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A multivariate hook formula for labelled trees

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ABSTRACT

Several hook summation formulae for binary trees have appeared recently in the literature. In this paper we present an analogous formula for unordered increasing trees of size r, which involves r parameters. The right-hand side can be written nicely as a product of linear factors. We study two specializations of this new formula, including Cayley's enumeration of trees with respect to vertex degree. We give three proofs of the hook formula. One of these proofs arises somewhat indirectly, from representation theory of the symmetric groups, and in particular uses Kerov's character polynomials. The other proofs are more direct, and of independent interest.

1. Introduction and the main result

Hook formulae first appeared in the context of representation theory of the symmetric groups: Frame, Robinson and Thrall [16, Theorem 1] proved that the dimension $\chi^{\lambda}((1^n))$ of the representation associated to a Young diagram λ with n boxes, (which is also the number of increasing labellings of the boxes of λ) is given by the simple ratio

$$\chi^{\lambda}((1^n)) = \frac{n!}{\prod_{\square \in \lambda} h(\square)},$$

where $h(\square)$ is the size of the hook attached to the Box $\square$.

It was subsequently pointed out by D. Knuth [24, §5.1.4 Exercise 20] that the number $L(T)$ of increasing labellings of the vertices of a rooted tree T can be expressed by using the same kind of formula. In particular,

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$$L(T) = \frac{|T|!}{\prod_{v \in T} h_T(v)}, \tag{1}$$

where $|T|$ is the number of vertices of T and $h_T(v)$ is the size of the hook $\mathfrak{h}_T(v)$ attached to the vertex v in T (see definition below).

At this point, we fix some terminology and notation. A *tree* is an acyclic connected graph. *Rooted* means that we distinguish a vertex; then each edge can be oriented towards the root and we call respectively *father* and *son* the head and tail of the edge. With this terminology, it is easy to guess what the *descendants* of a vertex are: they can be defined recursively as the sons and the descendants of the sons. The hook attached to the vertex v in the tree T, denoted by $\mathfrak{h}_T(v)$, is the set consisting of v and its descendants.

For another consequence of the rooted tree hook formula (1), recall that there is a well-known one-to-one correspondence between increasing binary trees with n vertices, and permutations of size n, see *e.g.* [32, pp. 23–25]. Hence, the total number of increasing labellings of all binary trees of size n is equal to the number of permutations of size n, which yields the formula

$$\sum_{\substack{T \text{ binary} \\ \text{tree of size } n}} \prod_{v \in T} \frac{1}{h_T(v)} = 1. \tag{2}$$

Despite their simplicity, both formulae (1) and (2) have been the subject of many research papers. We mention briefly five directions that these papers have taken:

- q-Analogues of formula (1) have been found where increasing labellings of a given tree are counted with respect to one (or more) statistics: see [3] and [9, Lemma 5.3];
- Formula (1) (and the q-analogues mentioned above) has been extended to more general classes of posets than trees (or forests): d-complete posets [27,28], shrubs [8, Proposition 3.6], forests with duplications [15, Theorem 1.4];
- In summation formula (2), the factor $\frac{1}{h_T(v)}$ can be replaced by some more complicated function of $h_T(v)$ such that the sum over binary trees remains nice. An example is the following formula [13, Eq. (1.2)]

$$\sum_{\substack{T \text{ binary} \\ \text{tree of size } n}} \prod_{v \in T} \left(x + \frac{1}{h_T(v)} \right) = \frac{1}{(n+1)!} \prod_{i=0}^{n-1} \big((n+1+i)x + n + 1 - i \big). \tag{3}$$

  The case $x = 0$ of course corresponds to (2), the case $x = 1$ is due to A. Postnikov [26, Corollary 17.3] and the general case is due to R. Du and F. Liu, who proved a conjecture of A. Lascoux, see [13] and the references therein. Subsequently, G. Han designed an algorithm to discover such equalities, finding a generalization of Du and Liu's result, as well as many other formulae [20];
- Another direction consists in replacing in summation formula (2) (or in the generalized version (3)) binary trees by other families of trees. Formulae of this kind for plane forests or m-ary trees have been given in several papers [13,34,33,10];
- Finally, formulae (1) and (2) admit a number of higher level interpretations. In [21], it is explained how (2) (and some generalizations) arises from solving differential equations and can be lifted to the level of combinatorial Hopf algebras. Probabilistic interpretations of (2) and generalizations are presented by B. Sagan in [31]. In a different direction, interpretations of (1) and some refinements/generalizations have been given in convex geometry [5, Section 6] and commutative algebra [15].

In this paper, we follow simultaneously both the third and fourth directions above. Indeed, we present a summation formula, in which the simple ratio $\frac{1}{h_T(v)}$ is replaced by a more complicated expression with several parameters. Besides, we do not work with binary trees, but instead with *unordered increasing rooted trees*:

- *unordered* means that the sons of a given vertex are not ordered;

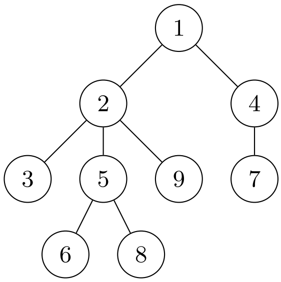

Fig. 1. An increasing unordered tree.

- *increasing* means that the vertices are labelled (each integer between 1 and r is used exactly once) and that the label of a son is always bigger than the label of its father (in particular, the root always gets label 1).

An example of an unordered increasing tree is given in Fig. 1. Since the sons of a given vertex are not ordered, we have chosen the convention of always drawing them in increasing order from left to right.

Our summation formula is given in the following theorem, which is the main result of this paper. We use the notation for falling factorials $(a)_m = a(a-1)\cdots(a-m+1)$ for positive integers m, with $(a)_0 = 1$, and $(a)_m = 1/(a-m)_{-m}$ for negative integers m.

Theorem 1.1. *Let $r \geqslant 1$ be an integer and $k_1, \dots, k_r$ be formal variables, with $K = \sum_{i=1}^{r} k_i$. For an unordered increasing tree T with r vertices, define the weight to be*

$$\mathrm{wt}(T) = \prod_{v=2}^{r} k_{f(v)} \left(\left(\sum_{u \in \mathfrak{h}_T(v)} k_u \right) - h_T(v) + 1 \right),$$

where $f(v)$ stands for the father of v in T. Then

$$\sum_T \mathrm{wt}(T) = k_1 \cdots k_r (K-1)_{r-2}, \tag{4}$$

where the sum runs over all unordered increasing trees on r vertices.

