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Shift Operators and Factorial Symmetric Functions

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A new class of symmetric functions called factorial Schur symmetric functions has recently been discovered in connection with a branch of mathematical physics. We align this theory more closely with the standard symmetric function theory, giving the factorial Schur function a tableau definition, introducing a shift operator and a new generating function with which we extend to factorial symmetric functions proofs of various determinantal identities for classical symmetric functions, and defining a new factorial symmetric function—the factorial elementary symmetric function.

1. INTRODUCTION

Recent work in the domain of mathematical physics has focused on a new inhomogeneous basis set of symmetric functions known as factorial Schur functions (see Biedenharn and Louck [2], [3] and Chen and Louck [5]). However, the factorial Schur functions are also interesting in their own right and a substantial theory surrounding them has begun to develop. Several equivalent definitions for factorial Schur functions are known, and factorial complete symmetric functions and skew factorial Schur functions have been identified. In this paper we seek to make this theory more closely aligned with standard material on symmetric functions as found, for example, in Chapter I of Macdonald [7]. Thus we give the factorial Schur functions a tableau definition, and, by means of generating functions involving a shift operator, show that the proofs of the various determinantal identities in the standard theory extend very nicely to the factorial versions. In addition, a factorial elementary symmetric function arises naturally by these means. Results similar to those in the paper but for α-paired factorial symmetric functions (defined in [2]) can also be proved; however, as the extension is routine, and space does not allow for a full exposition of these ideas, they are omitted.

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Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ are nonnegative integers and $\lambda_1 + \lambda_2 + \ldots + \lambda_m = n$. We then say λ is a *partition* of n (denoted $\lambda \vdash n$) with *m parts* (denoted $l(\lambda) = m$). Given any partition, we can represent it by a Ferrers diagram, that is, by an arrangement of squares which is left and top justified and which is such that there are λ_i squares in the ith row. The *content* of a square x in a Ferrers diagram is denoted by $c(x)$, and equals $j - i$ if x lies in column j from the left and row i from the top of the Ferrers diagram. The *conjugate* of a partition λ is defined to be the partition λ' whose Ferrers diagram is the transpose of the Ferrers diagram of λ. More explicitly, λ'_i is the number of squares in the ith column of λ, i.e., $\lambda'_i = \text{cardinality } \{j\colon \lambda_j \geq i\}$.

We can also define a *skew* partition. Given two partitions, λ and μ, we say $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all $i \geq 1$; i.e., the Ferrers diagram of λ contains the Ferrers diagram of μ in its upper left hand corner. If we remove the Ferrers diagram of μ from the upper left hand corner of the Ferrers diagram of λ, then we have the Ferrers diagram of the skew partition $\lambda - \mu$. The conjugate of $\lambda - \mu$ is defined to be the skew partition $\lambda' - \mu'$. If we insert positive integers into the squares of the Ferrers diagram of a skew partition $\lambda - \mu$ such that the entries strictly increase down each column and weakly increase left to right along each row, we say we have a *skew tableau* of shape $\lambda - \mu$. In a skew tableau T, we use $T(x)$ to denote the positive integer in square x of the Ferrers diagram of the shape of T. As a final comment, note that in what follows we assume a finite number of variables, $z_1, \ldots, z_m$, and adopt the conventions that $z = (z_1, \ldots, z_m)$ and $z + k = (z_1 + k, \ldots, z_m + k)$ for any integer k.

2. The Skew Factorial Schur Symmetric Function

For partitions λ, μ with $\mu \subseteq \lambda$ and $l(\lambda) \leq m$, the classical skew Schur function can be defined combinatorially as

$$s_{\lambda/\mu}(z) = \sum_{T} \prod_{x \in \lambda - \mu} z_{T(x)},$$

where the summation is over tableaux T of shape $\lambda - \mu$, and $x \in \lambda - \mu$ means that x ranges over all squares in the Ferrers diagram of $\lambda - \mu$.

