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A linear operator for symmetric functions and tableaux in a strip with given trace

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Abstract

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The sum of Schur symmetric functions in a countable set of variables, over partitions with given trace and upper bound on the number of parts, is evaluated. This generalizes results of Gordon (1971). The method is to extend an analogous sum in a finite set of variables by means of a linear operator.

1. Introduction

If $\lambda_1, \ldots, \lambda_n$ are nonnegative integers with $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, then λ is a *partition* of $|\lambda| = \lambda_1 + \cdots + \lambda_n$, and we write $\lambda \vdash |\lambda|$. The nonzero λ_i's are the *parts* of λ, so λ_1 is the largest part, and $l(\lambda)$ is the number of parts of λ. The *conjugate* of λ, denoted by $\tilde{\lambda}$, is the partition $(\mu_1, \ldots, \mu_k)$, in which μ_j is the number of λ_i's that are $\geqslant j$ for $j = 1, \ldots, k$, where $k = \lambda_1$. The *trace* of λ, denoted by $c(\lambda)$, is the number of odd parts of $\tilde{\lambda}$.

If $H(t) = \prod_{i \geqslant 1} (1 - x_i t)^{-1}$, then the *complete* symmetric functions $h_0(\boldsymbol{x}), h_1(\boldsymbol{x}), \ldots$ are given by

$$h_k(\boldsymbol{x}) = [t^k]H(t), \quad k \geqslant 0,$$

where $[t^k]$ denotes the *coefficient* of t^k in the expression to the right, and $\boldsymbol{x} = (x_1, x_2, \ldots)$. We usually suppress these arguments, and write h_k for $h_k(\boldsymbol{x})$. We adopt the convention that $h_k = 0$ for $k < 0$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with at most n parts, the *Schur* symmetric function $s_\lambda(\boldsymbol{x})$ is given by (the Jacobi–Trudi identity)

$$s_\lambda(\boldsymbol{x}) = \det(h_{\lambda_i - i + j})_{1 \leqslant i,j \leqslant n}.$$

Equivalently, $s_\lambda(\boldsymbol{x})$ is the generating function for tableaux of shape λ. A *tableau* of shape λ is an array

$$\begin{array}{cccc} t_{11} & t_{12} & \cdots & t_{1\lambda_1} \\ t_{21} & t_{22} & & t_{2\lambda_2} \\ \vdots & & & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{n\lambda_n} \end{array}$$

of positive integers t_{ij} arranged in rows and columns, with λ_i elements in the ith row, $i=1,\ldots,n$, such that the t_{ij} form a nondecreasing sequence from left to right along each row, and an increasing sequence down each column. In the generating function, each occurrence of i in the array is marked by x_i, for $i \geq 1$.

A *standard* tableau of shape λ is a tableau of shape λ containing each of the integers $1, 2, \ldots, |\lambda|$ once each. Thus, if f^λ is the number of standard tableaux of shape λ, and $\lambda \vdash n$, then we immediately have

$$f^\lambda = [x_1 \cdots x_n] s_\lambda(\boldsymbol{x}),$$

where $[x_1 \cdots x_n]$ denotes the coefficient of $x_1 \cdots x_n$ in the expression to the right.

For a finite set of variables, say $\boldsymbol{z}=(z_1,\ldots,z_n)$ the Schur function can be written as a ratio of alternants:

$$s_\lambda(\boldsymbol{z}) = \frac{a_{\lambda+\delta}(\boldsymbol{z})}{a_\delta(\boldsymbol{z})},$$

where $\delta=(n-1, n-2, \ldots, 1, 0)$ and, for $\mu=(\mu_1,\ldots,\mu_n)$ any vector of integers

$$a_\mu(\boldsymbol{z}) = \det(z_i^{\mu_j})_{1\leq i,j\leq n}.$$

See Macdonald [10, Chapter I] for more complete details about symmetric functions.

Gordon [3] has proved the following result (see also Bender and Knuth [1], Gessel [2], Goulden [5] and Gordon and Houten [4]).