For example, the weight of the tree given in Fig. 1 is

$$k_1(k_2+k_3+k_5+k_6+k_8+k_9-5)\cdot k_2k_3\cdot k_1(k_4+k_7-1)$$
$$\cdot k_2(k_5+k_6+k_8-2)\cdot k_5k_6\cdot k_4k_7\cdot k_5k_8\cdot k_2k_9.$$

Note that, if v is a leaf, its contribution to the weight is $k_{f(v)}k_v$. Since each vertex is either a leaf or the father of another vertex, the quantity $\mathrm{wt}(T)$ is always divisible by $k_1 \cdots k_r$ (except for $r = 1$).

We refer to (4) as our *hook formula*. We point out the fact that the formula for trees of size r involves r independent parameters, while formula (3) and all formulae in [20] involve a fixed number of parameters. As mentioned above, for $r > 1$, the monomial $k_1 \cdots k_r$ divides all terms of the sum, but the latter do not share any other factors. Thus it is quite remarkable that the right-hand side, which is a polynomial in r parameters, can be written as a product of simple linear factors. (Note that in the case $r = 1$, we have $(K-1)_{r-2} = k_1^{-1}$, which cancels the factor k_1.)

In Section 2 we present two specializations of our result: an analogue of the aforementioned hook formula of Postnikov, and the multivariate enumeration of Cayley trees with respect to vertex degree. In our opinion, this makes Theorem 1.1 interesting in itself.

Another interesting feature of this new hook formula is the connection with representation theory of the symmetric group. This link is explained in Section 3, where we give our first proof of

Theorem 1.1. This proof uses Kerov's character polynomials, and does not seem related to the Frame–Robinson–Thrall formula. The proof is quite involved, and reasonably indirect, so we also give two inductive proofs of the hook formula that are more direct. The first of these direct proofs, given in Section 4, uses elementary operators on polynomials. The second of these direct proofs is given in Section 5, and uses Lagrange's Implicit Function Theorem in many variables.

2. Two specializations of the hook formula

2.1. An analogue of Postnikov's formula

Here we consider the specialization of all variables $k_1, \dots, k_r$ to the same value k. Then the weight of an unordered increasing tree T in Theorem 1.1 becomes

$$\begin{aligned}\mathrm{wt}'(T) = \mathrm{wt}(T)|_{k_i=k} &= k^{r-1}\prod_{v=2}^{r}\big((k-1)h_T(v)+1\big)\\ &= \frac{k^{r-1}}{(k-1)r+1}\prod_{v\in T}\big((k-1)h_T(v)+1\big).\end{aligned}$$

Therefore, setting $x = k-1$, our hook formula becomes

$$\sum_{\substack{T \text{ increasing}\\ \text{unordered tree}\\ \text{of size } r}} \prod_{v\in T}\big(xh_T(v)+1\big) = (x+1)\prod_{i=1}^{r-1}(x\cdot r+i). \tag{5}$$

Using the fact (Eq. (1)) that there are $n!/(\prod_{v\in T} h_v(T))$ increasing labellings for each binary tree T, Eq. (3) can be rewritten as

$$\sum_{\substack{T \text{ increasing}\\ \text{binary tree}\\ \text{of size } n}} \prod_{v\in T}\big(xh_T(v)+1\big) = \frac{1}{n+1}\prod_{i=0}^{n-1}\big((n+1+i)x+n+1-i\big). \tag{6}$$

Thus the specialization with equal parameters of our formula is an analogue of Postnikov's formula for another family of trees. Unfortunately, a short computer exploration suggests that Eq. (6) does not seem to have such a nice multivariate refinement as Theorem 1.1.

2.2. Multivariate enumeration of Cayley trees

By definition, a Cayley tree is a tree[3] with distinguishable vertices. As early as 1860 [4], C.W. Borchardt proved that the number of trees with vertex set $[r] = \{1, \dots, r\}$ is r^{r-2}. As noticed by A. Cayley [7], his proof also leads to the following multivariate enumeration formula for what are now called Cayley trees:

$$\sum_{\substack{U \text{ Cayley tree}\\ \text{with vertex set } [r]}} k_1^{d_1(U)}\cdots k_r^{d_r(U)} = k_1\cdots k_r K^{r-2}, \tag{7}$$

where $d_i(U)$ denotes the degree of the vertex i in a tree U.

We will show that the specialization $k_1, \dots, k_r \to \infty$, that is the highest degree term in k of our hook formula, corresponds to (7). Hence our hook formula can be viewed as a non-homogeneous extension of the multivariate enumeration of Cayley trees.

[3] Cayley trees are not embedded in the plane and have no root, they are only specified by an adjacency matrix.

To do this, we define a mapping φ from Cayley trees with vertex set V to increasing unordered trees with label set V, where V is a finite nonempty set of positive integers. Consider a Cayley tree U with vertex set V. The definition is inductive and produces an increasing unordered tree $T=\varphi(U)$ as follows:

- Let $\ell=\min V$. If $|V|=1$, then T has a single vertex, with label ℓ. Otherwise, remove vertex ℓ and all incident edges from U, to obtain a forest whose connected components are Cayley trees $U_1, U_2, \dots;$
- Apply φ inductively to $U_1, U_2, \dots;$
- Take the disjoint union of all $T_i=\varphi(U_i)$, and add a vertex (which is the root vertex of T) with label ℓ, joined to the root vertices of all T_i.

The mapping φ is clearly not injective in general. If T is an increasing unordered tree with label set V, then the elements U of the preimage $\varphi^{-1}(T)$ can be obtained inductively as follows:

- Let $\ell=\min V$. If $|V|=1$, then U has the single vertex ℓ. Otherwise, remove the root vertex of T (which has label ℓ), to obtain the increasing unordered trees $T_1, T_2, \dots;$
- Select an element U_i in each set $\varphi^{-1}(T_i)$;
- Take the disjoint union of all U_i, choose one vertex in each U_i and add a vertex with label ℓ joined to all selected vertices.

