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# A BIJECTIVE PROOF OF THE $q$-SAALSCHÜTZ THEOREM

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A combinatorial proof of a $q$-analogue of Saalschütz's summation theorem is given by determining an explicit bijection between sets of integer partitions. The proof relies on the fact that the generating function for partitions with upper bounds on the number of parts and largest part is a $q$-analogue of the binomial coefficient.

## 1. Introduction

Saalschütz's theorem (see [11, p. 243]) is a summation formula involving the ordinary hypergeometric function ${}_3F_2$. A number of binomial coefficient identities equivalent to Saalschütz's theorem have been obtained by various means. Combinatorial proofs of such identities have been given by Cartier and Foata [4, p. 63], Knuth [9, p. 30], Andrews [1] and Foata [5]. Andrews' [1] proof was for Nanjundiah's [10] form, which is

$$\sum_{r\geq 0}\binom{m-\mu}{r}\binom{n+\mu}{\mu+r}\binom{\mu+\nu+r}{m+n}=\binom{\mu+\nu}{m}\binom{\nu}{n}. \tag{1.1}$$

Many biomial coefficient identities possess $q$-analogues involving the $q$-binomial (or Gaussian) coefficient $\left[{i \atop j}\right]$, defined for positive integers $j$ by

$$\begin{bmatrix} i \\ j \end{bmatrix}=\frac{(1-q^i)\cdots(1-q^{i-j+1})}{(1-q^i)\cdots(1-q)},$$

and $\left[{i \atop 0}\right]=1$ for all $i$. Saalschütz's theorem has a $q$-analogue due to Jackson ([8]), which we call the $q$-Saalschütz theorem. This can be expressed as a $q$-binomial coefficient identity in a number of ways (see [2, p. 37], [7]). For example, Andrews ([2, p. 37]) has proved

$$\sum_{r\geq 0} q^{(n-r)(m-r-\mu)}\begin{bmatrix} m-\mu \\ r \end{bmatrix}\begin{bmatrix} n+\mu \\ \mu+r \end{bmatrix}\begin{bmatrix} \mu+\nu+r \\ m+n \end{bmatrix}=\begin{bmatrix} \mu+\nu \\ m \end{bmatrix}\begin{bmatrix} \nu \\ n \end{bmatrix}. \tag{1.2}$$

(We obtain (1.1) from (1.2) by letting $q$ approach 1, so we say that (1.2) is a $q$-analogue of (1.1).) Gould's [7] form was applied by Stanley [12] to deduce an enumerative result for permutations, with respect to greater index (defined in Section 2).

In this paper we give a direct combinatorial proof of

$$\sum_{k\geq u} q^{k(k-u)}\begin{bmatrix} m-u \\ k-u \end{bmatrix}\begin{bmatrix} n+u \\ k \end{bmatrix}\begin{bmatrix} m+n+t-k \\ m+n \end{bmatrix}=\begin{bmatrix} n+t \\ n \end{bmatrix}\begin{bmatrix} m-u+t \\ m \end{bmatrix}, \tag{1.3}$$

which is another form of the $q$-Saalschütz theorem. (Note that (1.2) is obtained from (1.3) by setting $t=\nu-n$, $u=m-n-\mu$ and $k=m-\mu-r$.) The proof relies on the interpretation of the $q$-binomial coefficient as a generating function for partitions.

Let $\mathscr{P}(i,j)$ consist of all $j$-tuples $p=(p_1,\ldots,p_j)$, where $i\geq p_1\geq\cdots\geq p_j\geq 0$. If $p_1+\cdots+p_j=s(p)$, then $p$ is called a *partition* of $s(p)$. The non-zero $p_k$'s are called the *parts* of $p$, so $\mathscr{P}(i,j)$ is the set of partitions with largest part at most $i$, and at most $j$ parts. Define $P(i,j)=\sum_{p\in\mathscr{P}(i,j)} q^{s(p)}$, so the coefficient of $q^n$ in $P(i,j)$ is the number of partitions of $n$ with parts at most $i$, and at most $j$ parts.