Theorem 1.1. *Let* $g_l = \sum_{i\geq 0} h_i h_{i+l}$ *for* $l \geq 0$, *and* $g_{-l} = g_l$. *Then*

(1) $\sum_{l(\lambda)\leq 2m} s_\lambda(\boldsymbol{x}) = \det(g_{i-j} + g_{i+j-1})_{1\leq i,j\leq m}$

(2) $\sum_{l(\lambda)\leq 2m+1} s_\lambda(\boldsymbol{x}) = (\sum_{k\geq 0} h_k)\det(g_{i-j} - g_{i+j})_{1\leq i,j\leq m}$.

In this paper these results are extended, by evaluating the summations further restricted to partitions λ with a given value of trace.

The method of proof is quite different from Gordon's. The starting point is the following well-known result of Littlewood [9].

Theorem 1.2. $\sum_{c(\lambda)=0} s_\lambda(\boldsymbol{z}) = \prod_{1\leq i<j\leq n} (1-z_i z_j)^{-1}$.

From the interpretation of $s_\lambda(\boldsymbol{z})$ as a generating function for tableaux, it is clear that $s_\lambda(\boldsymbol{z})=0$ if $l(\lambda)>n$, since a tableau of shape λ has a first column consisting of

$l(\lambda)$ distinct integers. Thus the summation of Theorem 1.2 is naturally terminating, and we could add the restriction that $l(\lambda) \leq n$ (and if n is odd, we can further restrict to $l(\lambda) \leq n-1$, since $c(\lambda) = 0$), so we can evaluate

$$\sum_{l(\lambda) \leq n, c(\lambda)=0} s_\lambda(\mathbf{z}) \tag{1}$$

by means of Theorem 1.2. However the corresponding summation in an infinite set of variables

$$\sum_{l(\lambda) \leq n, c(\lambda)=0} s_\lambda(\mathbf{x}) \tag{2}$$

is not immediately deducible from Theorem 1.2 since the number of variables in $\mathbf{z}$ is the same as the upper bound on $l(\lambda)$, so we cannot simply change the number of variables without changing this bound.

In Section 2 we introduce an operator ϕ that avoids this difficulty, allowing us to pass from (1) to (2), increasing the number of variables from n to infinite without raising the upper bound on $l(\lambda)$. Thus we evaluate (2) by means of Theorem 1.2. The result is given as Theorem 2.3. A similar operator has been used by Goulden [5] and Macdonald [10, p. 32, Ex. 12]. The operator that raises the number of variables one at a time has been used by Lascoux and Pragacz [7] and Lascoux and Schützenberger [8].

The extensions of Gordons' results are deduced directly from Theorem 2.3 by means of two different constructions for Schur functions. The odd case is given in Theorem 2.4 and the even case is given in Theorem 2.6. The analogous sums for numbers of standard tableaux have been evaluated for the cases $l(\lambda) \leq 4$ and 5 by Gouyou-Beauchamps [6]. These analogous sums are evaluated for the general case in Theorem 2.7.

In Section 3 we use the operator ϕ for two sets of variables and the Cauchy determinant formula to evaluate

$$\sum_{l(\lambda) \leq n} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}),$$

where $\mathbf{y} = (y_1, y_2, \ldots)$. This has been previously given by Gessel [2], using a different method.

2. A linear operator for symmetric functions and the extension of Gordon's result

Consider the operator ϕ, acting on Laurent series in $\mathbf{z}$, defined for integers $b_1, \ldots, b_n$ by

$$\phi(z_1^{b_1} \cdots z_n^{b_n}) = h_{b_1} \cdots h_{b_n}$$

extended linearly. The action of ϕ is particularly striking for Schur functions.

Lemma 2.1. *For any partition* λ *with* $l(\lambda) \leqslant n$,

$$\phi\Big(\prod_{1\leqslant i\leqslant j\leqslant n}(1-z_i^{-1}z_j)s_\lambda(z)\Big)=s_\lambda(x).$$

Proof. We have $\prod_{1\leqslant i<j\leqslant n}(1-z_i^{-1}z_j)=\prod_{i=1}^n z_i^{-(n-i)}a_\delta(z)$, and writing $s_\lambda(z)$ as a ratio of alternants gives

$$\begin{aligned}\mathrm{LHS}&=\phi\Big(\prod_{i=1}^n z_i^{-(n-i)}\alpha_\delta(z)a_{\lambda+\delta}(z)/\alpha_\delta(z)\Big)\\&=\phi\Big(\prod_{i=1}^n z_i^{-(n-i)}a_{\lambda+\delta}(z)\Big)\\&=\phi(\det(z_i^{\lambda_j+n-j-(n-i)})_{1\leqslant i,j\leqslant n})\\&=\det(h_{\lambda_j-j+i})_{1\leqslant i,j\leqslant n}\end{aligned}$$

and the result follows by the Jacobi–Trudi identity. □

In applying ϕ, it is convenient to note the following property.