For a given increasing unordered tree T, denote

$$\mathrm{wt}''(T)=\sum_{U:\varphi(U)=T}\prod_{v\in V} k_v^{d_v(U)}.$$

The above description of $\varphi^{-1}(T)$ implies that

$$\mathrm{wt}''(T)=\prod_{T_i}\mathrm{wt}''(T_i)\left(k_\ell\sum_{v\in T_i}k_v\right),$$

where ℓ is the label of the root and the product is taken over the trees $T_1, T_2, \dots$ obtained by removing the root of T. An immediate induction yields

$$\mathrm{wt}''(T)=\prod_{v=2}^{r}k_{f(v)}\left(\sum_{u\in\mathfrak{h}_T(v)}k_u\right),$$

with the same notation as in Theorem 1.1. We observe that $\mathrm{wt}''(T)$ is exactly the highest degree term in $\mathrm{wt}(T)$ and therefore, as an immediate corollary of Theorem 1.1, we get

$$\sum_{\substack{T \text{ increasing}\\ \text{unordered tree}\\ \text{of size } r}}\mathrm{wt}''(T)=k_1\cdots k_r K^{r-2},$$

which is the multivariate enumeration formula (7) for Cayley trees.

3. Kerov character polynomials

In this section, we explain how Theorem 1.1 arises from computations in representation theory of the symmetric group. In fact, the two sides of our hook formula correspond to the same coefficient of the so-called *Kerov character polynomials*, computed in two different ways.

In Section 3.1, we explain Kerov character polynomials and which coefficient we want to compute. Then, in Sections 3.2, 3.3 and 3.4, we give different ways to compute this coefficient, which lead to our hook formula. The first two approaches lead to the same result, but we have chosen to present both to be more comprehensive on the subject.

3.1. Definitions

Let us consider, for each n, the family of symmetric groups S_n. It is well-known (see, *e.g.*, [30, Chapter 2]) that both conjugacy classes and irreducible representations of S_n can be indexed canonically by partitions of n, so the character table of S_n is a collection of numbers $\chi^\lambda(\mu)$, where λ and μ run over partitions of n and are, respectively, the indices of the irreducible representation and the conjugacy class.

Following S.V. Kerov and G.I. Olshanski [23], for any partition μ of size k, we shall consider the function Ch_μ on the set $\mathcal{Y}$ of all Young diagrams (or equivalently of all partitions of all sizes) defined by

$$\mathrm{Ch}_\mu(\lambda) = \begin{cases} 0 & \text{if } n < k; \\ n(n-1)\cdots(n-k+1)\frac{\chi^\lambda(\mu\cup(1^{n-k}))}{\chi^\lambda((1^n))} & \text{otherwise,} \end{cases}$$

where n is the size of λ.

We also consider another family of functions on Young diagrams: the free cumulants $(R_k)_{k\geqslant 2}$ of the transition measure (for their definition we refer to [1, Section 1]). It has been shown by S. Kerov [2, Theorem 1] (the reference given deals only with the case of a one-part partition μ, but the proof can be readily extended to the general case) that there exist polynomials K_μ such that, as functions on all Young diagrams,

$$\mathrm{Ch}_\mu = K_\mu(R_2, R_3, \ldots). \tag{8}$$

These polynomials are called Kerov character polynomials. Their coefficients have been the subject of many research articles in the last few years, see [11] and references therein. Here we focus on the coefficient of a single R_j (*linear* coefficient) for the maximal value of j, that is

$$j = |\mu| - \ell(\mu) + 2.$$

This coefficient has a very compact expression that we prove in the next paragraph (we use throughout the notation $[A]B$ to denote the *coefficient* of A in the expansion of B).

Proposition 3.1. *Let μ be a partition and $j = |\mu| - \ell(\mu) + 2$. Then*

$$[R_j]K_\mu = (-1)^{\ell(\mu)-1}\left(\prod_{i=1}^{\ell(\mu)} \mu_i\right)\frac{(|\mu|-1)!}{(|\mu|-\ell(\mu)+1)!}.$$

3.2. Combinatorial interpretation of Kerov polynomials

Linear coefficients in Kerov polynomials have a quite simple combinatorial interpretation, established by P. Biane [2, Theorem 5.1] for one-part partitions μ, and by A. Rattan and P. Śniady [29, Theorem 19] for arbitrary partitions μ:

$(-1)^{\ell(\mu)-1}[R_j]K_\mu$ is the number of pairs (σ_1, σ_2) such that

- σ_1 and σ_2 are permutations in $S_{|\mu|}$ with

$$\sigma_1\sigma_2 = \sigma_\mu, \tag{9}$$

  where $\sigma_\mu = (1\cdots\mu_1)(\mu_1+1\cdots\mu_2)\cdots$;
- σ_2 is a long cycle;
- σ_1 has $j-1$ cycles.

Note that the absolute lengths[4] of σ_1 and σ_μ are $|\mu| - (j-1) = \ell(\mu) - 1$ and $|\mu| - \ell(\mu)$. These two numbers sum up to $|\mu| - 1$. This allows to use a theorem of F. Bédard and A. Goupil, who counted the number of factorizations (9) where σ_1 has a given cycle-type λ (here, $|\lambda| = |\mu|$ and $\ell(\lambda) = j-1$). They obtained the following number [6, Theorem 3.1] (see also [18, Theorem 2.2]):

$$\frac{(\ell(\mu)-1)!(j-2)!\prod_i \mu_i}{m_1(\lambda)!m_2(\lambda)!\cdots},$$

where $m_i(\lambda)$ is the number of parts of λ equal to i, $i \geqslant 1$. To obtain $[R_j]K_\mu$, we have to sum over all possible cycle-types λ:

$$(-1)^{\ell(\mu)-1}[R_j]K_\mu = \frac{(\ell(\mu)-1)!}{j-1}\prod_i \mu_i \sum_{\substack{\lambda \vdash |\mu|,\\ \ell(\lambda)=|\mu|-\ell(\mu)+1}} \frac{(j-1)!}{m_1(\lambda)!m_2(\lambda)!\cdots}.$$

The term indexed by λ in the sum counts the number of sequences $i_1, \dots, i_{j-1}$ that are permutations of λ. Hence the sum is the number of sequences $i_1, \dots, i_{j-1}$ of positive integers of sum $|\mu|$, that is $\binom{|\mu|-1}{j-2}$. It is then straightforward to see that the expression above simplifies to the one in Proposition 3.1.

3.3. *Macdonald symmetric functions*

In this paragraph, we present another approach to Proposition 3.1, which relies on a basis of the symmetric function ring introduced by I.G. Macdonald.