**Lemma 1.1.** $P(i,j)=P(j,i)=\begin{bmatrix} i+j \\ j \end{bmatrix}$.

**Proof.** See Andrews [2] for a proof (p. 35) and references (p. 51). $\square$

Thus we actually prove (in Section 3) that

$$\sum_{k\geq u} q^{k(k-u)}P(m-k,k-u)P(n+u-k,k)P(m+n,t-k)=P(t,n)P(t-u,m), \tag{1.4}$$

which is equivalent to (1.3) by Lemma 1.1.

Part of our proof of the $q$-Saalschütz theorem involves a direct evaluation of Stanley's [12] generating function for permutations with respect to greater index in a special case. In this sense we are reversing Stanley's approach, since he gave an indirect evaluation of this generating function by applying the $q$-Saalschütz theorem.

Andrews and Bressoud [3] have given a completely different combinatorial proof of the $q$-Saalschütz theorem.

## 2. Definitions and notation

Let $\mathscr{N}_n=\{1,\ldots,n\}$, and $\alpha=1\cdots n$ be the increasing permutation on $\mathscr{N}_n$, and $\beta=n+u+1\cdots n+1\ n+u+2\cdots n+m$, for some $0\leq u\leq m-1$. Let $\mathscr{S}$ be the set of permutations on $\mathscr{N}_{n+m}$ which contain both $\alpha$ and $\beta$ as subsequences. Thus $|\mathscr{S}|=\binom{n+m}{n}$, since an element of $\mathscr{S}$ is uniquely determined by the subset of its $n+m$ positions containing the elements of $\alpha$ in the specified (increasing) order (the complementary positions contain the elements of $\beta$ in their specified order).

For $\sigma=\sigma_1\cdots\sigma_{n+m}\in\mathcal{S}$, we say that $\sigma_i$ is a *descent* if $\sigma_i>\sigma_{i+1}$, and we define $\mathcal{D}(\sigma)=\{i\mid\sigma_i>\sigma_{i+1}\}$, the set of positions in $\sigma$ that are descents, and the *greater index* $I(\sigma)=\sum_{i\in\mathcal{D}(\sigma)}i$, the sum of positions of descents in $\sigma$. We denote the number of descents in $\sigma$ by $d(\sigma)=|\mathcal{D}(\sigma)|$, and let $\mathcal{S}_k=\{\sigma\mid\sigma\in\mathcal{S},\ d(\sigma)=k\}$, the set of permutations in $\mathcal{S}$ with $k$ descents. Finally, let $S_k=\sum_{\sigma\in\mathcal{S}_k}q^{I(\sigma)-\binom{u+1}{2}}$. Since $n+u+1,\ldots,n+2$ are descents in every $\sigma\in\mathcal{S}$, then $I(\sigma)\geqslant 1+\cdots+u=\binom{u+1}{2}$, so $I(\sigma)-\binom{u+1}{2}\geqslant 0$, and $S_k=0$ for $k<u$.

Fix a nonnegative integer $t$. We now consider a set $\mathcal{R}$ of sequences obtained by inserting $t$ copies of 0 into each element $\sigma=\sigma_1\cdots\sigma_{n+m}$ of $\mathcal{S}_k$ for $u\leqslant k\leqslant t$, wherever we please, at either end of $\sigma$ or between adjacent elements of $\sigma$, with the only restriction being that we must insert at least one 0 between $\sigma_i$ and $\sigma_{i+1}$ if $\sigma_i$ is a descent. For any $\rho\in\mathcal{R}$ constructed in this way from $\sigma$, then $\sigma$ is called the *base permutation* of $\rho$. For $\rho\in\mathcal{R}$, let $N(\rho)=N_1(\rho)+\cdots+N_t(\rho)$ and $M(\rho)=M_1(\rho)+\cdots+M_t(\rho)$, where $N_i(\rho)$ (respectively $M_i(\rho)$) is the number of elements of $\alpha$ (respectively $\beta$) that are to the left of the $i$th 0 in $\rho$ (numbered from left to right). Now let $Z(\rho)=N(\rho)+M(\rho)$, and define $R=\sum_{\rho\in\mathcal{R}}q^{Z(\rho)-\binom{u+1}{2}}$. Note that $M(\rho)\geqslant 1+\cdots+u=\binom{u+1}{2}$ and $N(\rho)\geqslant 0$, by definition, so $Z(\rho)-\binom{u+1}{2}\geqslant 0$.

