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Lower order terms for the moments of symplectic and orthogonal families of L-functions ☆

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We derive formulas for the terms in the conjectured asymptotic expansions of the moments, at the central point, of quadratic Dirichlet L-functions, $L(1/2,\chi_d)$, and also of the L-functions associated to quadratic twists of an elliptic curve over $\mathbb{Q}$. In so doing, we are led to study determinants of binomial coefficients of the form $\det\left(\binom{2k-i-\lambda_{k-i+1}}{2k-2j}\right)$.

Contents

1. Introduction . . . 640
 1.1. The CFKRS conjecture for $L(1/2,\chi_d)$ . . . 641
 1.2. Symmetric function theory . . . 646
2. Terms of the asymptotic expansion . . . 651
 2.1. The leading term . . . 652
 2.2. Further lower order terms . . . 656
3. The coefficients $b_\lambda^\pm(k)$ . . . 661
4. Determinant of a matrix of binomial coefficients . . . 662
5. Family of quadratic twists of elliptic curve L-functions . . . 669
References . . . 673

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1. Introduction

In this paper we describe formulas, derived from conjectures of Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS], for the moments of quadratic Dirichlet L-functions at the central point, and the moments of L-functions associated to quadratic twists of an elliptic curve.

We are motivated to study moments in these two families of L-functions because of their apparent connection to the moments of characteristic polynomials of unitary symplectic and orthogonal matrices.

Montgomery was the first to discover a link between an L-function and characteristic polynomials of unitary matrices [Mo]. He computed, with restrictions on the allowed test functions, the limiting pair correlation of the zeros of the Riemann zeta function, and found that it coincides with the average pair correlation of the eigenvalues of large random (according to Haar measure) unitary matrices that had been computed earlier by Dyson [Dy]. Odlyzko later confirmed this agreement numerically, without restriction [O]. Rudnick and Sarnak generalized Montgomery's result to higher correlations and to any primitive L-function [RS].

Katz and Sarnak then made precise connections between various families of L-functions and matrices from specific classical compact groups, based on results linking the density of zeros L-functions and analogous zeta functions over function fields, to the eigenvalue densities of random matrices in the classical compact groups [KS,KS2]. For instance, their work showed a statistical connection between the zeros of quadratic Dirichlet L-functions and eigenvalues of unitary symplectic matrices, and between the zeros of L-functions of quadratic twists of an elliptic curve and eigenvalues of orthogonal matrices. The papers [R] and [R2], provided further theoretical and numerical support for the relevance of these matrix groups to our two families of L-functions.

Subsequently, Keating and Snaith were able to predict the leading term in the asymptotics for the moments of the Riemann zeta function on the critical line by carrying out an analogous computation for random unitary matrices [KeS]. In a companion paper [KeS2], they also conjectured the leading term in the asymptotics for the moments in our two families of L-functions by computing the moments of the characteristic polynomials of random unitary symplectic and even orthogonal matrices. See also the paper of Conrey and Farmer [CF] which contains some arithmetic information needed for the Keating and Snaith approach to moments.

The method of Keating and Snaith for predicting moments of L-functions relies on computations in random matrix theory, for example it uses Weyl's integration formula and the Selberg integral, and some guesswork to make the heuristic leap to number theoretic moments. It also has the drawback of requiring, as input, the relevant classical compact group as predicted by Katz and Sarnak.

On the other hand, the approach, referred to above, of Conrey, Farmer, Keating, Rubinstein, and Snaith does not rely on random matrix theory to derive, heuristically, the moments of various families of L-functions. Their method is based strictly on number theoretic techniques involving the approximate functional equation, the traditional equation that is used to study moments of L-functions [T,J]. While random matrix theory is not needed in their approach, the formulas that their heuristic approach yields for L-functions have provable analogues in random matrix theory. CFKRS were also able to make progress by introducing 'shifts' into the moments, a strategy that was inspired by Motohashi's evaluation of the fourth moment of the zeta function [Mot] and also by an analogous problem in random matrix theory. Their method, therefore, produces an answer that can be compared against various moment computations in random matrix theory, and, instead of using the predictions of Katz and Sarnak, it provides evidence for them. Furthermore, the conjectured formulas of CFKRS go well beyond the leading asymptotic of Keating and Snaith, providing, implicitly, a full asymptotic expansion for a variety of L-function moment problems. Because their conjectured formulas provide a full asymptotic expansion for moments, one can test them numerically by comparing the predicted moments against those computed from L-function data. See for instance [CFKRS,CFKRS2,AR,RY].

Our goal is to turn the implicit formulas of CFKRS into asymptotic expansions with explicitly given coefficients. We elaborate on the CFKRS formulas, for the family of quadratic Dirichlet L-functions, in Section 1.1 and for quadratic elliptic curve L-functions in Section 5.

Besides the approaches of Keating and Snaith and of CFKRS, two additional methods have yielded interesting results for the moments of L-functions.

Gonek, Hughes, and Keating [GHK], and Bui and Keating [BK] use the explicit formula for an L-function to realize the L-function as a hybrid between partial Hadamard and Euler products. They assume statistical independence between these two products and study the moments of the partial Euler product using number theoretic heuristics. The moments of the partial Hadamard product are studied by modelling the zeros of the Hadamard product based on the predicted classical compact group. Their approach therefore suffers the same disadvantage of the Keating and Snaith method of requiring the predictions of Katz and Sarnak as input. The main advantage of their method over the Keating and Snaith method is that it explains, rather than guesses, the appearance of an 'arithmetic factor' in moment formulas for L-functions. Another disadvantage is that it only seems to correctly predict the leading asymptotic for the L-function moments that they consider, and thus only agrees with the CFKRS prediction to leading order. Presumably this is because their assumptions are too strong, for example the statistical independence between the partial Hadamard and Euler products, and their use of matrix eigenvalues to model the partial Hadamard product.