Consider the center $Z(\mathbb{C}[S_n])$ of the symmetric group algebra of size n. A basis is given by the conjugacy class sums, that is

$$C\ell_\lambda = \sum_{\text{cycle-type}(\sigma)=\lambda} \sigma.$$

Since $Z(\mathbb{C}[S_n])$ is an algebra, there exist constants $c_{\mu,\nu}^\lambda$ such that, for any two partitions μ and ν of size n,

$$C\ell_\mu \, C\ell_\nu = \sum_{\lambda \vdash n} c_{\mu,\nu}^\lambda \, C\ell_\lambda.$$

These constants are called *structure constants* or *connection coefficients* of $Z(\mathbb{C}[S_n])$ and have been widely studied in the literature.

Macdonald [25, Exercises I.7.24, I.7.25] gave an explicit construction of a basis u_λ of the symmetric function ring, which can be characterized as follows:

- u_λ is homogeneous of degree $|\lambda|$;
- if λ has only one part, then u_λ is given by

  $$u_{(n)} = -p_n,$$

  where p_n is the n-th power sum;
- for a partition λ, denote by $\bar{\lambda}$ the partition obtained from λ by adding one to every part. Then, for any partitions μ, ν and $n \geqslant |\bar{\mu}| + |\bar{\nu}|$,

  $$u_\mu u_\nu = \sum_{\lambda \vdash |\mu|+|\nu|} c_{\bar{\mu}1^{n-|\bar{\mu}|}, \bar{\nu}1^{n-|\bar{\nu}|}}^{\bar{\lambda}1^{n-|\bar{\lambda}|}} u_\lambda \tag{10}$$

  where c is the structure constant of the center of the symmetric group algebra defined above.

[4] The absolute length of a permutation is the minimal number of factors needed to write it as a product of transpositions. It should note be confused with its Coxeter length.

This construction can be found in paper [19] (see in particular Theorem 3.2 and Proposition 4.1, which corresponds to the properties above).

Note that it is well-known [14, Lemma 3.9] that the coefficients in the right-hand side of (10) do not depend on n (because $|\lambda| = |\mu| + |\nu|$).

We will see that Kerov polynomials contain in some sense Macdonald symmetric functions. To do this, consider, as in [12] the gradation $\deg_2$ on the algebra Λ generated by R_k (for $k \geqslant 2$) defined

$$\deg_2(R_k) = k - 2.$$

One can show that free cumulants are algebraically independent so the definition makes sense. Then, one has the following properties:

- The top component of K_k is R_{k+1}. Indeed consider a monomial $\prod_{i=1}^t R_{j_i}$ appearing to the top component of K_k for $\deg_2$, *i.e.* such that

$$\sum_{i=1}^{t} (j_i - 2) = k - 1.$$

  Then we must also have $\sum j_i \leqslant k + 1$ [2, Section 6]. These two equations imply $t \leqslant 1$, which means that only R_{k+1} appears in the top component of K_k (and its coefficient is known to be 1);
- Let μ and ν be two partitions. Then one has

$$\frac{K_{\bar{\mu}}}{z_{\bar{\mu}}} \cdot \frac{K_{\bar{\nu}}}{z_{\bar{\nu}}} = \sum_{\lambda \vdash |\mu|+|\nu|} c^{\bar{\lambda} 1^{n-|\bar{\lambda}|}}_{\bar{\mu} 1^{n-|\bar{\mu}|}, \bar{\nu} 1^{n-|\bar{\nu}|}} \frac{K_{\bar{\lambda}}}{z_{\bar{\lambda}}} + \text{smaller degree terms for } \deg_2,$$

  where z_π is the classical constant $\prod_i i^{m_i} m_i!$ if π is written as $1^{m_1} 2^{m_2} \cdots$ in exponential notation [25, Chapter 1]. This second property can be deduced from [22, Proposition 4.5]: we skip details here.

Consider the algebra isomorphism between the subalgebra $\mathbb{Q}[R_3, R_4, \ldots]$ of Λ and the symmetric function ring sending R_{j+2} to $-(j+1)p_j$. Then the top component of $\frac{K_{\bar{\lambda}}}{z_{\bar{\lambda}}}$ is sent to u_λ because of the two properties above.

Hence, this top component can be computed using results on u_λ, in particular [19, Lemmas 7.1 and 7.2]. If $j - 2 = |\bar{\nu}| - \ell(\bar{\nu}) = |\nu|$, then

$$\begin{aligned}
[R_j]K_{\bar{\nu}} &= \frac{-z_{\bar{\nu}}}{j-1}[p_{j-2}]u_\nu = \frac{-z_{\bar{\nu}}}{(j-1)(j-2)}[h_\nu]\left[s^{j-2}\right]\frac{1}{(\sum_{m \geqslant 0} h_m s^m)^{j-2}} \\
&= \frac{-z_{\bar{\nu}}}{(j-1)(j-2)}\binom{-(j-2)}{m_1(\nu), m_2(\nu), \ldots} \\
&= \frac{-z_{\bar{\nu}}}{(j-1)(j-2)}(-1)^{\ell(\nu)}\binom{j-2+\ell(\nu)-1}{m_1(\nu), m_2(\nu), \ldots}.
\end{aligned}$$

Simplifying the expression above and setting $\mu = \bar{\nu}$, we obtain Proposition 3.1.

3.4. *Using the generalized Frobenius formula*

The most efficient way to compute the polynomials K_μ with a computer is to use the generalized Frobenius formula [29, Theorem 5]. To state it, we need the notion of boolean cumulants B_k (for $k \geqslant 2$) of the transition measure. They are functions on the set of all Young diagrams and they form another algebraic basis of Λ such that

$$B_k = R_k + \text{non-linear terms}.$$

This implies that $[B_k]\,\mathrm{Ch}_\mu = [R_k]\,\mathrm{Ch}_\mu$, which is by definition $[R_k]K_\mu$ (see Eq. (8)). Lastly, we denote by $H(z)$ the generating function of boolean cumulants (which has coefficients in the ring Λ):

$$H(z) = z - B_2 z^{-1} - B_3 z^{-2} - \cdots.$$

The following result of A. Rattan and P. Śniady expresses the normalized character values Ch_μ in terms of boolean cumulants:

Theorem 3.2. *(See [29].) For any integers $\mu_1 \geqslant \cdots \geqslant \mu_r \geqslant 1$,*

$$(-1)^r \mu_1 \cdots \mu_r \,\mathrm{Ch}_{\mu_1,\dots,\mu_r} = \left[z_1^{-1}\right] \cdots \left[z_r^{-1}\right]\left[\left(\prod_{1 \leqslant u \leqslant r} H(z_u) H(z_u - 1) \cdots H(z_u - \mu_u + 1)\right)\right.$$
$$\left. \times \prod_{1 \leqslant s < t \leqslant r} \frac{(z_s - z_t)(z_s - z_t + \mu_t - \mu_s)}{(z_s - z_t - \mu_s)(z_s - z_t + \mu_t)}\right]. \tag{11}$$

The right-hand side of (11) should be understood as follows: we expand the expression appearing there as a power series in decreasing powers of z_r with coefficients being Λ-valued functions of $z_1, \dots, z_{r-1}$ and select the appropriate coefficient. We repeat this procedure with respect to $z_{r-1}, z_{r-2}, \dots, z_1$.