In the next section we evaluate the generating function $R$ in two ways to obtain identity (1.4). This will involve three bijections, and each will be demonstrated in the specific case $m=6$, $n=3$, $u=2$, $t=7$, $k=3$ for $\rho_0=0600501470023089\in\mathcal{R}$, with base permutation $\sigma_0=651472389$. Note that $N(\rho_0)=0+0+0+0+1+1+3=5$, $M(\rho_0)=0+1+1+2+4+4+4=16$, so $Z(\rho_0)-\binom{u+1}{2}=5+16-3=18$, and $I(\sigma_0)=1+2+5=8$, so $I(\sigma_0)-\binom{u+1}{2}=5$.

## 3. The $q$-Saalschütz theorem

First we show that $R$ is equal to the right hand side of (1.4).

**Theorem 3.1.** $R=P(t,n)P(t-u,m)$.

**Proof.** In $\rho\in\mathcal{R}$, suppose that there are $a_i$ zero's to the right of the $i$th element (from the left) of $\alpha$, for $i=1,\ldots,n$, and $b_j$ zero's to the right of the $j$th element of $\beta$, for $j=1,\ldots,m$. Then, by definition of $\mathcal{R}$, we have

$$t\geqslant a_1\geqslant\cdots\geqslant a_n\geqslant 0\quad\text{and}\quad t\geqslant b_1>\cdots>b_{u+1}\geqslant b_{u+2}\geqslant\cdots\geqslant b_m\geqslant 0.$$

Moreover $N(\rho)=a_1+\cdots+a_n$, since $a_1,\ldots,a_{N_i(\rho)}$ count the $i$th zero from the left exactly once each, so $a_1+\cdots+a_n=N_1(\rho)+\cdots+N_t(\rho)$. Similarly $M(\rho)=b_1+\cdots+b_m$.

Let $c_i=b_i$ for $i=u+1,\ldots,m$ and $c_i=b_i-(u+1-i)$ for $i=1,\ldots,u$. Then $t-u\geqslant c_1\geqslant\cdots\geqslant c_m\geqslant 0$ and $b_1+\cdots+b_m=c_1+\cdots+c_m+\binom{u+1}{2}$. Thus $a=(a_1,\ldots,a_n)\in\mathcal{P}(t,n)$, $c=(c_1,\ldots,c_m)\in\mathcal{P}(t-u,m)$ and $Z(\rho)-\binom{u+1}{2}=s(a)+s(c)$.

But this procedure is reversible, since knowledge of $a$ and $c$ tells us how many elements of $\alpha$ and how many elements of $\beta$ to place between consecutive zero's in $\rho$. There is a unique such $\rho$ since the elements between consecutive zero's must be in increasing order, by definition of $\mathcal{R}$. Thus we have a bijection between $\mathcal{R}$ and $\mathcal{P}(t, n)\times\mathcal{P}(t-u, m)$, so

$$R = \sum_{\rho\in\mathcal{R}} q^{Z(\rho)-\binom{u+1}{2}} = \sum_{a\in\mathcal{P}(t,n)} q^{s(a)} \sum_{c\in\mathcal{P}(t-u,m)} q^{s(c)}$$
$$= P(t, n)P(t-u, m). \qquad \square$$

As an example of the bijection in Theorem 3.1, consider $\rho_0$ given at the end of Section 2. Corresponding to $\rho_0\in\mathcal{R}$ we have $a_0=(3, 1, 1)\in\mathcal{P}(7, 3)$ and $c_0=(4, 3, 3, 3, 0, 0)\in\mathcal{P}(5, 6)$, and indeed $s(a_0)+s(c_0)=5+13=18=Z(\rho_0)-\binom{u+1}{2}$.