Another method for studying moments of L-functions has been developed by Diaconu, Goldfeld, and Hoffstein [GH,DGH] and uses the theory of multiple Dirichlet series. It has the advantage of proving asymptotic formulas for some L-function moments, for example the first three moments of quadratic Dirichlet L-functions at the central point. However, it has the disadvantage of involving an elaborate sieving process (in the case of quadratic characters), that makes it unwieldy for producing explicit formulas for the asymptotic expansion. Interestingly, their method predicts the existence of additional lower order terms of smaller magnitude that go beyond those of the asymptotic expansion of CFKRS. See the paper of DGH as well as that of Zhang [Z], and Alderson and Rubinstein [AR] for discussions and computations regarding these additional lower terms.

1.1. The CFKRS conjecture for $L(1/2, \chi_d)$

We begin by describing the CFKRS conjecture for quadratic Dirichlet L-functions. Let D be a squarefree integer, $D \neq 0, 1$, and let $K = \mathbb{Q}(\sqrt{D})$ be the corresponding quadratic field. The fundamental discriminant d of K equals D if $D = 1 \bmod 4$, and $4D$ if $D = 2, 3 \bmod 4$. Let $\chi_d(n)$ be the Kronecker symbol $(\frac{d}{n})$, and $L(s, \chi_d)$ the quadratic Dirichlet L-function given by the Dirichlet series

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}, \qquad \Re(s) > 0, \tag{1.1}$$

satisfying the functional equation

$$L(s, \chi_d) = |d|^{\frac{1}{2}-s} X(s, a) L(1-s, \chi_d), \tag{1.2}$$

where

$$X(s, a) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s+a}{2})}{\Gamma(\frac{s+a}{2})}, \quad a = \begin{cases} 0 & \text{if } d > 0, \\ 1 & \text{if } d < 0. \end{cases} \tag{1.3}$$

Let $S(X)$ denote the set of fundamental discriminants with $|d| < X$. The Gamma factor in functional equation for $L(s, \chi_d)$ depends on whether $d < 0$ or $d > 0$. Thus, define further

$$\begin{aligned} S_+(X) &= \{d \in S(X)\colon d > 0\}, \\ S_-(X) &= \{d \in S(X)\colon d < 0\}, \end{aligned} \tag{1.4}$$

to be, respectively, the sets of positive and negative fundamental discriminants with $|d| < X$.

CFKRS conjectured [CFKRS] the asymptotic expansion:

$$\sum_{d\in S\pm(X)} L(1/2,\chi_d)^k \sim \frac{3}{\pi^2} X\mathcal{Q}_{\pm}(k,\log X), \tag{1.5}$$

where $\mathcal{Q}_+(k,x)$ and $\mathcal{Q}_-(k,x)$ are polynomials of degree $k(k+1)/2$ in x that we will describe below. The fraction $3/\pi^2$ accounts for the density of fundamental discriminants amongst all the integers.

The polynomial $\mathcal{Q}_{\pm}(k,\log X)$ is expressed in terms of a more fundamental polynomial $Q_{\pm}(k,x)$ of the same degree that captures the moments locally:

$$\mathcal{Q}_{\pm}(k,\log X) = \frac{1}{X}\int_1^X Q_{\pm}(k,\log t)\,dt. \tag{1.6}$$

One of the main achievements of CFKRS was to give a general recipe/heuristic for producing formulas for moments of various families of L-functions. Their formula (see Conjecture 1.5.3 in [CFKRS]) for the polynomial $Q_{\pm}(k,x)$ is given implicitly in terms of a k-fold multivariate residue:

$$Q_{\pm}(k,x) = \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint \frac{G_{\pm}(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} e^{\frac{x}{2}\sum_{j=1}^k z_j}\,dz_1\ldots dz_k, \tag{1.7}$$

where $\Delta(w_1,\ldots,w_k)$ is the Vandermonde determinant

$$\Delta(w_1,\ldots,w_k) = \det\bigl(w_i^{j-1}\bigr)_{k\times k} = \prod_{1\leqslant i<j\leqslant k}(w_j-w_i), \tag{1.8}$$

and

$$G_{\pm}(z_1,\ldots,z_k) = A_k(z_1,\ldots,z_k)\prod_{j=1}^k X\left(\frac{1}{2}+z_j,a\right)^{-1/2}\prod_{1\leqslant i\leqslant j\leqslant k}\zeta(1+z_i+z_j). \tag{1.9}$$

Here, $a=0$ for G_+ and $a=1$ for G_-, $X(s,a)$ is given in (1.3), and A_k equals the Euler product, absolutely convergent in a neighbourhood of $(z_1,\ldots,z_k)=(0,\ldots,0)$, defined by

$$\begin{aligned} A_k(z_1,\ldots,z_k) &= \prod_p \prod_{1\leqslant i\leqslant j\leqslant k}\left(1-\frac{1}{p^{1+z_i+z_j}}\right) \\ &\quad\times\left(\frac{1}{2}\left(\prod_{j=1}^k\left(1-\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}+\prod_{j=1}^k\left(1+\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\right)+\frac{1}{p}\right)\left(1+\frac{1}{p}\right)^{-1}. \end{aligned} \tag{1.10}$$

One advantage of Eq. (1.7) is that it allows one to easily see that $Q_{\pm}(k,x)$ is a polynomial of degree $k(k+1)/2$ in x. That is because the denominator of the multivariate residue picks up terms in the numerator involving $\prod_{j=1}^k z_j^{2k-2}$, which is of degree $2k(k-1)$. Now, the factor $\Delta(z_1^2,\ldots,z_k^2)^2$ is a homogeneous polynomial, also of degree $2k(k-1)$. However, the factor $G_{\pm}(z_1,\ldots,z_k)$ cancels $k(k+1)/2$ of the factors of the Vandermonde, because each $\zeta(1+z_i+z_j)$ has a Laurent expansion that begins $1/(z_i + z_j)$ coming from the pole at $s = 1$ of $\zeta(s)$. Therefore, in considering the multivariate Taylor expansion of the numerator about $z_1 = \cdots = z_k = 0$, we only need to take terms in the series

$$\exp\left(\frac{x}{2}\sum_{j=1}^{k} z_j\right) = \sum_{0}^{\infty} \frac{x^n}{2^n n!}(z_1 + \cdots + z_k)^n \tag{1.11}$$

up to $n = k(k+1)/2$. Hence, in the x aspect, the k-fold residue only involves terms up to $x^{k(k+1)/2}$.