This is a symmetric homogeneous polynomial in z with many interesting properties (see Macdonald [7] for a complete treatment). Among these are the Jacobi–Trudi identity

$$s_{\lambda/\mu}(z) = \det\left(h_{\lambda_i - \mu_j - i + j}(z)\right)_{m \times m}, \tag{1}$$

and its dual form

$$s_{\lambda/\mu}(z) = \det\left(e_{\lambda'_i-\mu'_j-i+j}(z)\right)_{\lambda_1\times\lambda_1}, \tag{2}$$

where $h_k(z)$ and $e_k(z)$, $k \geq 0$ are, respectively, the *complete* and *elementary* symmetric functions given by $\sum_{k\geq 0} h_k(z)t^k = \prod_{j=1}^m (1 - z_j t)^{-1}$ and $\sum_{k\geq 0} e_k(z)t^k = \prod_{j=1}^m (1 + z_j t)$.

The classical Schur function $s_\lambda(z)$ is simply $s_{\lambda/\varnothing}(z)$, and has the determinantal property

$$s_\lambda(z) = \frac{|z_i^{\lambda_j+m-j}|_{m\times m}}{|z_i^{m-j}|_{m\times m}}. \tag{3}$$

Consider now the related polynomials, with combinatorial definition

$$t_{\lambda/\mu}(z) = \sum_T \prod_{x\in\lambda-\mu} \left(z_{T(x)} - T(x) + 1 - c(x)\right) \tag{4}$$

where the summation is over tableaux T of shape $\lambda - \mu$. We let $t_\lambda(z) = t_{\lambda/\varnothing}(z)$. The polynomial $t_{\lambda/\mu}(z)$ has been considered by Chen and Louck [5] in an equivalent form—as a sum over skew Gel'fand patterns—and is called the *skew factorial Schur function*; $t_\lambda(z)$ has been considered by Biedenharn and Louck [2]—also in terms of Gel'fand patterns—and is called the factorial Schur function. This name has been chosen because of the following determinantal property, analogous to (3) above for Schur functions, where $(z_i)_k$ denotes the falling factorial $z_i(z_i - 1)\ldots(z_i - k + 1)$:

$$t_\lambda(z) = \frac{|(z_i)_{\lambda_j+m-j}|_{m\times m}}{|(z_i)_{m-j}|_{m\times m}}. \tag{5}$$

The equivalence of (5) and the combinatorial definition (4) above with $\mu = \varnothing$ (in its equivalent Gel'fand pattern form) has been established by Biedenharn, Louck, and Macdonald [5]. Clearly, from (5), $t_\lambda(z)$ is a symmetric polynomial in z, and, although it is not homogeneous, Biedenharn and Louck [2] have proved that $\{t_\lambda(z)\}$ forms a **Z**-basis for the ring of symmetric polynomials in z.

Chen and Louck [5] defined the sequence of polynomials $w_n(z)$, $n \geq 0$ by

$$w_n(z) = \sum_{1\leq i_1\leq i_2\leq \ldots \leq i_n\leq m} y_{i_1}(y_{i_2} - 1)\ldots(y_{i_n} - n + 1), \tag{6}$$

where $y_j = z_j - j + 1, j = 1, \ldots, m$, and proved an analogue of (1), namely

$$t_{\lambda/\mu}(z) = \det\Big(w_{\lambda_i - \mu_j - i + j}(z + j - 1 - \mu_j)\Big)_{m \times m}. \tag{7}$$

In this sense the $w_n(z)$ are analogues of the complete symmetric functions. As noted in Chen and Louck [5], they are themselves symmetric functions, so from (7), the skew factorial Schur function, $t_{\lambda/\mu}(z)$, is a symmetric function. We have been unable to find an elementary proof of this symmetry based on the combinatorial definition (4). For example, such a proof for the symmetry of the skew Schur function is given in Bender and Knuth [1] and is based on an involution for tableaux to which an adjacent transposition of the elements has been applied. However, this procedure does not seem to extend to the factorial case, so the fact that perturbation of $z_{T(x)}$ by $T(x) - 1 + c(x)$ retains symmetry is perhaps deeper than it might first appear.