To show that $R$ is equal to the left hand side of (1.4), we give a construction in two stages.

**Theorem 3.2.** $R=\sum_{k\geqslant u} S_kP(m+n, t-k)$.

**Proof.** Suppose that $\rho\in\mathcal{R}$ has base permutation $\sigma=\sigma_1\cdots\sigma_{n+m}\in\mathcal{S}_k$. Then for $i\in\mathcal{D}(\sigma)$, $\sigma_i$ and $\sigma_{i+1}$ are separated by at least one zero in $\rho$, and we call the left-most of these zeros an *essential* zero. The remaining $t-k$ zeros in $\rho$ are called nonessential. Let the $i$th of the nonessential zeros (numbered from left to right) have $e_i$ elements of $\sigma$ to its left, for $i=1, \ldots, t-k$. Then $0\leqslant e_1\leqslant\cdots\leqslant e_{t-k}\leqslant m+n$, so $e=(e_{t-k}, \ldots, e_1)\in\mathcal{P}(m+n, t-k)$. Moreover, the contribution of the nonessential zeros to $Z(\rho)$ is $s(e)$, and the contribution of the essential zeros to $Z(\rho)$ is $I(\sigma)$, by definition, so $Z(\rho)-\binom{u+1}{2}=s(e)+I(\sigma)-\binom{u+1}{2}$.

But this procedure is reversible, since knowledge of $\sigma$ determines $k=d(\sigma)$, and thus allows us to place the essential zeros, one immediately following each of the $k$ descents in $\sigma$. Then $e$ tells us uniquely how to distribute the nonessential zeros. Thus we have a bijection between $\mathcal{R}$ and $\bigcup_{k\geqslant u}\mathcal{S}_k\times\mathcal{P}(m+n, t-k)$. Accordingly

$$R = \sum_{\rho\in\mathcal{R}} q^{Z(\rho)-\binom{u+1}{2}} = \sum_{k\geqslant u}\sum_{\sigma\in\mathcal{S}_k} q^{I(\sigma)-\binom{u+1}{2}} \sum_{e\in\mathcal{P}(m+n,t-k)} q^{s(e)}$$
$$= \sum_{k\geqslant u} S_kP(m+n, t-k). \qquad \square$$

As an example of the bijection in Theorem 3.2, corresponding to $\rho_0\in\mathcal{R}$ we have $\sigma_0=651472389\in\mathcal{S}_3$ and $e_0=(7, 5, 1, 0)\in\mathcal{P}(9, 4)$, and indeed $s(e_0)+I(\sigma_0)-\binom{u+1}{2}=13+5=18=Z(\rho_0)-\binom{u+1}{2}$.

The final result involves a direct construction of the elements of $\mathcal{S}_k$.