In Proposition 3.1, we are interested in the coefficient of a single R_j of maximal degree. As mentioned above, it is equivalent to look at the coefficient of a single B_j of maximal degree. In this paragraph, we try to understand this coefficient using Theorem 3.2.

Let us first see what happens in the case $r = 2$: we consider the coefficient of $B_{\mu_1+\mu_2}$ in $\mathrm{Ch}_{\mu_1,\mu_2}$. The right-hand side of (11) can then be written as

$$\left[z_1^{-1}\right] H(z_1) \cdots H(z_1 - \mu_1 + 1) \left[z_2^{-1}\right] H(z_2) \cdots H(z_2 - \mu_2 + 1)$$
$$\times \frac{(z_1 - z_2)(z_1 - z_2 + \mu_2 - \mu_1)}{(z_1 - z_2 - \mu_1)(z_1 - z_2 + \mu_1)}. \tag{12}$$

When we expand the fraction in decreasing powers of z_2, no positive powers appear. In a factor H, the maximal exponent of z_2 is 1. Hence, the term $B_h z_2^{-(h-1)}$ for $h \geqslant \mu_2 + 2$ will not contribute to the coefficient in z_2^{-1}. In particular, one cannot obtain $B_{\mu_1+\mu_2}$, which is what we are looking for. Therefore each term $H(z_2 - c)$ can be replaced by $z_2 - c$.

That being said, to obtain at the end the B_j of maximal index, we have to keep the biggest possible power of z_1 in the coefficient of z_2^{-1}. To do that, we notice, that if we consider the total degree in the z-variable set

$$z_2 - c = z_2 + \text{smaller degree terms};$$
$$\frac{(z_1 - z_2)(z_1 - z_2 + \mu_t - \mu_s)}{(z_1 - z_2 - \mu_s)(z_1 - z_2 + \mu_t)} = 1 + \frac{\mu_2 \mu_1 / z_2^2}{(1 - z_1/z_2)^2} + \text{smaller degree terms}.$$

Hence we have

$$\left[z_2^{-1}\right] H(z_2) \cdots H(z_2 - \mu_2 + 1) \frac{(z_1 - z_2)(z_1 - z_2 + \mu_2 - \mu_1)}{(z_1 - z_2 - \mu_1)(z_1 - z_2 + \mu_1)}$$
$$= \left[z_2^{-1}\right]\left(z_2^{\mu_2} \cdot \frac{\mu_2 \mu_1 / z_2^2}{(1 - z_1/z_2)^2}\right) + \text{smaller degree terms in } z_1$$
$$= \mu_1 \mu_2^2 z_1^{\mu_2 - 1} + o\left(z_1^{\mu_2 - 1}\right).$$

Plugging this into Eq. (12) and setting all B_j to 0, except $B_{\mu_1+\mu_2}$, we obtain

$$[B_{\mu_1+\mu_2}]\mu_1\mu_2\,\mathrm{Ch}_{\mu_1,\mu_2}$$
$$= [B_{\mu_1+\mu_2}]\big[z_1^{-1}\big]\prod_{i=0}^{\mu_1-1}\big(z_1 - i - B_{\mu_1+\mu_2}(z_1-i)^{-(\mu_1+\mu_2-1)}\big)\big(\mu_1\mu_2^2 z_1^{\mu_2-1} + o\big(z_1^{\mu_2-1}\big)\big).$$

When we expand the product on the right-hand side, the term containing $B_{\mu_1+\mu_2}$ of maximal degree in z_1 is obtained by picking $\mu_1 - 1$ factors z_1, one factor $-B_{\mu_1+\mu_2} z_1^{-(\mu_1+\mu_2-1)}$ and finally the factor $\mu_1\mu_2^2 z_1^{\mu_2-1}$ in the last parenthesis. We have μ_1 ways to do so (corresponding to the choice of the index i from which we take the term $B_{\mu_1+\mu_2} z_1^{\mu_1+\mu_2-1}$) and thus

$$[B_{\mu_1+\mu_2}]\mu_1\mu_2\,\mathrm{Ch}_{\mu_1,\mu_2}$$
$$= \big[z_1^{-1}\big]\big(-\mu_1 z_1^{\mu_1-1} z_1^{\mu_1+\mu_2-1}\big(\mu_1\mu_2^2 z_1^{\mu_2-1}\big) + \text{smaller degree terms in } z_1\big) = -\mu_1^2\mu_2^2.$$

Since $[B_{\mu_1+\mu_2}]\mathrm{Ch}_{\mu_1,\mu_2} = [R_{\mu_1+\mu_2}]\mathrm{Ch}_{\mu_1,\mu_2}$, we recover Proposition 3.1 in the case $\ell(\mu) = 2$.

Let us consider now the general case. We want to compute the coefficient of B_j in $\mathrm{Ch}_{\mu_1,\dots,\mu_r}$ for $j - 2 = \sum_i(\mu_i - 1) = K - r$. As in the case $\ell(\mu) = 2$, when we extract the coefficient of some z_t (for $t > 1$), we have to keep only the highest degree term in the z-variable set. Therefore, for a fixed index $t > 1$, we can replace $H(z_t - c)$ by z_t and use the approximation

$$\prod_{1\leqslant s<t}\frac{(z_s - z_t)(z_s - z_t + \mu_t - \mu_s)}{(z_s - z_t - \mu_s)(z_s - z_t + \mu_t)} = 1 + \sum_{1\leqslant s<t}\frac{\mu_t\mu_s/z_t^2}{(1 - z_s/z_t)^2} + \text{smaller degree terms.} \tag{13}$$

So the highest degree term in z_1 after successive extractions of the coefficients of $z_r^{-1}, z_{r-1}^{-1}, \dots, z_2^{-1}$ is

$$\big[z_2^{-1}\big]\cdots\big[z_r^{-1}\big]\left(\prod_{t=2}^{r} z_t^{\mu_t}\left[1 + \sum_{1\leqslant s<t}\frac{\mu_t\mu_s/z_t^2}{(1 - z_s/z_t)^2}\right]\right).$$

Exchanging the product and summation symbol, we get a sum over the following set: for each $t > 1$, we have to choose an integer $s < t$ (we cannot choose the summand 1 in the bracket, because we would get z_t with a positive power, while we want to extract the coefficient of z_t^{-1}). These choices can be represented as an unordered increasing tree T with r vertices, in which s is the father of t. In the case $r = 2$, we only had one summand.