Eq. (1.7) has the disadvantage of expressing $Q_\pm(k, x)$ implicitly. Let us therefore write

$$Q_\pm(k, x) = c_\pm(0, k)x^{k(k+1)/2} + c_\pm(1, k)x^{k(k+1)/2-1} + \cdots + c_\pm\big(k(k+1)/2, k\big). \tag{1.12}$$

Our main result, described in the following theorem, gives explicit formulae for the coefficients $c_\pm(r, k)$. We first define

$$a_k := A_k(0, \ldots, 0) = \prod_p \frac{(1 - \frac{1}{p})^{\frac{k(k+1)}{2}}}{1 + \frac{1}{p}}\left(\frac{(1 - \frac{1}{\sqrt{p}})^{-k} + (1 + \frac{1}{\sqrt{p}})^{-k}}{2} + \frac{1}{p}\right). \tag{1.13}$$

Theorem 1.1. *In (1.12), the leading coefficient $c_\pm(0, k)$ of $Q_+(k, x)$ or $Q_-(k, x)$ are both equal to*

$$\frac{a_k}{2^k}\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!} =: c(0, k), \tag{1.14}$$

and, for given $r \geqslant 1$, we have

$$c_\pm(r, k) = c(0, k)\sum_{|\lambda|=r} b_\lambda^\pm(k)N_\lambda(k), \tag{1.15}$$

where $N_\lambda(k)$ is a polynomial in k of degree at most $2|\lambda|$, defined in (2.52), a_k is defined in (1.13), and the $b_\lambda^\pm(k)$'s are the Taylor coefficients of a holomorphic function, defined in (2.4) and (2.5). The sum is over all partitions $|\lambda| = r$, with $\sum \lambda_i = r$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots > 0$.

We remark that formula (1.14) for the leading term agrees with the prediction of Keating and Snaith. See (34), (45), and (47) of Keating and Snaith [KS2] (replacing $\log D$ by x in their Eq. (45)). Their derivation is heuristic and based on the Selberg integral. Compare also to the leading term of Eq. (1.5.17) of [CFKRS], with $N = x/2$ in that equation. To verify the agreement between these, one can check, inductively, that:

$$\frac{1}{2^k}\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!} = \frac{1}{2^{k(k+1)/2}}\prod_{j=1}^{k}\frac{1}{(2j-1)!!} = \prod_{j=1}^{k}\frac{j!}{(2j)!}. \tag{1.16}$$

Note that (1.15) is analogous to formula (1.16) of [CFKRS] which provides a formula for the coefficients of the moment polynomials of the Riemann zeta function. See also Dehaye's paper [D], also for the Riemann zeta function, where he gives a combinatorial formula for the analogue of our polynomial $N_\lambda(k)$.

We work out examples, for $r = 1$ and $r = 2$. Table 1 provides $N_{(1)}(k) = k(k+1)$, $N_{(1,1)}(k) = \frac{1}{2}k(k-1)(k+1)(k+2)$, and $N_{(2)}(k) = 0$. Thus,

Table 1
We display the polynomials $N_\lambda(k)$, for all $|\lambda| \leqslant 7$. Because each monomial of $m_\lambda(z)$ contributes the same to (2.6), $N_\lambda(k)$ has, as a factor, the polynomial: $r_\lambda(k) := \binom{k}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} = (k)_{l(\lambda)}/(m_1(\lambda)! m_2(\lambda)! \ldots)$, where $(k)_m = k(k-1)\ldots(k-m+1)$. Therefore, rather than display $N_\lambda(k)$, here we list $N_\lambda(k)/r_\lambda(k)$.