Lemma 2.2. *For any* $k=1,\ldots,n$, *suppose that* α *is independent of* z_k *and* P *is a polynomial whose coefficients are independent of* z_k. *Then*

$$\phi((1-\alpha z_k)^{-1}P(z_k^{-1}))=\phi((1-\alpha z_k)^{-1}P(\alpha)).$$

Proof. Let $P(w)=\sum_{j=0}^m p_j w^j$. Then

$$\begin{aligned}\mathrm{LHS}&=\phi\Big(\sum_{i\geqslant 0}(\alpha z_k)^i\sum_{j=0}^m p_j z_k^{-j}\Big)=\phi\Big(\sum_{j=0}^m p_j\sum_{i\geqslant 0}\alpha^i z_k^{i-j}\Big)\\&=\phi\Big(\sum_{j=0}^m p_j\sum_{i\geqslant j}\alpha^i h_{i-j}\Big),\quad \text{since } h_l=0 \text{ for } l<0,\end{aligned}$$

and the result follows. □

The first Schur function summation we evaluate is for those with at most $2m$ rows, and zero trace. The symmetric functions g_k, defined in Theorem 1.1, arise because $g_k=\phi(z_i^k(1-z_iz_j)^{-1})$ for $i\neq j$.

Theorem 2.3 *For any* $m\geqslant 1$,

$$\sum_{l(\lambda)\leqslant 2m,\,c(\lambda)=0}s_\lambda(x)=\det(g_{i-j}-g_{i+j})_{1\leqslant i,j\leqslant m}.$$

Proof. From Lemma 2.1, with $n=2m$,

$$\mathrm{LHS}=\phi\Big(\prod_{1\leqslant i<j\leqslant n}(1-z_i^{-1}z_j)\sum_{l(\lambda)\leqslant n,\,c(\lambda)=0}s_\lambda(z)\Big).$$

But $s_\lambda(z)=0$ for $l(\lambda)>n$, so the restriction $l(\lambda)\leq n$ can be removed in the above sum. Thus, from Theorem 1.2,

$$\mathrm{LHS}=\phi(F_n(z))$$

where

$$F_n(z)=\prod_{1\leq i<j\leq n}(1-z_i^{-1}z_j)(1-z_iz_j)^{-1}.$$

Now, by a partial fraction expansion, we have

$$\prod_{j=2}^{n}(1-z_1z_j)^{-1}=\sum_{j=2}^{n}(1-z_1z_j)^{-1}\prod_{i=2,i\neq j}^{n}(z_j-z_i)^{-1}z_j^{n-2}.$$

Thus

$$\phi(F_n(z))=\sum_{j=2}^{n}\phi\Big((1-z_1z_j)^{-1}\prod_{l=2}^{n}(1-z_1^{-1}z_l)\beta_j\Big)$$

where

$$\beta_j=z_j^{n-2}\prod_{i=2,i\neq j}^{n}(z_j-z_i)^{-1}F_{n-1}(z\setminus\{z_1\}).$$

Now apply Lemma 2.2 with $k=1$ to obtain

$$\begin{aligned}\phi(F_n(z))&=\sum_{j=2}^{n}\phi\Big((1-z_1z_j)^{-1}\prod_{l=2}^{n}(1-z_1^{-1}z_l)\beta_j\Big)\\&=\sum_{j=2}^{n}(-1)^j\phi((1-z_1z_j)^{-1}z_j^{j-2}(1-z_j^2))\\&\quad\times\phi(z_2^{-1}\cdots z_{j-1}^{-1}F_{n-2}(z\setminus\{z_1,z_j\})).\end{aligned}$$