If $f(t)$ denotes the father of t in a tree T, the summand associated to T is

$$A_T := \big[z_2^{-1}\big]\cdots\big[z_r^{-1}\big]\left(\prod_{t=2}^{r} z_t^{\mu_t}\frac{\mu_t\mu_{f(t)}/z_t^2}{(1 - z_{f(t)}/z_t)^2}\right). \tag{14}$$

We then use the expansion

$$\frac{1}{(1 - z_{f(t)}/z_t)^2} = \sum_{m_t\geqslant 1} m_t(z_{f(t)}/z_t)^{m_t-1}$$

and rewrite Eq. (14) as

$$A_T = \big[z_2^{-1}\big]\cdots\big[z_r^{-1}\big]\left(\prod_{t=2}^{r} z_t^{\mu_t}\mu_t\mu_{f(t)}z_t^{-2}\sum_{m_t\geqslant 1} m_t(z_{f(t)}/z_t)^{m_t-1}\right). \tag{15}$$

A straightforward induction beginning at the leaves of T and going up to the root shows that the coefficient of $z_2^{-1}\cdots z_r^{-1}$ corresponds to the summand

$$m_t = \sum_{u\in\mathfrak{h}_T(t)}\mu_u - h_T(t) + 1,$$

where $h_T(t) = |\mathfrak{h}_T(t)|$, and $\mathfrak{h}_T(t)$ is the hook of t, as defined in the introduction. So, finally Eq. (15) reduces to

$$A_T = z_1^{K-\mu_1+r-1} \prod_{t=2}^{r} \mu_t \mu_{f(t)} \left(\sum_{u \in \mathfrak{h}_T(t)} \mu_u - h_T(t) + 1 \right).$$

Coming back to formula (11), the coefficient $[B_j]\,\mathrm{Ch}_{\mu_1,\dots,\mu_r}$ is given by

$$[B_{K-r+2}](-1)^r \mu_1 \cdots \mu_r\, \mathrm{Ch}_{\mu_1,\dots,\mu_r} = [B_{K-r+2}]\big[z_1^{-1}\big] H(z_1) \cdots H(z_1 - \mu_1 + 1) \left(\sum_T A_T \right).$$

As in the case $r = 2$, the extraction of the coefficient of $B_{K-r+2} z_1^{-1}$ yields an extra factor μ_1 and the equation above simplifies to

$$(-1)^{r-1} [B_j]\, \mathrm{Ch}_{\mu_1,\dots,\mu_r} = \sum_T \left(\prod_{t=2}^{r} \mu_{f(t)} \left(\sum_{u \in \mathfrak{h}_T(t)} \mu_u - h_T(t) + 1 \right) \right).$$

Together with Proposition 3.1 and the remark above that

$$[B_j]\, \mathrm{Ch}_{\mu_1,\dots,\mu_r} = [R_j]\, \mathrm{Ch}_{\mu_1,\dots,\mu_r} = [R_j] K_{\mu_1,\dots,\mu_r},$$

this proves (in a very indirect way) Theorem 1.1.

4. Elementary operators on polynomials

The purpose of this section is to give the first of our two direct proofs of the hook formula (Theorem 1.1), which uses operators on polynomials. We proceed by induction on r, with base case $r = 1$, for which the theorem is trivially true.

In the induction, we will consider trees whose label sets are not necessarily an interval $[r] = \{1, \dots, r\}$. Thus we use the notation $X(T)$ for the label set of a tree T. We shall use the following construction on trees.

Definition 4.1. Let T_1 and T_2 be two unordered increasing trees with disjoint sets of labels. Assume that the label of the root of T_1 is smaller than the label of the root of T_2. Then, we can construct a new unordered increasing tree, called *grafting of* T_2 *on* T_1, denoted $T_2 \bullet T_1$, defined as follows:

- its set of labels is $X(T_1) \sqcup X(T_2)$;
- its root label is the root label of T_1;
- the vertex with the root label of T_2 is a son of the root;
- every non-root vertex of T_1 (resp. T_2) has the same father in $T_2 \bullet T_1$ as in T_1 (resp. T_2).

This construction is illustrated in Fig. 2.

Now consider an arbitrary (unordered increasing) tree T of size $r > 1$. The vertices labelled 1 and 2 must be joined by an edge because T is increasing, so T can be obtained in a unique way by grafting a tree T_2 with root 2 on a tree T_1 with root 1.

Let us denote, for a subset X of $[r]$, $K_X = \sum_{i \in X} k_i$. The weight of the tree $T_2 \bullet T_1$ obtained by grafting is given by the formula

$$\mathrm{wt}(T_2 \bullet T_1) = \mathrm{wt}(T_2)\,\mathrm{wt}(T_1) k_1 \big(K_{X(T_2)} - \big|X(T_2)\big| + 1\big),$$

so summing over all trees $T = T_2 \bullet T_1$, we obtain

$$\sum_{\substack{T \text{ tree,} \\ X(T)=[r]}} \mathrm{wt}(T) = \sum_{T_1, T_2} \mathrm{wt}(T_2)\,\mathrm{wt}(T_1) k_1 \big(K_{X(T_2)} - \big|X(T_2)\big| + 1\big).$$

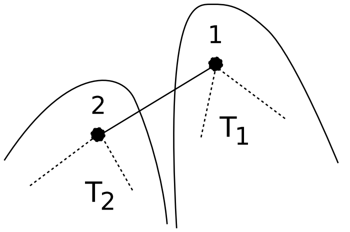

Fig. 2. A tree T as a grafting of T_2 on T_1.