λ	$N_\lambda(k)/r_\lambda(k)$	$r_\lambda(k)$
[1]	$k+1$	$(k)_1$
[1, 1]	$(k+2)(k+1)$	$(k)_2/2$
[2]	0	$(k)_1$
[1, 1, 1]	$(k+3)(k+2)(k+1)$	$(k)_3/6$
[2, 1]	$(k+2)(k+1)$	$(k)_2$
[3]	$-(k-1)(k+2)(k+1)$	$(k)_1$
[1, 1, 1, 1]	$(k+4)(k+3)(k+2)(k+1)$	$(k)_4/24$
[2, 1, 1]	$2(k+3)(k+2)(k+1)$	$(k)_3/2$
[2, 2]	0	$(k)_2/2$
[3, 1]	$-(k-2)(k+3)(k+2)(k+1)$	$(k)_2$
[4]	0	$(k)_1$
[1, 1, 1, 1, 1]	$(k+5)(k+4)(k+3)(k+2)(k+1)$	$(k)_5/120$
[2, 1, 1, 1]	$3(k+4)(k+3)(k+2)(k+1)$	$(k)_4/6$
[2, 2, 1]	$4(k+3)(k+2)(k+1)$	$(k)_3/2$
[3, 1, 1]	$-(k-3)(k+4)(k+3)(k+2)(k+1)$	$(k)_3/2$
[3, 2]	$-2(k-2)(k+3)(k+2)(k+1)$	$(k)_2$
[4, 1]	$-2(k-2)(k+3)(k+2)(k+1)$	$(k)_2$
[5]	$2(k-1)(k-2)(k+3)(k+2)(k+1)$	$(k)_1$
[1, 1, 1, 1, 1, 1]	$(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)$	$(k)_6/720$
[2, 1, 1, 1, 1]	$4(k+5)(k+4)(k+3)(k+2)(k+1)$	$(k)_5/24$
[2, 2, 1, 1]	$10(k+4)(k+3)(k+2)(k+1)$	$(k)_4/4$
[2, 2, 2]	0	$(k)_3/6$
[3, 1, 1, 1]	$-(k-4)(k+5)(k+4)(k+3)(k+2)(k+1)$	$(k)_4/6$
[3, 2, 1]	$-(k+3)(k+2)(k+1)(3k^2+3k-40)$	$(k)_3$
[3, 3]	$(k-2)(k-4)(k+5)(k+3)(k+2)(k+1)$	$(k)_2/2$
[4, 1, 1]	$-4(k+3)(k+2)(k+1)(k^2+k-10)$	$(k)_3/2$
[4, 2]	0	$(k)_2$
[5, 1]	$2(k-2)(k+3)(k+2)(k+1)(k^2+k-10)$	$(k)_2$
[6]	0	$(k)_1$
[1, 1, 1, 1, 1, 1, 1]	$(k+7)(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)$	$(k)_7/5040$
[2, 1, 1, 1, 1, 1]	$5(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)$	$(k)_6/120$
[2, 2, 1, 1, 1]	$18(k+5)(k+4)(k+3)(k+2)(k+1)$	$(k)_5/12$
[2, 2, 2, 1]	$30(k+4)(k+3)(k+2)(k+1)$	$(k)_4/6$
[3, 1, 1, 1, 1]	$-(k-5)(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)$	$(k)_5/24$
[3, 2, 1, 1]	$-2(k+4)(k+3)(k+2)(k+1)(2k^2+2k-45)$	$(k)_4/2$
[3, 2, 2]	$-10(k-3)(k+4)(k+3)(k+2)(k+1)$	$(k)_3/2$
[3, 3, 1]	$(k-3)(k-5)(k+6)(k+4)(k+3)(k+2)(k+1)$	$(k)_3/2$
[4, 1, 1, 1]	$-6(k+4)(k+3)(k+2)(k+1)(k^2+k-15)$	$(k)_4/6$
[4, 2, 1]	$-10(k-3)(k+4)(k+3)(k+2)(k+1)$	$(k)_3$
[4, 3]	$5(k-2)(k-3)(k+4)(k+3)(k+2)(k+1)$	$(k)_2$
[5, 1, 1]	$2(k-3)(k+4)(k+3)(k+2)(k+1)(k^2+k-15)$	$(k)_3/2$
[5, 2]	$5(k-2)(k-3)(k+4)(k+3)(k+2)(k+1)$	$(k)_2$
[6, 1]	$5(k-2)(k-3)(k+4)(k+3)(k+2)(k+1)$	$(k)_2$
[7]	$-5(k-1)(k-2)(k-3)(k+4)(k+3)(k+2)(k+1)$	$(k)_1$

$$\begin{aligned} c_\pm(1,k) &= c(0,k) b^\pm_{(1)}(k) N_{(1)}(k) \\ &= \frac{a_k}{2^k} \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j)!} k(k+1) b^\pm_{(1)}(k) \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} c_\pm(2,k) &= c(0,k)\big(b^\pm_{(1,1)}(k) N_{(1,1)}(k) + b^\pm_{(2)}(k) N_{(2)}(k)\big) \\ &= \frac{a_k}{2^k} \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j)!} \times \frac{1}{2} k(k-1)(k+1)(k+2) b^\pm_{(1,1)}(k). \end{aligned} \tag{1.18}$$

Let

$$\zeta(1+s) = \frac{1}{s} + \sum_{n=0}^{\infty} (-1)^n \gamma_n \frac{s^n}{n!} \tag{1.19}$$

be the Laurent expansion about 0 of $\zeta(1+s)$ (γ_0 is Euler's constant), and define

$$f_j(p) := \frac{(-1)^j (p^{1/2}-1)^{-j-k} + (p^{1/2}+1)^{-j-k}}{(p^{1/2}-1)^{-k} + (p^{1/2}+1)^{-k} + 2p^{-1-k/2}}. \tag{1.20}$$

Formulas for the coefficients $b^{\pm}_{(1)}(k)$, $b^{\pm}_{(1,1)}(k)$ can be derived using the method described in Section 3, and are given by

$$b^{\pm}_{(1)}(k) = -\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4 + a/2) + (k+1)\gamma_0 + \sum_p \left(\frac{(k+1)}{p-1} + f_1(p)\right)\log p, \tag{1.21}$$

where $a = 0$ for $b^{+}_{(1)}$, and $a = 1$ for $b^{-}_{(1)}$, and

$$b^{\pm}_{(1,1)}(k) = b^{\pm}_{(1)}(k)^2 - \gamma_0^2 - 2\gamma_1 - \sum_p \left(\frac{p}{(p-1)^2} + f_1(p)^2 - f_2(p)\right)\log(p)^2. \tag{1.22}$$

In Section 2 we derive formula (1.14) for the leading coefficient of $Q_{\pm}(k,x)$. Our tools are then applied, in Section 2.2, to the general term $c_{\pm}(r,k)$, where we obtain a formula for $N_\lambda(k)$ expressed as a sum of determinants of the form:

$$D_\lambda(k) = \det\left(\binom{2k-i-\lambda_{k-i+1}}{2k-2j}\right)_{1\leqslant i,j\leqslant k}, \tag{1.23}$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$ is a partition with length $l(\lambda) \leqslant k$ (see Section 1.2 for definitions).

In Section 4 we derive some interesting formulas for these determinants. To describe our formulas, let $y = (y_1, \dots, y_m)$. We define the *coefficient operator* $[y^\beta]$ on the set of formal multivariate Taylor or Laurent series in y, which picks the coefficient of the monomial y^β in the series. More precisely, if

$$f(y_1, \dots, y_m) = \sum_{r_1,\dots,r_m \in \mathbb{Z}} a_{r_1,\dots,r_m} y_1^{r_1} \dots y_m^{r_m}, \tag{1.24}$$

define

$$\left[y_1^{u_1} \dots y_m^{u_m}\right] f = a_{u_1,\dots,u_m}. \tag{1.25}$$

We prove the following theorem.