In this paper we give a compact treatment of (5) and (7), and introduce an analogue of the elementary symmetric functions which leads to an analogue of (2). These results involve the shift operator, S, which acts homomorphically on polynomials in $z_1, \ldots, z_m$ and which is defined as $S(P(z)) = P(z - 1)$ for P a polynomial in $z_1, \ldots, z_m$. For example, using the shift operator, S, (5) may be rewritten as

$$t_\lambda(z) = \frac{|(z_i S)^{\lambda_j + m - j}(1)|_{m \times m}}{|(z_i S)^{m-j}(1)|_{m \times m}},$$

which provides a striking relationship between $t_\lambda(z)$ and $s_\lambda(z)$ when compared to (3).

At this juncture it is perhaps appropriate to note that $|(z_i S)^{m-j}(1)|_{m \times m} = |z_i^{m-j}|_{m \times m}$, the Vandermonde determinant, for if we replace z_i^k in $|z_i^{m-j}|_{m \times m}$ by a monic polynomial in z_i of degree k, then we can add and subtract multiples of other columns to recover z_i^k.

In what follows we will actually have two kinds of objects defined for each factorial symmetric function: an operator and a polynomial. The operators are polynomials in $z_1, \ldots, z_m$ and S. Products are, of course, not commutative, and are distinguished by the placement of an S in the argument list. For example, $f(z, S)$ is such an operator, while $f(z)$ is the polynomial defined by $f(z) = f(z, S)(1)$. Multiplication of operators is, of course, compositional multiplication, whereas multiplication of polynomials is ordinary multiplication. Also, any S occurring in a product is assumed to act on *any and all* zs *which follow it*. This is what one would expect in a compositional product but it is nonetheless a point worth stressing. Finally, suppose P and Q are operators and suppose every term in P contains d occurrences of S in some order. Then $(P(z,S)Q(z,S))(1) = (P(z)S^dQ(z,S))(1) = (P(z)Q(z-d,S)S^d)(1) = P(z)Q(z-d)$.

3. The Factorial Complete and Elementary Symmetric Functions

Two important symmetric functions in classical theory are the complete symmetric function and the elementary symmetric function. We now consider factorial operator versions of these, using the shift operator S. Note that all products are noncommutative, and that the indexing of each product below runs in a different direction.

Definition 3.1. (1) *The factorial complete symmetric generating function operators, $w_n(z,S)$, $n \geq 0$, are given by*

$$\sum_{n\geq 0} w_n(z,S)t^n = W(t,z,S) = \prod_{i=1}^{m} (1-(z_i-i+1)tS)^{-1}.$$

(2) *The factorial elementary symmetric function operators, $u_n(z,S)$, $n \geq 0$, are given by*

$$\sum_{n\geq 0} u_n(z,S)t^n = U(t,z,S) = \prod_{i=m}^{1} (1+(z_i-i+1)tS).$$

The factorial complete symmetric functions $w_n(z)$, $n \geq 0$, are defined as $w_n(z) = w_n(z,S)(1)$ and the factorial elementary symmetric functions $u_n(z)$, $n \geq 0$, are defined as $u_n(z) = u_n(z,S)(1)$.

Clearly, from Definition 3.1,

$$w_n(z,S) = \sum_{1\leq i_1\leq i_2\leq \ldots \leq i_n\leq m} (y_{i_1}S)(y_{i_2}S)\ldots(y_{i_n}S),$$

so,

$$w_n(z) = \sum_{1\leq i_1\leq i_2\leq \ldots \leq i_n\leq m} y_{i_1}(y_{i_2}-1)\ldots(y_{i_n}-n+1), \tag{8}$$

where $y_j = z_j - j + 1$, $j = 1,\ldots,m$, and, by (6) our definition of $w_n(z)$ is therefore equivalent to that of Chen and Louck [5]. A similar explicit form for $u_n(z)$ is given by

$$u_n(z) = \sum_{1\leq i_1< i_2< \ldots < i_n\leq m} y_{i_n}(y_{i_{n-1}}-1)\ldots(y_{i_1}-n+1).$$

Although perhaps not immediately obvious for the above expressions, $u_n(z, S)$, $u_n(z)$, $w_n(z, S)$, and $w_n(z)$ are indeed symmetric in the zs as can be seen from their generating functions in the following result. Chen and Louck [5] have shown that the $w_n(z)$ are symmetric by a divided difference argument.