The sum on the right-hand side runs over pairs of trees such that $X(T_1)$ contains 1, $X(T_2)$ contains 2 and the sets $X(T_1)$ and $X(T_2)$ form a partition of $[r]$. Splitting the sum according to the sets $X_h = X(T_h)\backslash\{h\}$ (for $h = 1, 2$), we obtain

$$\sum_{\substack{T \text{ tree}, \\ X(T)=[r]}} \mathrm{wt}(T) = \sum_{\substack{X_1, X_2, \\ X_1 \sqcup X_2 = \{3,\dots,r\}}} k_1\big(k_2 + K_{X_2} - |X_2|\big) \times \Bigg(\sum_{\substack{T_1, \\ X(T_1)=\{1\}\sqcup X_1}} \mathrm{wt}(T_1)\Bigg)\Bigg(\sum_{\substack{T_2, \\ X(T_2)=\{2\}\sqcup X_2}} \mathrm{wt}(T_2)\Bigg). \tag{16}$$

We now apply the induction hypothesis on the right-hand side to get, for $h = 1, 2$,

$$\sum_{\substack{T_h, \\ X(T_h)=\{h\}\sqcup X_h}} \mathrm{wt}(T_h) = k_h\Bigg(\prod_{i \in X_h} k_i\Bigg)(k_h + K_{X_h} - 1)_{|X_h|-1}.$$

Plugging this into (16), we obtain

$$\sum_{\substack{T \text{ tree}, \\ X(T)=[r]}} \mathrm{wt}(T) = \Bigg(\prod_{i=1}^{r} k_i\Bigg) P(k_1, \dots, k_r),$$

where

$$P(k_1, \dots, k_r) := \sum_{\substack{X_1, X_2, \\ X_1 \sqcup X_2 = \{3,\dots,r\}}} k_1(k_1 + K_{X_1} - 1)_{|X_1|-1}(k_2 + K_{X_2} - 1)_{|X_2|}. \tag{17}$$

In order to complete the inductive proof of our hook formula, we now prove that, for $r \geqslant 2$, $P(k_1, \dots, k_r)$ is equal to

$$Q(k_1, \dots, k_r) = (K - 1)_{r-2}.$$

It is clear that both $\{P(k_1, \dots, k_r)\}_{r \geqslant 2}$ and $\{Q(k_1, \dots, k_r)\}_{r \geqslant 2}$ are families of multivariate polynomials, and that, for each $r \geqslant 2$, Q satisfies the following two properties:

- As a polynomial in k_1, the constant term is

$$Q(0, k_2, \dots, k_3) = (K_{\{2,\dots,r\}} - 1)_{r-2}; \tag{18}$$

- It satisfies the finite difference equation

$$\Delta_{k_1} Q(k_1, \dots, k_r) = \sum_{i=3}^{r} Q(k_1 + k_i, k_2, \dots, \widehat{k_i}, \dots, k_r). \tag{19}$$

Here Δ_{k_1} stands for the finite difference operator with respect to k_1, that is, $\Delta_{k_1} f(k_1) = f(k_1+1) - f(k_1)$, and the notation $\widehat{k_i}$ means that k_i does *not* appear as an argument.

These two properties completely determine the family of multivariate polynomials $\{Q(k_1,\dots,k_r)\}_{r\geqslant 2}$ (by immediate induction on r). We now complete the proof that $P=Q$ by proving that the family $\{P(k_1,\dots,k_r)\}_{r\geqslant 2}$ also has these two properties.

Constant term: If $X_1 \neq \emptyset$, then $(k_1+K_{X_1}-1)_{|X_1|-1}$ is a polynomial in k_1, which implies that the summand corresponding to X_1 in Eq. (17) is a multiple of k_1. Thus, the constant term of P corresponds to the summand indexed by $X_1=\emptyset$, which implies immediately that P satisfies Eq. (18).

Finite difference equation: A simple computation gives

$$\Delta_{k_1}\bigl(k_1(k_1+K_{X_1}-1)_{|X_1|-1}\bigr) = \bigl(|X_1|k_1+K_{X_1}\bigr)(k_1+K_{X_1}-1)_{|X_1|-2}.$$

Therefore, from (17) we obtain

$$\begin{aligned}\Delta_{k_1} P(k_1,\dots,k_r) \\ = \sum_{\substack{X_1,X_2,\\ X_1\sqcup X_2=\{3,\dots,r\}}} \bigl(|X_1|k_1+K_{X_1}\bigr)(k_1+K_{X_1}-1)_{|X_1|-2}(k_2+K_{X_2}-1)_{|X_2|}.\end{aligned} \tag{20}$$

Also, directly from (17), we have

$$\begin{aligned}&\sum_{i=3}^{r} P(k_1+k_i,k_2,\dots,\widehat{k_i},\dots,k_r) \\ &= \sum_{i=3}^{r} \sum_{\substack{Y_1,Y_2,\\ Y_1\sqcup Y_2=\{3,\dots,r\}\setminus\{i\}}} (k_1+k_i)(k_1+k_i+K_{Y_1}-1)_{|Y_1|-1}(k_2+K_{Y_2}-1)_{|Y_2|} \\ &= \sum_{i=3}^{r} \sum_{\substack{X_1,X_2,\\ X_1\sqcup X_2=\{3,\dots,r\},\ i\in X_1}} (k_1+k_i)(k_1+K_{X_1}-1)_{|X_1|-2}(k_2+K_{X_2}-1)_{|X_2|} \\ &= \sum_{\substack{X_1,X_2,\\ X_1\sqcup X_2=\{3,\dots,r\}}} \Bigl(\sum_{i\in X_1}(k_1+k_i)\Bigr)(k_1+K_{X_1}-1)_{|X_1|-2}(k_2+K_{X_2}-1)_{|X_2|},\end{aligned}$$

where we have changed summation indices from the first equation above to the second by setting $X_1 = Y_1 \sqcup \{i\}$ and $X_2 = Y_2$. Comparing this with (20) implies immediately that P satisfies Eq. (19), which completes the proof that $P=Q$, and hence the first direct proof of our hook formula.

5. Multivariate Lagrange inversion

For the second direct proof of our hook formula (Theorem 1.1), we apply Lagrange inversion in many variables. We again proceed by induction on r, with base case $r=1$, for which the theorem is trivially true. Now consider an arbitrary (unordered increasing) tree T of size $r>1$. The root vertex labelled 1 has degree j for some $j\geqslant 1$, and the tree decomposes into j sub-trees, whose vertex sets form a partition of $\{2,\dots,r\}$. From this analysis we immediately obtain the following recurrence relationship for the combinatorial sum on the left-hand side of the hook formula in Theorem 1.1:

$$\sum_T \mathrm{wt}(T) = \sum_{j\geqslant 1}\frac{k_1^j}{j!}\sum_{X_1\sqcup\cdots\sqcup X_j=\{2,\dots,r\}}\prod_{i=1}^{j}\bigl(K_{X_i}-|X_i|+1\bigr)\sum_{T_i:X(T_i)=X_i}\mathrm{wt}(T_i). \tag{21}$$

We complete the proof by showing that the algebraic expression on the right-hand side of the hook formula in Theorem 1.1 also satisfies this recurrence equation. To do so, we apply the following multivariate form of Lagrange's Implicit Function Theorem, as given in Goulden and Jackson [17, Theorem 1.2.9(1)].