Theorem 1.2. *Let λ be a partition and μ be the conjugate partition. Let $m = l(\lambda)$, and $n = l(\mu) = \lambda_1$. For $k \geqslant \max(l(\lambda), \lambda_1)$, we have*

$$D_\lambda(k) = 2^{\binom{k}{2} - |\lambda|} \times \left[y_1^{\lambda_1+m-1} \dots y_m^{\lambda_m}\right]\left(\prod_{1\leqslant i<j\leqslant m} (y_i - y_j)(1 - y_i - y_j) \prod_{l=1}^{m} (1-y_l)^{-k-m}\right), \tag{1.26}$$

and also

$$D_\lambda(k) = 2^{\binom{k}{2}-|\lambda|} \times \left[z_1^{\mu_1+n-1}\dots z_n^{\mu_n}\right]\left(\prod_{1\leqslant i<j\leqslant n}(z_i-z_j)(1+z_i+z_j)\prod_{l=1}^{n}(1+2z_l)(1+z_l)^{k-n}\right). \tag{1.27}$$

Corollary 1.3. *Let λ be a partition, with $l(\lambda)=m$. There is a polynomial $P_\lambda(k)$, integer valued at integers, of degree $|\lambda|$ such that for $k \geqslant \max(l(\lambda),\lambda_1)$,*

$$D_\lambda(k) = 2^{\binom{k}{2}-|\lambda|} \times P_\lambda(k). \tag{1.28}$$

The leading coefficient of $P_\lambda(k)$ is

$$\frac{\prod_{1\leqslant i<j\leqslant m}(\lambda_i-\lambda_j-i+j)}{\prod_{1\leqslant i\leqslant m}(\lambda_i+m-i)!} = \chi^\lambda(1)/|\lambda|!, \tag{1.29}$$

where $\chi^\lambda(1)$ is the degree of the irreducible representation of the symmetric group $S_{|\lambda|}$ indexed by λ. In particular,

$$D_0(k) = 2^{\binom{k}{2}}, \tag{1.30}$$

where $D_0(k)$ is the determinant associated to the empty partition.

Table 2 gives a list of the polynomials $P_\lambda(k)$ for partitions up to weight 7. Observe, in the table, that $P_\lambda(k)$ often has many linear factors. This fact plays a role in our formula for $N_\lambda(k)$ so we encode it in the following corollary.

Corollary 1.4. *Let λ be a partition. Then $P_\lambda(k)$ is divisible by*

$$(k-\lambda_1)(k-\lambda_1-1)\dots\big(k-l(\lambda)+1\big)\times(k+\lambda_1)(k+\lambda_1-1)\dots\big(k+l(\lambda)\big), \tag{1.31}$$

where we take the first product to be 1 if $\lambda_1 \geqslant l(\lambda)$, and the second product to be 1 if $\lambda_1 < l(\lambda)$.

Finally, in Section 5 we discuss the application of our techniques to the related problem of the moments of the L-functions associated to quadratic twists of an elliptic curve.

1.2. Symmetric function theory

We collect here some definitions and results from the theory of symmetric functions that we use in our paper. The details can be found in [M, Chapter 1]. We have used the notations of [M].

A *partition* λ is a sequence of non-negative integers $(\lambda_1,\lambda_2,\dots)$ such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots, \tag{1.32}$$

and only finitely many λ_is are non-zero.

The *length* of the partition λ is defined to be the number of non-zero λ_is. We denote it by $l(\lambda)$. The *weight* of a partition λ, denoted by $|\lambda|$ is

$$|\lambda| = \sum_{i\geqslant 1}\lambda_i. \tag{1.33}$$

Table 2
Table of $P_\lambda(k)$.

Partition	$P_\lambda(k)$
[1]	$k+1$
[2]	$(1/2)(k+1)(k+2)$
[1, 1]	$(1/2)(k-1)(k+2)$
[3]	$(1/6)(k+1)(k+2)(k+3)$
[2, 1]	$(1/3)(k+2)(k^2+k-3)$
[1, 1, 1]	$(1/6)(k-2)(k-1)(k+3)$
[4]	$(1/24)(k+1)(k+2)(k+3)(k+4)$
[3, 1]	$(1/8)(k+2)(k+3)(k^2+k-4)$
[2, 2]	$(1/12)(k-2)(k+1)(k+2)(k+3)$
[2, 1, 1]	$(1/8)(k-2)(k+3)(k^2+k-4)$
[1, 1, 1, 1]	$(1/24)(k-3)(k-2)(k-1)(k+4)$
[5]	$(1/120)(k+1)(k+2)(k+3)(k+4)(k+5)$
[4, 1]	$(1/30)(k+2)(k+3)(k+4)(k^2+k-5)$
[3, 2]	$(1/24)(k+1)(k+2)(k+3)(k^2+k-8)$
[3, 1, 1]	$(1/20)(k+3)(k^4+2k^3-11k^2-12k+40)$
[2, 2, 1]	$(1/24)(k-2)(k+1)(k+3)(k^2+k-8)$
[2, 1, 1, 1]	$(1/30)(k-3)(k-2)(k+4)(k^2+k-5)$
[1, 1, 1, 1, 1]	$(1/120)(k-4)(k-3)(k-2)(k-1)(k+5)$
[6]	$(1/720)(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)$
[5, 1]	$(1/144)(k-2)(k+2)(k+4)(k+5)(k+3)^2$
[4, 2]	$(1/80)(k+1)(k+2)(k+3)(k+4)(k^2+k-10)$
[4, 1, 1]	$(1/72)(k+3)(k+4)(k^4+2k^3-13k^2-14k+60)$
[3, 3]	$(1/144)(k-3)(k+1)(k+3)(k+4)(k+2)^2$
[3, 2, 1]	$(1/45)(k+1)(k+3)(k^4+2k^3-16k^2-17k+75)$
[3, 1, 1, 1]	$(1/72)(k-3)(k+4)(k^4+2k^3-13k^2-14k+60)$
[2, 2, 2]	$(1/144)(k-3)(k-2)(k-1)(k+2)(k+3)(k+4)$
[2, 2, 1, 1]	$(1/80)(k-3)(k-2)(k+1)(k+4)(k^2+k-10)$
[2, 1, 1, 1, 1]	$(1/144)(k-4)(k-3)(k+3)(k+5)(k-2)^2$
[1, 1, 1, 1, 1, 1]	$(1/720)(k-5)(k-4)(k-3)(k-2)(k-1)(k+6)$
[7]	$(1/5040)(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)(k+7)$
[6, 1]	$(1/840)(k+2)(k+3)(k+4)(k+5)(k+6)(k^2+k-7)$
[5, 2]	$(1/360)(k-3)(k+1)(k+2)(k+3)(k+5)(k+4)^2$
[5, 1, 1]	$(1/336)(k+3)(k+4)(k+5)(k^4+2k^3-15k^2-16k+84)$
[4, 3]	$(1/360)(k+1)(k+3)(k+4)(k+2)^2(k^2+k-15)$
[4, 2, 1]	$(1/144)(k+1)(k+3)(k+4)(k^4+2k^3-19k^2-20k+108)$
[4, 1, 1, 1]	$(1/252)(k+4)(k^6+3k^5-26k^4-57k^3+277k^2+306k-1260)$
[3, 3, 1]	$(1/240)(k-3)(k+1)(k+2)(k+3)(k+4)(k^2+k-10)$
[3, 2, 2]	$(1/240)(k-3)(k-1)(k+2)(k+3)(k+4)(k^2+k-10)$
[3, 2, 1, 1]	$(1/144)(k-3)(k+1)(k+4)(k^4+2k^3-19k^2-20k+108)$
[3, 1, 1, 1, 1]	$(1/336)(k-4)(k-3)(k+5)(k^4+2k^3-15k^2-16k+84)$
[2, 2, 2, 1]	$(1/360)(k-3)(k-2)(k-1)(k+2)(k+4)(k^2+k-15)$
[2, 2, 1, 1, 1]	$(1/360)(k-4)(k-2)(k+1)(k+4)(k+5)(k-3)^2$
[2, 1, 1, 1, 1, 1]	$(1/840)(k-5)(k-4)(k-3)(k-2)(k+6)(k^2+k-7)$
[1, 1, 1, 1, 1, 1, 1]	$(1/5040)(k-6)(k-5)(k-4)(k-3)(k-2)(k-1)(k+7)$