PROPOSITION 3.2. (1) $(1 - (z_i - 1)tS)(1 - z_jtS) = (1 - (z_j - 1)tS)(1 - z_itS)$.

(2) *$u_n(z, S)$ and $u_n(z)$, $n \geq 0$ are symmetric in z.*

(3) *$w_n(z, S)$ and $w_n(z)$, $n \geq 0$ are symmetric in z.*

Proof. (1)

$$\begin{aligned}
&(1 - (z_i - 1)tS)(1 - z_jtS) \\
&\quad = 1 - (z_i - 1)tS - z_jtS + (z_i - 1)(z_j - 1)t^2S^2 \\
&\quad = 1 - (z_i + z_j - 1)tS + (z_i - 1)(z_j - 1)t^2S^2 \\
&\quad = 1 - (z_j - 1)tS - z_itS + (z_j - 1)(z_i - 1)t^2S^2 \\
&\quad = \big(1 - (z_j - 1)tS\big)(1 - z_itS).
\end{aligned}$$

(2) Replace i by $j + 1$, t by $-t$, z_j by $z_j - j + 1$, and z_{j+1} by $z_{j+1} - j + 1$ in (1). This proves that $U(t, z, S)$ is invariant under the adjacent transposition $(j, j + 1)$ applied to z for any $j = 1, \ldots, n - 1$. It follows immediately that $U(t, z, S)$ is symmetric in z, so $u_n(z, S)$, $n \geq 0$ are symmetric in z and thus $u_n(z)$, $n \geq 0$ are symmetric in z.

(3) From (1) we obtain immediately

$$\left(\frac{1}{1 - z_itS}\right)\left(\frac{1}{1 - (z_j - 1)tS}\right) = \left(\frac{1}{1 - z_jtS}\right)\left(\frac{1}{1 - (z_i - 1)tS}\right)$$

and the result follows similarly to (2). ∎

Since the $u_n(z)$ and $w_n(z)$ are symmetric, it is reasonable to ask if the us and the ws defined by $u_\lambda = u_{\lambda_1}u_{\lambda_2}\ldots u_{\lambda_m}$ and $w_\lambda = w_{\lambda_1}w_{\lambda_2}\ldots w_{\lambda_m}$, are each bases for the ring of symmetric polynomials. The veracity of this is routine to show and can be done using the method outlined in both Macdonald [7, pp. 13, 54, 55] and Biedenharn and Louck [2, pp. 413, 414]. For completeness we state it as a proposition.

PROPOSITION 3.3. *The sets of symmetric polynomials, $\{u_\lambda\}$ and $\{w_\lambda\}$, are each Z-bases for the ring of symmetric polynomials.*

Note that $U(-t, z, S)W(t, z, S)$ is the identity operator. This leads to the following analogue of the relation $\sum_{i=0}^{n}(-1)^i e_i(z)h_{n-i}(z) = \delta_{0,n}$.

PROPOSITION 3.4. *For $n \geq 0$,*

$$\sum_{i=0}^{n}(-1)^i u_i(z) w_{n-i}(z-i) = \delta_{0,n}.$$

Proof. This identity is similar to that of Macdonald [7, (2.6), p. 14] and follows directly from $(U(-t, z, S)W(t, z, S))(1) = 1$ since $u_n(z, S)$ is homogeneous of degree n in S. Thus $[t^n](U(-t, z, S)W(t, z, S))(1) = (\sum_{i=0}^{n}(-1)^i u_i(z, S)w_{n-i}(z, S))(1) = \sum_{i=0}^{n}(-1)^i u_i(z)w_{n-i}(z-i)$. ∎

4. THE SKEW FACTORIAL JACOBI—TRUDI IDENTITY AND ITS DUAL

Equations (1) and (7) introduced the classical skew and skew factorial Jacobi–Trudi identities. Here we give a more concise proof of this second result, exploiting the combinatorial definition of the skew factorial Schur functions and using the Gessel–Viennot lattice path techniques [6]. Note that this method of proof was suggested in Chen and Louck [5].