Theorem 5.1. *Suppose that $w_i = t_i\phi_i(\mathbf{w})$, where ϕ_i is a formal power series with constant term 1, for $i = 1, \dots, r$, with $\mathbf{w} = (w_1, \dots, w_r)$. Then for integers $n_1, \dots, n_r$ and formal Laurent series f, we have*

$$\left[t_1^{n_1} \cdots t_r^{n_r}\right] f(\mathbf{w}) = \left[\lambda_1^{n_1} \cdots \lambda_r^{n_r}\right] f(\lambda)\phi_1(\lambda)^{n_1} \cdots \phi_r(\lambda)^{n_r} \det\left(\delta_{ij} - \frac{\lambda_j}{\phi_i(\lambda)} \frac{\partial \phi_i(\lambda)}{\partial \lambda_j}\right)_{1 \leqslant i, j \leqslant r},$$

where $\lambda = (\lambda_1, \dots, \lambda_r)$.

Applying this form of Lagrange's Theorem, we obtain the following identity.

Theorem 5.2. *For $r \geqslant 2$, we have*

$$k_1 \cdots k_r (K-1)_{r-2} = \sum_{j \geqslant 1} \frac{k_1^j}{j!} \sum_{X_1 \sqcup \cdots \sqcup X_j = \{2, \dots, r\}} \prod_{i=1}^{j} \left(\prod_{\ell \in X_i} k_\ell\right) (K_{X_i} - 1)_{|X_i| - 1}.$$

Proof. Consider $\phi_i(\mathbf{w}) = (1 + w_1 + \cdots + w_r)^{k_i}$, for $i = 1, \dots, r$. Then we have

$$\begin{aligned}
\det\left(\delta_{ij} - \frac{\lambda_j}{\phi_i(\lambda)} \frac{\partial \phi_i(\lambda)}{\partial \lambda_j}\right) &= \det\left(\delta_{ij} - \frac{\lambda_j k_i}{1 + \lambda_1 + \cdots + \lambda_r}\right) \\
&= 1 - \frac{\sum_{i=1}^r \lambda_i k_i}{1 + \sum_{i=1}^r \lambda_i},
\end{aligned}$$

since $\det(I + M) = 1 + \operatorname{trace} M$ when $\operatorname{rank} M \leqslant 1$.

We now calculate $[t_1 \cdots t_r]w_1$ in two ways. First, directly from Theorem 5.1, with $n_1 = \cdots = n_r = 1$, and $f(\mathbf{w}) = w_1$, we obtain

$$\begin{aligned}
[t_1 \cdots t_r]w_1 &= [\lambda_1 \cdots \lambda_r]\lambda_1 \left(1 + \sum_{i=1}^r \lambda_i\right)^K \left(1 - \frac{\sum_{i=1}^r \lambda_i k_i}{1 + \sum_{i=1}^r \lambda_i}\right) \\
&= (r-1)! \binom{K}{r-1} - (K - k_1)(r-2)! \binom{K-1}{r-2} \\
&= k_1 (K-1)_{r-2}.
\end{aligned}$$

Second, applying the functional equation $w_1 = t_1\phi_1(\mathbf{w})$, we obtain

$$\begin{aligned}
[t_1 \cdots t_r]w_1 &= [t_1 \cdots t_r]t_1 \left(1 + \sum_{i=1}^r w_i\right)^{k_1} \\
&= [t_2 \cdots t_r] \sum_{j \geqslant 0} \frac{k_1^j}{j!} \left(\log\left(1 + \sum_{i=1}^r w_i\right)\right)^j \\
&= \sum_{j \geqslant 1} \frac{k_1^j}{j!} \sum_{X_1 \sqcup \cdots \sqcup X_j = \{2, \dots, r\}} \prod_{i=1}^{j} \left(\left[\prod_{x \in X_i} t_x\right] \log\left(1 + \sum_{i=1}^r w_i\right)\right).
\end{aligned}$$

But, for any $X \subseteq \{2, \dots, r\}$, with $|X| = m \geqslant 1$, Theorem 5.1 gives

$$\begin{aligned}
&\left[\prod_{x\in X} t_x\right]\log\left(1+\sum_{i=1}^{r} w_i\right)\\
&\quad=\left[\prod_{x\in X}\lambda_x\right]\log\left(1+\sum_{i=1}^{r}\lambda_i\right)\left(1+\sum_{i=1}^{r}\lambda_i\right)^{K_X}\left(1-\frac{\sum_{i=1}^{r}\lambda_i k_i}{1+\sum_{i=1}^{r}\lambda_i}\right)\\
&\quad=\left[\prod_{x\in X}\lambda_x\right]\log\left(1+\sum_{x\in X}\lambda_x\right)\left(1+\sum_{x\in X}\lambda_x\right)^{K_X}\left(1-\frac{\sum_{x\in X}\lambda_x k_x}{1+\sum_{x\in X}\lambda_x}\right)\\
&\quad=m!\left[z^m\right]\log(1+z)(1+z)^{K_X}-K_X(m-1)!\left[z^{m-1}\right]\log(1+z)(1+z)^{K_X-1}\\
&\quad=(m-1)!\left[z^{m-1}\right]\left\{\frac{d}{dz}\left(\log(1+z)(1+z)^{K_X}\right)-\log(1+z)\frac{d}{dz}(1+z)^{K_X}\right\}\\
&\quad=(m-1)!\left[z^{m-1}\right]\frac{1}{1+z}(1+z)^{K_X}=(m-1)!\binom{K_X-1}{m-1}=(K_X-1)_{m-1}.
\end{aligned}$$

The result follows by equating the two expressions for $[t_1 \cdots t_r]w_1$, and then multiplying by $k_2 \cdots k_r$. $\square$

It follows immediately from Theorem 5.2 that the algebraic expression on the right-hand side of the hook formula in Theorem 1.1 also satisfies recurrence Eq. (21), and this completes the second direct proof of our hook formula.

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