The *diagram* of a partition is the set of points

$$\{(i,j) \mid 1 \leqslant i \leqslant l(\lambda),\ 1 \leqslant j \leqslant \lambda_i\}. \tag{1.34}$$

The *conjugate partition* λ' of a partition λ is the partition whose diagram is

$$\{(i,j) \mid (j,i) \text{ is in the diagram of } \lambda\}. \tag{1.35}$$

Equivalently, the conjugate partition of λ is a partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ where

$$\lambda'_i = \#\{\lambda_j \mid \lambda_j \geqslant i\}. \tag{1.36}$$

The symmetric group S_n acts on the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ by permuting the independent variables $x_1, \dots, x_n$. The *ring of symmetric polynomials* in n-variables, Λ_n, is the set of polynomials in $\mathbb{Z}[x_1, \dots, x_n]$ which are invariant under this action of S_n. The ring Λ_n is a graded ring:

$$\Lambda_n = \bigoplus_{k \geqslant 0} \Lambda_n^k, \tag{1.37}$$

where Λ_n^k is the set of homogeneous symmetric polynomials of degree k.

For $m > n$, there is a ring homomorphism

$$\rho_{m,n} : \mathbb{Z}[x_1, \dots, x_m] \to \mathbb{Z}[x_1, \dots, x_n], \tag{1.38}$$

where $\rho_{m,n}(x_i) = x_i$ for $i \leqslant n$, and $\rho_{m,n}(x_i) = 0$ for $i > n$. This restricts to a map

$$\rho_{m,n} : \Lambda_m^k \to \Lambda_n^k. \tag{1.39}$$

The maps given by (1.39) define an inverse system. Let

$$\Lambda^k = \varprojlim \Lambda_n^k, \tag{1.40}$$

and

$$\Lambda = \bigoplus_{k \geqslant 0} \Lambda^k. \tag{1.41}$$

The ring Λ is called the *ring of symmetric functions*. This is a graded ring. The definition of Λ gives us maps

$$\rho_n : \Lambda \to \Lambda_n. \tag{1.42}$$

In this paper, we shall use four $\mathbb{Z}$-bases, parametrized by partitions, of the ring Λ: the monomial symmetric functions (m_λ), elementary symmetric functions (e_λ), complete symmetric functions (h_λ) and the Schur symmetric functions (s_λ). In addition, we shall be using power symmetric functions (p_λ). The power symmetric functions form a $\mathbb{Q}$ basis of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. We shall use the same symbols to denote their image under ρ_n in Λ_n.

Given $\alpha = (\alpha_1, \dots, \alpha_n)$, we write x^α to denote $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let λ be a partition of length less than or equal to n. We define the *monomial symmetric function* m_λ by its image under ρ_n for every n. If $n \geqslant l(\lambda)$, then

$$m_\lambda(x_1, \dots, x_n) = \sum_\alpha x^\alpha, \tag{1.43}$$

where the α ranges over distinct permutations of $(\lambda_1, \dots, \lambda_n)$. If $l(\lambda) > n$, then $m_\lambda(x_1, \dots, x_n) = 0$. For the only partition of 0, the empty partition, we define $m_0 = 1$.