Here a lattice path has two types of steps: vertical steps, which increase the y-coordinate by 1, and horizontal steps, which increase the x-coordinate by 1. We use the weight function θ for a lattice path P defined by $\theta(P) = \prod_{(i,j)}(z_j - j - i)$, where the product is over points (i, j), which are starting points of the horizontal steps. First we identify the factorial complete symmetric function as a lattice path generating function with this weight function.

PROPOSITION 4.1. *The generating function for lattice paths which start at $(\alpha, 1)$ and end at (β, m) is $w_{\beta-\alpha}(z - \alpha - 1)$.*

Proof. The horizontal steps in a lattice path from $(\alpha, 1)$ to (β, m) start at $(\alpha, i_1), (\alpha + 1, i_2), \ldots, (\beta - 1, i_{\beta-\alpha})$ for $1 \leq i_1 \leq i_2 \leq \ldots \leq i_{\beta-\alpha} \leq m$, so the required generating function is

$$\sum_{1 \leq i_1 \leq \ldots \leq i_{\beta-\alpha} \leq m} \left(z_{i_1} - i_1 - \alpha\right) \ldots \left(z_{i_{\beta-\alpha}} - i_{\beta-\alpha} - \beta + 1\right)$$

and the result follows immediately from (8). ∎

THEOREM 4.2 (Skew Factorial Jacobi–Trudi Identity). *For partitions* $\lambda = (\lambda_1, \dots, \lambda_m)$ *and* $\mu = (\mu_1, \dots, u_m)$, *we have*

$$t_{\lambda/\mu}(z) = \det\Big(w_{\lambda_i - \mu_j - i + j}(z + j - 1 - \mu_j)\Big)_{m \times m}.$$

Proof. We modify the Gessel–Viennot lattice path bijection method of proof for the skew Jacobi–Trudi identity to account for the modified weight in the skew tableaux generating function (4). This results in weighting each horizontal step in the corresponding non-intersecting lattice paths by $z_j - j - i$ where the step begins at point (i, j) in the plane. The result is a consequence of Theorem 1.2 of Stembridge [9], with $u = ((\mu_m - m, 1), \dots, (\mu_1 - 1, 1))$ and $v = ((\lambda_m - m, \infty), \dots, (\lambda_1 - 1, \infty))$ and invoking Proposition 4.1. ∎

It is well-known in classical theory that the Jacobi–Trudi identity has a dual version, namely (2) above. Similarly the factorial Jacobi–Trudi identity has a dual version.

THEOREM 4.3. *For partitions* λ *and* μ *with* $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ *and* $\mu' = (\mu'_1, \dots, \mu'_m)$, $t_{\lambda/\mu}(z) = \det(u_{\lambda'_i - \mu'_j - i + j}(z - i + \lambda'_i))_{\lambda_1 \times \lambda_1}$.

Proof. We achieve this result by showing

$$\det\Big(w_{\lambda_i - \mu_j - i + j}(z + j - 1 - \mu_j)\Big) = \det\Big(u_{\lambda'_i - \mu'_j - i + j}(z - i + \lambda'_i)\Big).$$

We mimic Macdonald [7, p. 15].

Let N be a positive integer and consider the matrices of $N + 1$ rows and columns,

$$W = \big(w_{i-j}(z - j + 1)\big) \quad \text{and} \quad U = \Big((-1)^{i-j} u_{i-j}(z - j + 1)\Big).$$

Both W and U are lower triangular, with 1s down the diagonal, so that $\det W = \det U = 1$; moreover, Proposition 3.4 shows that W and U are inverses of each other. It follows that each minor of W is equal to the complementary cofactor of U^T, the transpose of U.