Comparing this with the recurrence for a Pfaffian pf, we get

$$\phi(F_n(z))=\mathrm{pf}(a_{ij})_{1\leq i,j\leq n},$$

where $a_{ij}=0$ for $i=j$, $a_{ij}=-a_{ji}$ for $i>j$, and

$$\begin{aligned}a_{ij}&=\phi((1-z_iz_j)^{-1}z_j^{j-i-1}(1-z_j^2))\\&=g_{j-i-1}-g_{j-i+1},\quad\text{for } i<j.\end{aligned}$$

The result follows from Gordon [3, Lemma 1]. □

Note that, for $l\geq 0$

$$\begin{aligned}g_l-g_{l+2}&=\sum_{i\geq 0}h_ih_{i+l}-\sum_{j\geq 0}h_jh_{j+l+2}\\&=\sum_{i\geq 0}(h_ih_{i+l}-h_{i-1}h_{i+l+1})=\sum_{i\geq 0}s_{(i+l,i)}(\boldsymbol{x})\end{aligned}$$

by the Jacobi–Trudi identity, so g_l-g_{l+2} is the generating function for tableaux with at most two rows, whose lengths differ by exactly l. Thus the entries of

$\mathrm{pf}(g_{j-i-1} - g_{j-i+1})$ have a nice combinatorial interpretation, which suggests that a direct combinatorial proof of Theorem 2.3 is possible.

First we extend the odd case of Gordon's result, deducing it from Theorem 2.3.

Theorem 2.4. *For $m, k \geqslant 0$,*

$$\sum_{l(\lambda)\leqslant 2m+1, c(\lambda)=k} s_\lambda(\boldsymbol{x}) = h_k \det(g_{i-j} - g_{i+j})_{1\leqslant i,j\leqslant m}.$$

Proof. From Macdonald [10, p. 42, 5.16] we deduce that

$$\sum_{l(\lambda)\leqslant 2m+1} s_\lambda(\boldsymbol{x}) t^{c(\lambda)} = \Big(\sum_{r\geqslant 0} h_r t^r\Big) \sum_{l(\lambda)\leqslant 2m, c(\lambda)=0} s_\lambda(\boldsymbol{x}).$$

The result follows from Theorem 2.3. □

Now we extend the even case of Gordon's result. In the proof it is convenient to use the operator ψ_t that replaces some x_i by t in a symmetric function in $\boldsymbol{x}$. More precisely, if $f(\boldsymbol{x})$ is symmetric in $\boldsymbol{x}$ and

$$f(\boldsymbol{x}) = \sum_{j\geqslant 0} f_j(\boldsymbol{x}\setminus\{x_i\}) x_i^j,$$

then

$$\psi_t(f(\boldsymbol{x})) = \sum_{j\geqslant 0} f_j(\boldsymbol{x}) t^j.$$

Since $f(\boldsymbol{x})$ is symmetric, this is independent of the choice of i, and is itself a symmetric function in $\boldsymbol{x}$. The important fact that is needed about ψ_t is given in the next result.

Proposition 2.5. *For any $m \geqslant 1$,*

$$\psi_t\Big(\sum_{l(\lambda)\leqslant 2m, c(\lambda)=0} s_\lambda(\boldsymbol{x})\Big) = \sum_{l(\lambda)\leqslant 2m} s_\lambda(\boldsymbol{x}) t^{c(\lambda)}.$$

Proof. Since $s_\lambda(\boldsymbol{x})$ is the generating function for the tableaux of shape λ, then $\psi_t(s_\lambda(\boldsymbol{x}))$ is the generating function for tableaux of shape λ, with t marking the occurrence of any positive integer we choose. Suppose t marks the 'largest' integer, say ∞. Then ∞ appears only as the last element in a column, to the right of all smaller symbols in the same row. If there are k occurrences of ∞ and these are removed, the remaining symbols form a tableaux, k of whose columns have length one less than the corresponding columns in the original tableau of shape λ.

Specifically, if $c(\lambda)=0$ and $l(\lambda)\leqslant 2m$, then a tableau of shape λ with k occurrences of ∞ yields a tableau of shape λ' when the ∞s are removed, where $l(\lambda')\leqslant 2m$ and $c(\lambda')=k$. The result follows. □

The even case of Gordon's result can now be extended.

