\det \mathbf{D}[\alpha|\alpha])(\operatorname{per}\mathbf{D}(\alpha|\alpha))$$

where the sum is over all $\alpha \subseteq \mathcal{N}_\kappa\backslash\{c\}$, for any $c\in\mathcal{N}_\kappa$, and $\det \mathbf{D}[\varnothing|\varnothing]=1$.

PROOF. In the notation of Lemma 2.1, the generating function for sequences on $\mathcal{N}_\kappa$ which begin and end with $c\in\mathcal{N}_\kappa$ and have exactly one occurrence of each of the remaining elements of $\mathcal{N}_\kappa$ is $[x_c\mathbf{x}^{\mathbf{1}}]f_c$, where in this case f_c is regarded as a power series in $x_1,\ldots,x_\kappa$, with coefficients which are polynomials in the indeterminates a_{ij}. From Lemma 2.1 and Theorem 2.2 we obtain

$$[x_c\mathbf{x}^{\mathbf{1}}]f_c=[f_c\mathbf{f}^{\mathbf{1}}]f_c\phi_c\|\delta_{ij}\phi_i-a_{ij}f_j\|$$

where $\phi_i(\mathbf{f})=\delta_{ic}+\psi_i(\mathbf{f})$, $\psi_i(\mathbf{f})=\sum_{j=1}^{\kappa}a_{ij}f_j$, for $i=1,\ldots,\kappa$. Thus

$$\begin{aligned}[x_c\mathbf{x}^{\mathbf{1}}]f_c&=[\mathbf{f}^{\mathbf{1}}]\phi_c\|\delta_{ij}\phi_i-a_{ij}f_j\|\\&=[\mathbf{f}^{\mathbf{1}}]\phi_c\{\|\delta_{ij}\psi_i-a_{ij}f_j\|+\operatorname{cof}_{cc}[\delta_{ij}\psi_i-a_{ij}f_j]_{\kappa\times\kappa}\},\end{aligned}$$

by expanding the determinant using row c. But all the row sums in $[\delta_{ij}\psi_i-a_{ij}f_j]_{\kappa\times\kappa}$ are zero, so that

$$\begin{aligned}[x_c\mathbf{x}^{\mathbf{1}}]f_c&=[\mathbf{f}^{\mathbf{1}}]\phi_c\{0+\operatorname{cof}_{cc}[\delta_{ij}\psi_i-a_{ij}f_j]_{\kappa\times\kappa}\}\\&=[\mathbf{f}^{\mathbf{1}}](\psi_c+1)\operatorname{cof}_{cc}[\delta_{ij}\psi_i-a_{ij}f_j]_{\kappa\times\kappa}\\&=[\mathbf{f}^{\mathbf{1}}]\psi_c\operatorname{cof}_{cc}[\delta_{ij}\psi_i-a_{ij}f_j]_{\kappa\times\kappa},\end{aligned}$$

since $\operatorname{cof}_{cc}[\delta_{ij}\psi_i-a_{ij}f_j]_{\kappa\times\kappa}$ is homogeneous of degree $(\kappa-1)$ in $f_1,\ldots,f_\kappa$, and $\mathbf{f}^{\mathbf{1}}$ is of degree κ. Finally, expanding the cofactor in terms involving a given subset of the ψ_is on the diagonal, we obtain

$$\begin{aligned}[x_c\mathbf{x}^{\mathbf{1}}]f_c&=[\mathbf{f}^{\mathbf{1}}]\sum_{\alpha\subseteq\mathcal{N}_\kappa\backslash\{c\}}(-1)^{|\alpha|}(\det\mathbf{A}[\alpha|\alpha])\Big(\prod_{j\in\alpha}f_j\Big)\Big(\prod_{i\in\mathcal{N}_\kappa\backslash\alpha}\psi_i\Big)\\&=\sum_{\alpha\subseteq\mathcal{N}_\kappa\backslash\{c\}}(-1)^{|\alpha|}(\det\mathbf{A}[\alpha|\alpha])(\operatorname{per}\mathbf{A}(\alpha|\alpha)),\end{aligned}$$

where $\det\mathbf{A}[\varnothing|\varnothing]=1$.

Now a directed Hamiltonian circuit in a digraph can be represented as a sequence, by listing the names of the vertices which are traversed in succession, starting at an arbitrary vertex c, and terminating there as well. Such a sequence starts and ends with c, contains no other occurrence of c and has exactly one occurrence of each of the remaining elements of $\mathcal{N}_\kappa$. The generating function for such sequences is accordingly $[x_c\mathbf{x}^{\mathbf{1}}]f_c$, which is evaluated above, and in which the occurrence of the substring ij is marked by the indeterminate a_{ij}, for $i,j=1,\ldots,k$. The occurrence of the substring ij in a sequence means that a directed edge from vertex i to vertex j is used in the corresponding directed Hamiltonian circuit, and since such a substring occurs at most once, there are d_{ij} choices for which edge is used in the graph. Thus h is obtained by replacing a_{ij} in $[x_c\mathbf{x}^{\mathbf{1}}]f_c$ by d_{ij}.

Acknowledgements

This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

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Received 30 October 1980

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