Let $r \geqslant 0$ be an integer. The *elementary symmetric function* $e_r \in \Lambda$ is given by

$$e_r = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r} x_{i_1} \dots x_{i_r} = m_{(1,\dots,1)}, \tag{1.44}$$

and $e_0 = 1$. For a partition λ, we define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots . \tag{1.45}$$

The generating function for e_r is

$$E(t) = \sum_{r \geqslant 0} e_r t^r = \prod_{i \geqslant 1} (1 + x_i t). \tag{1.46}$$

Let $r \geqslant 0$ be an integer. The *complete symmetric function* h_r is defined to be

$$h_r = \sum_{|\lambda| = r} m_\lambda. \tag{1.47}$$

Given a partition λ, we define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots. \tag{1.48}$$

The generating function for h_r is

$$H(t) = \sum_{r \geqslant 0} h_r t^r = \prod_{i \geqslant 1} (1 - x_i t)^{-1}. \tag{1.49}$$

Eqs. (1.46) and (1.49) give us the identity,

$$H(t)E(-t) = 1. \tag{1.50}$$

For $r \geqslant 1$, the *power symmetric function* p_r is defined as

$$p_r = \sum_{i \geqslant 1} x_i^r = m_{(r)}. \tag{1.51}$$

For a partition λ, we define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots. \tag{1.52}$$

Let $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. We define $a_\alpha \in \mathbb{Z}[x_1, \ldots, x_n]$ by

$$a_\alpha(x_1, \ldots, x_n) = \det\left(x_i^{\alpha_j}\right)_{1 \leqslant i, j \leqslant n}. \tag{1.53}$$

Clearly a_α is skew-symmetric; that is, for $w \in S_n$, $w(a_\alpha) = \operatorname{sgn}(w) a_\alpha$, where $\operatorname{sgn}(w)$ is the sign of permutation w. Let δ_n be the partition

$$\delta_n = (n-1, n-2, \ldots, 1, 0). \tag{1.54}$$

For a partition λ of length less than or equal to n, we append 0's as necessary to λ to create an n-tuple, and define

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\delta_n + \lambda}(x_1, \ldots, x_n)}{a_{\delta_n}(x_1, \ldots, x_n)}. \tag{1.55}$$

This is a polynomial. Since $s_\lambda(x_1, \ldots, x_n)$ is a ratio of skew-symmetric polynomials, it is a symmetric polynomial. These symmetric polynomials are called *Schur symmetric polynomials*. For $m > n$, $\rho_{m,n}(s_\lambda(x_1, \ldots, x_m)) = s_\lambda(x_1, \ldots, x_n)$, hence they are represented by a function $s_\lambda \in \Lambda$.

Using the definitions of a_λ and s_λ, it is easy to check that

$$a_{\delta_n}(x_1,\dots,x_n)=\prod_{1\leqslant i<j\leqslant n}(x_i-x_j), \tag{1.56}$$

and

$$s_{\delta_n}(x_1,\dots,x_n)=\prod_{1\leqslant i<j\leqslant n}(x_i+x_j). \tag{1.57}$$

Let λ be a partition and λ' be the conjugate partition. Then, for $n\geqslant l(\lambda)$ [M, p. 41]

$$s_\lambda=\det(h_{\lambda_i-i+j})_{1\leqslant i,j\leqslant n}, \tag{1.58}$$

and for $m\geqslant l(\lambda')$

$$s_\lambda=\det(e_{\lambda'_i-i+j})_{1\leqslant i,j\leqslant m}. \tag{1.59}$$

Identity (1.58) is called the Jacobi–Trudi identity, and (1.59) is called the dual Jacobi–Trudi identity. Schur symmetric functions satisfy [M, p. 63]

$$\prod_{i,j\geqslant 1}(1-x_iy_j)^{-1}=\sum_\lambda s_\lambda(x)s_\lambda(y), \tag{1.60}$$

and

$$\prod_{i,j\geqslant 1}(1+x_iy_j)=\sum_\lambda s_\lambda(x)s_{\lambda'}(y). \tag{1.61}$$

The sum in (1.60) and (1.61) is over all partitions λ. Identity (1.60) is called the Cauchy identity, and (1.61) is called the dual Cauchy identity.

There is a *fundamental involution* ω, a ring automorphism, defined on the ring of symmetric functions:

$$\omega(e_r)=h_r. \tag{1.62}$$

Using (1.50), we can prove that

$$\omega(h_r)=e_r. \tag{1.63}$$

We also have

$$\omega(s_\lambda)=s_{\lambda'},\quad \text{and}\quad \omega(p_n)=(-1)^{n-1}p_n. \tag{1.64}$$

2. Terms of the asymptotic expansion

We begin by rewriting the integrand on the right hand side of (1.7) as a ratio of a holomorphic function and a monomial. The function $G(z_1, \dots, z_k)$ in (1.9) has a pole in each z_j at $(0, \dots, 0)$ coming from the product of the zeta functions. These poles are eliminated by a portion of the Vandermonde determinants. Note that

$$\Delta\left(z_1^2, \dots, z_k^2\right)^2 = \left(\prod_{1 \leqslant i \leqslant j \leqslant k} (z_i + z_j)\right) \frac{\Delta(z_1, \dots, z_k)\Delta(z_1^2, \dots, z_k^2)}{2^k \prod_{j=1}^k z_j}. \tag{2.1}$$

Specifically each factor $(z_i + z_j)$ occurring here cancels a pole coming from $\zeta(1 + z_i + z_j)$. We obtain (2.1) by observing

$$\begin{aligned} \Delta\left(z_1^2, \dots, z_k^2\right) &= \prod_{i<j}\left(z_j^2 - z_i^2\right) = \Delta(z_1, \dots, z_k) \prod_{j>i} (z_i + z_j) \\ &= \Delta(z_1, \dots, z_k) \frac{\prod_{j \geqslant i} (z_i + z_j)}{2^k \prod_{j=1}^k z_j}. \end{aligned} \tag{2.2}$$

Substituting (2.1) into (1.7), we have

$$\begin{aligned} Q_\pm(k, x) = {} & \frac{(-1)^{k(k-1)/2}}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint A_k(z_1, \dots, z_k) \\ & \times \prod_{j=1}^k X\left(\frac{1}{2} + z_j, a\right)^{-\frac{1}{2}} \prod_{1 \leqslant i \leqslant j \leqslant k} (z_i + z_j)\zeta(1 + z_i + z_j) \\ & \times \frac{\Delta(z_1, \dots, z_k)\Delta(z_1^2, \dots, z_k^2)}{\prod_{j=1}^k z_j^{2k-1} \prod_{j=1}^k z_j} \exp\left(\frac{x}{2}\sum_{j=1}^k z_j\right) dz_1 \dots dz_k. \end{aligned} \tag{2.3}$$

Now the integrand is written as a ratio of a function which is holomorphic in a neighbourhood of $(0, \dots, 0)$ and a monomial.