Theorem 2.6. *For any $m \geqslant 1$ and $k \geqslant 0$,*

$$\sum_{l(\lambda)\leqslant 2m, c(\lambda)=k} s_\lambda(\boldsymbol{x}) = \det\begin{pmatrix} (g_{i-j} - g_{i+j})_{1\leqslant i\leqslant m-1, 1\leqslant j\leqslant m} \\ (g_{m-j+k} - g_{m+j+k})_{1\leqslant j\leqslant m} \end{pmatrix}$$

Proof. From Proposition 2.5,

$$\begin{aligned} \text{LHS} &= [t^k]\psi_t\Bigg(\sum_{l(\lambda)\leqslant 2m, c(\lambda)=0} s_\lambda(\boldsymbol{x})\Bigg) \\ &= [t^k]\psi_t(\det(g_{i-j} - g_{i+j})_{1\leqslant i,j\leqslant m}), \quad \text{from Theorem 2.3} \\ &= [t^k]\det(\psi_t(g_{i-j} - g_{i+j}))_{1\leqslant i,j\leqslant m}. \end{aligned}$$

But, as noted above for $l \geqslant 0$,

$$g_l - g_{l+2} = \sum_{\lambda_1 - \lambda_2 = l} s_{(\lambda_1, \lambda_2)}(\boldsymbol{x}),$$

and, for $\lambda_1 - \lambda_2 = l$,

$$\psi_t(s_{(\lambda_1,\lambda_2)}(\boldsymbol{x})) = \sum_{d=0}^{l} \sum_{n\geqslant 0} t^{d+n} s_{(\lambda_1 - d, \lambda_2 - n)}(\boldsymbol{x}),$$

by letting t mark the largest symbol in the tableaux counted by $s_{(\lambda_1,\lambda_2)}(\boldsymbol{x})$. Thus

$$\begin{aligned} \psi_t(g_l - g_{l+2}) &= \sum_{d=0}^{l} t^d \sum_{n\geqslant 0} t^n \sum_{\mu_1 - \mu_2 = l+n-d} s_{(\mu_1,\mu_2)}(\boldsymbol{x}) \\ &= \sum_{d=0}^{l} t^d \sum_{n\geqslant 0} (g_{l+n-d} - g_{l+n-d+2})t^n \\ &= \sum_{d=0}^{l} t^d\,(G_{l-d} - G_{l-d+2}) \end{aligned}$$

where $G_i = \sum_{n\geqslant 0} g_{i+n}t^n$, so

$$\begin{aligned} \psi_t(g_l - g_{l+2n}) &= \psi_t\Bigg(\sum_{b=0}^{n-1} (g_{l+2b} - g_{l+2b+2})\Bigg) \\ &= \sum_{b=0}^{n-1} \sum_{d=0}^{l+2b} t^d(G_{l+2b-d} - G_{l+2b-d+2}). \end{aligned}$$

Substituting this in the determinant yields

$$\text{LHS} = [t^k]\det(a_{ij})_{1\leqslant i,j\leqslant m}$$

where

$$a_{ij} = \sum_{b=0}^{j-1} \sum_{d=0}^{i-j+2b} t^d(G_{i-j+2b-d} - G_{i-j+2b-d+2}), \quad i \geqslant j,$$

and

$$a_{ij} = a_{ji}, \quad i < j.$$

We now simplify $\det(a_{ij})$ by a sequence of row operations:

(I) Replace row i by row $i - t$ row $(i-1)$, $i = 2, \ldots, m$

(II) Replace row i by row $i - t^{-1}$ row $(i-1)$, $i = 2, \ldots, m$

(III) Replace row 1 by $\sum_{i=1}^{m} t^i$ row i

(IV) Divide row 1 by t^{m-1}, and multiply row i by t, $i = 2, \ldots, m$.

(V) Multiply row i by -1 and interchange row i and row $(i-1)$, $i = 2, \ldots, m$.

These row operations do not change the value of the determinant, and transform (a_{ij}), using the identity $G_l = g_l + tG_{l+1}$, $l \geq 0$, repeatedly, to

$$\left(\frac{(g_{i-j} - g_{i+j})_{1\leq i\leq m-1, 1\leq j\leq m}}{(G_{m-j} - G_{m+j})_{1\leq j\leq m}}\right).$$

The result follows since $\text{LHS} = [t^k]\det(a_{ij})$ and $[t^k](G_{m-j} - G_{m+j}) = g_{m-j+k} - g_{m+j+k}$. $\square$

The corresponding results for degree sums follow immediately.