Let λ and μ be partitions of length $\leq p$ such that λ' and μ' have length $\leq q$ where $p + q = N + 1$. Consider the minor of W with row indices $\lambda_i + p - i$ $(1 \leq i \leq p)$ and column indices $\mu_i + p - i$ $(1 \leq i \leq p)$. By (1.7), p. 3 of [7], the complementary cofactor of U^T has row indices $p - 1 + j - \lambda'_j$ $(1 \leq j \leq q)$ and column indices $p - 1 + j - \mu'_j$ $(1 \leq j \leq q)$. Hence we have

$$\det\Big(w_{\lambda_i - \mu_j - i + j}(z - p + j - \mu_j)\Big)$$
$$= (-1)^{|\lambda| + |\mu|} \det\Big((-1)^{\lambda'_i - \mu'_i - i + j} u_{\lambda'_i - \mu'_j - i + j}(z - p - i + \lambda'_i + 1)\Big).$$

The minus signs cancel out, and thus we have

$$\det\left(w_{\lambda_i-\mu_j-i+j}(z-p+j-\mu_j)\right)=\det\left(u_{\lambda'_i-\mu'_j-i+j}(z-p-i+\lambda'_i+1)\right).$$

If we replace z by $z+p-1$ (valid since all statements made are true for all z), we have

$$\det\left(w_{\lambda_i-\mu_j-i+j}(z+j-1-\mu_j)\right)_{m\times m}=\det\left(u_{\lambda'_i-\mu'_j-i+j}(z-i+\lambda'_i)\right)_{\lambda_1\times\lambda_1}. \quad \blacksquare$$

5. The Factorial Jacobi—Trudi Identity: An Alternate Proof

As an aside, we invoke the alternate definition of $t_\lambda(z)$ to prove the factorial Jacobi–Trudi identity algebraically. This proof follows closely the techniques of Macdonald [7, p. 25], and involves manipulations of the factorial elementary symmetric functions.

THEOREM 5.1. *For partition $\lambda=(\lambda_1,\dots,\lambda_m)$, we have*

$$\frac{\left|(z_i)_{\lambda_j+m-j}\right|_{m\times m}}{\left|(z_i)_{m-j}\right|_{m\times m}}=\det\left(w_{\lambda_i-i+j}(z+j-1)\right)_{m\times m}.$$

Proof. For $k=1,\dots,m$ and $n\geq 0$, let $U^{[k]}(t,z,S), u_n^{[k]}(z,S), u_n^{[k]}(z)$ denote $U(t,z\setminus\{z_k\},S), u_n(z\setminus\{z_k\},S), u_n(z\setminus\{z_k\})$, respectively. Define $\overline{M}=((-1)^{m-j}u_{m-j}^{[k]}(z-m-1))_{1\leq k,j\leq m}$ and, for $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbf{N}^m$, define

$$\overline{A}_\alpha=\left((z_i)_{\alpha_j}\right) \qquad \text{and} \qquad W_\alpha=\left(w_{\alpha_j-m+i}(z+i-1)\right).$$

Now from the symmetry of W we have

$$U^{[k]}(-t,z,S)W(t,z,S)=\left(1-(z_k-m+1)tS\right)^{-1}.$$

Apply $S^{-(m-1)}$ to both sides of this equation and equate coefficients of t^{α_j} to obtain

$$\sum_{l=1}^{m}(-1)^{m-l}u_{m-l}^{[k]}(z+m-1,S)S^{-m+l}w_{\alpha_j-m+l}(z,S)=(z_kS)^{\alpha_j}.$$

Apply both sides to 1, giving

$$\sum_{l=1}^{m} (-1)^{m-l} u_{m-l}^{[k]}(z+m-1) w_{\alpha_j-m+l}(z+l-1) = (z_k)_{\alpha_j},$$

so $\overline{M}W_\alpha = \overline{A}_\alpha$. If we take determinants we obtain

$$\bar{a}_\alpha = \det(\overline{A}_\alpha) = \det(\overline{M})\det(W_\alpha).$$

Let $\delta = (m-1, m-2, \ldots, 1, 0)$, and note that W_δ is upper unitriangular, so $\det(W_\delta) = 1$. Thus substituting $\alpha = \delta$ in the above expression gives $\det(\overline{M}) = \bar{a}_\delta$. Hence $\bar{a}_\alpha = \bar{a}_\delta \det(W_\alpha)$ and the result follows if we let $\alpha = \lambda + \delta$ and then divide both sides by $\bar{a}_\delta$. ∎

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