**Theorem 3.3.** $S_k=q^{k(k-u)}P(m-k, k-u)P(n+u-k, k)$.

**Proof.** Let $x=(x_{k-u}, \ldots, x_1)\in\mathcal{P}(m-k, k-u)$ and $y=(y_k, \ldots, y_1)\in\mathcal{P}(n+u-k, k)$. Then we construct $\sigma\in\mathcal{S}_k$ from $x$ and $y$, as follows, by choosing the elements of $\sigma$ from left to right. We begin by taking the first $y_1$ ($\geqslant 0$) elements of $\alpha$, followed by the first element of $\beta$. Then, for $i=2,\ldots,u$, alternate successive blocks of the next $y_i - y_{i-1}$ ($\geqslant 0$) elements of $\alpha$, followed by the $i$th element of $\beta$. Thus we have obtained the first $y_u + u$ elements of $\sigma$ from the first $y_u$ elements of $\alpha$ and the first $u$ elements (namely $n+u+1,\ldots,n+2$) of $\beta$. We continue by taking the next $y_{u+1}-y_u$ ($\geqslant 0$) elements of $\alpha$, followed by the next $x_1+1$ ($>0$) elements of $\beta$. Then for $j=2,\ldots,k-u$, alternate blocks of $y_{u+j}-y_{u+j-1}+1$ ($>0$) elements of $\alpha$ and $x_j - x_{j-1}+1$ ($>0$) elements of $\beta$. To finish, take the final $n+u+1-k-y_k$ ($>0$) elements of $\alpha$, followed by the final $m-k-x_{k-u}$ ($\geqslant 0$) elements of $\beta$.

Now for $\sigma$ constructed in this way, we have $\mathcal{D}(\sigma)=\{y_1+1,\ \ldots,\ y_u+u,\ y_{u+1}+u+1+x_1,\ y_{u+2}+u+2+x_2+1,\ \ldots,\ y_k+k+x_{k-u}+k-u-1\}$ so $d(\sigma)=k$ and $I(\sigma)=s(x)+s(y)+(1+\cdots+k)+(1+\cdots+(k-u-1))$. Simplifying gives $I(\sigma)-\binom{u+1}{2}=s(x)+s(y)+k(k-u)$.

But this procedure is reversible since, for any $\sigma\in\mathcal{S}_k$, the $k$ descents of $\sigma$ must include the first $u$ elements of $\beta$, as well as $k-u$ of the other $n-u$ elements of $\beta$, which are descents in $\sigma$ if and only if they are immediately followed by an element of $\alpha$. Thus $x$ and $y$ are uniquely recoverable from $\sigma$, so we have a bijection, between $\mathcal{S}_k$ and $\mathcal{P}(m-k,k-u)\times\mathcal{P}(n+u-k,k)$, which yields

$$\begin{aligned} S_k &= \sum_{\sigma\in\mathcal{S}_k} q^{I(\sigma)-\binom{u+1}{2}} = q^{k(k-u)} \sum_{x\in\mathcal{P}(m-k,k-u)} q^{s(x)} \sum_{y\in\mathcal{P}(n+u-k,k)} q^{s(y)} \\ &= q^{k(k-u)}P(m-k,k-u)P(n+u-k,k). \quad \square \end{aligned}$$

As an example of the bijection in Theorem 3.3, consider $\sigma_0$ given at the end of Section 2. Corresponding to $\sigma_0\in\mathcal{S}_3$ we have $x_0=(1)\in\mathcal{P}(3,1)$ and $y_0=(1,0,0)\in\mathcal{P}(2,3)$, and indeed $s(x_0)+s(y_0)+k(k-u)=1+1+3=5=I(\sigma_0)-\binom{u+1}{2}$.

The three results of this section yield an immediate proof of (1.4).

**Proof of the *q*-Saalschütz theorem.** Theorems 3.1, 3.2 and 3.3 demonstrate that the left- and right-hand sides of (1.4) are both expressions for $R$. The $q$-Saalschütz theorem (1.3) follows by applying Lemma 1.1. $\square$

We say that this is a bijective proof because it follows from the bijection between

$$\bigcup_{k\geqslant u} \mathcal{P}(m-k,k-u)\times\mathcal{P}(n+u-k,k)\times\mathcal{P}(m+n,t-k) \tag{3.1}$$

and

$$\mathcal{P}(t,n)\times\mathcal{P}(t-u,m), \tag{3.2}$$

obtained by combining the bijections used to prove Theorems 3.1, 3.2 and 3.3. For example, combining the examples following Theorems 3.1, 3.2 and 3.3, we find that ((1), (1, 0, 0), (7, 5, 1, 0)) in (3.1) corresponds under this bijection to ((3, 1, 1), (4, 3, 3, 3, 0, 0)) in (3.2).

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