Theorem 2.7. *Let $w_l = \sum_{i\geq 0} (x^{2i+l}/i!\,(i+l)!)$ for $l \geq 0$, and $w_{-l} = w_l$. Then*

(1) $$\sum_{l(\lambda)\leq 2m+1, c(\lambda)=k} f^\lambda = \binom{n}{k}\left[\frac{x^{n-k}}{(n-k)!}\right]\det(w_{i-j} - w_{i+j})_{1\leq i,j\leq m}$$

(2) $$\sum_{l(\lambda)\leq 2m, c(\lambda)=k} f^\lambda = \left[\frac{x^n}{n!}\right]\det\left(\frac{(w_{i-j} - w_{i+j})_{1\leq i\leq m-1, 1\leq j\leq m}}{(w_{m-j+k} - w_{m+j+k})_{1\leq j\leq m}}\right).$$

Proof. The results follow from Theorems 2.4 and 2.6, using $f^\lambda = [x_1 \cdots x_n]\, s_\lambda(\boldsymbol{x})$, and the easily established fact that

$$[x_1 \cdots x_n]\Phi(h_1, h_2, \ldots) = \left[\frac{x^n}{n!}\right]\Phi\left(\frac{x^1}{1!}, \frac{x^2}{2!}, \ldots\right)$$

for any formal power series in the h_i's. $\square$

Suitably simplified, this result has been given for $m = 2$ by Gouyou-Beauchamps [6].

Note that the p-recursiveness (see Gessel [2], Stanley [11]) of the number of tableaux with at most i rows, with trace k, and containing j copies of each of $1, 2, \ldots, n$, where i, j, k are fixed, also follows immediately from the above results.

3. Pairs of Schur functions and Cauchy's determinant

Let $\boldsymbol{v} = (v_1, \ldots, v_n)$, $\boldsymbol{y} = (y_1, \ldots)$ and the operator $\phi_{(2)}$ be defined for Laurent series in $\boldsymbol{z}$ and $\boldsymbol{v}$ by

$$\phi_{(2)}(z_1^{b_1} \cdots z_n^{b_n} v_1^{c_1} \cdots v_n^{c_n}) = h_{b_1}(\boldsymbol{x}) \cdots h_{b_n}(\boldsymbol{x}) h_{c_1}(\boldsymbol{y}) \cdots h_{c_n}(\boldsymbol{y}),$$

extended linearly, where $b_1, \ldots, b_n$, $c_1, \ldots, c_n$ are arbitrary integers. The analogue of Lemma 2.1 is

$$\phi_{(2)}\Big(\prod_{1\le i<j\le n}(1-z_i^{-1}z_j)(1-v_i^{-1}v_j)s_\lambda(\boldsymbol{z})s_\lambda(\boldsymbol{v})\Big)=s_\lambda(\boldsymbol{x})s_\lambda(\boldsymbol{y}). \tag{3}$$

To calculate the summation

$$\sum_{l(\lambda)\le n} s_\lambda(\boldsymbol{x})s_\lambda(\boldsymbol{y})$$

we apply (3) to the following well-known result of Cauchy

$$\sum_\lambda s_\lambda(\boldsymbol{z})s_\lambda(\boldsymbol{v})=a_\delta(\boldsymbol{z})^{-1}a_\delta(\boldsymbol{v})^{-1}\det\Big(\frac{1}{1-z_iv_j}\Big)_{1\le i,j\le n}.$$

This yields

$$\begin{aligned}\sum_{l(\lambda)\le n} s_\lambda(\boldsymbol{x})s_\lambda(\boldsymbol{y}) &= \phi_{(2)}\Big(\det\Big(\frac{z_i^{-(n-i)}v_j^{-(n-j)}}{1-z_iv_j}\Big)_{1\le i,j\le n}\Big)\\ &=\det(u_{i-j})_{1\le i,j\le n}\end{aligned}$$

where $u_l=\sum_{k\ge 0}h_{k+l}(\boldsymbol{x})h_k(\boldsymbol{y})$, in agreement with Gessel [2].

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