Recall that $a_k = A_k(0, \dots, 0)$. Let $z = (z_1, \dots, z_k)$ and $m_\lambda(z)$ be the monomial symmetric polynomial defined in (1.43). Let

$$\sum_{i=0}^\infty \sum_{|\lambda|=i} b_\lambda^\pm(k) m_\lambda(z) \tag{2.4}$$

be the power series expansion of

$$\frac{1}{a_k} A_k(z_1, \dots, z_k) \prod_{j=1}^k X\left(\frac{1}{2} + z_j, a\right)^{-\frac{1}{2}} \prod_{1 \leqslant i \leqslant j \leqslant k} (z_i + z_j)\zeta(1 + z_i + z_j). \tag{2.5}$$

Here, the coefficients b_λ^+ are associated to the $a = 1$ case, and b_λ^- with $a = -1$.

In (2.4), the sum is over all partitions $\lambda_1 + \cdots + \lambda_k = i$, with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0$. We divide the expression by a_k to ensure that the constant term in the power series is 1. We shall calculate the Taylor series of (2.5) by calculating the Taylor series of its logarithm. This calculation is simpler if the constant term is 1, i.e. $b_{\mathbf{0}}^\pm(k) = 1$ in (2.4). So (2.3) becomes

$$Q_\pm(k,x) = \frac{(-1)^{k(k-1)/2}}{k!}\frac{a_k}{(2\pi i)^k}\sum_{i=0}^{\infty}\sum_{|\lambda|=i} b_\lambda^\pm(k)\oint\cdots\oint m_\lambda(z_1,\ldots,z_k)$$
$$\times\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{\prod_{j=1}^k z_j^{2k}}\exp\left(\frac{x}{2}\sum_{j=1}^k z_j\right)dz_1\ldots dz_k. \tag{2.6}$$

Only finitely many integrals in the sum (2.6) are non-zero. Each of the integrals in (2.6) picks up the coefficient of $z_1^{2k-1}\ldots z_k^{2k-1}$ in the Taylor expansion of the numerator of the corresponding integrand. If $\deg m_\lambda(z_1,\ldots,z_k)+\deg\Delta(z_1,\ldots,z_k)+\deg\Delta(z_1^2,\ldots,z_k^2)>\deg(z_1^{2k-1}\ldots z_k^{2k-1})$, that is $|\lambda|>k(k+1)/2$, then in the Taylor expansion of the numerator of (2.6) the coefficient of $z_1^{2k-1}\ldots z_k^{2k-1}$ is 0.

Given k, and a λ in the sum (2.6), the coefficient of the monomial $z_1^{2k-1}\ldots z_k^{2k-1}$ in the Taylor expansion of the numerator of the integrand is a constant, depending on λ and k, times $x^{\frac{k(k+1)}{2}-|\lambda|}$.

2.1. The leading term

In this section, we shall calculate the leading coefficient of $Q_\pm(k,x)$, i.e. the coefficient $c_\pm(0,k)$ of $x^{\frac{k(k+1)}{2}}$. The calculation will also provide insight into how to calculate the lower order terms of $Q_\pm(k,x)$. The leading coefficient is the same for $Q_+(k,x)$ and for $Q_-(k,x)$, and is given in the following proposition.

Proposition 2.1. *The leading coefficient $c_\pm(0,k)$ of $Q_\pm(k,x)$ in (1.12) is*

$$\frac{a_k}{2^k}\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!}. \tag{2.7}$$

The leading term in (1.12) corresponds to the $i=0$ term of (2.6). In this case there is only one integral within the inner summation sign, giving

$$c_\pm(0,k)x^{k(k+1)/2} = \frac{(-1)^{\frac{k(k-1)}{2}}}{k!(2\pi i)^k}a_k\oint\cdots\oint\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{\prod_{j=1}^k z_j^{2k}}\exp\left(\frac{x}{2}\sum_{j=1}^k z_j\right)dz_1\ldots dz_k. \tag{2.8}$$

Substituting $u_j = xz_j/2$, simplifying, and then relabelling u_j with z_j, we obtain

$$c_\pm(0,k)x^{k(k+1)/2} = \frac{(-1)^{\frac{k(k-1)}{2}}}{k!(2\pi i)^k}a_k\left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}}\oint\cdots\oint\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{\prod_{j=1}^k z_j^{2k}}$$
$$\times\exp\left(\sum_{j=1}^k z_j\right)dz_1\ldots dz_k. \tag{2.9}$$

The presence of the Vandermonde determinants prevents us from separating the integrals. However, we apply the following trick to move the Vandermonde determinants outside the integral. Introduce new variables $x_1,\ldots,x_k$ and consider the more general integral

$$I(x_1,\ldots,x_k) := \frac{1}{(2\pi i)^k}\oint\cdots\oint\frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{\prod_{j=1}^k z_j^{2k}}\exp\left(\sum_{j=1}^k x_j z_j\right)dz_1\ldots dz_k. \tag{2.10}$$

Thus, the evaluation of $c_\pm(0,k)$ boils down to determining $I(1,\ldots,1)$.

Next, we introduce a partial differential operator which will help us move the Vandermonde determinants outside the integral. Note that for a polynomial $P(x_1,\dots,x_k)$ in k variables, we have

$$P\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\exp\left(\sum_{j=1}^{k}x_jz_j\right)=P(z_1,\dots,z_k)\exp\left(\sum_{j=1}^{k}x_jz_j\right). \tag{2.11}$$

We set

$$q(z_1,\dots,z_k):=\Delta(z_1,\dots,z_k)\Delta\left(z_1^2,\dots,z_k^2\right). \tag{2.12}$$

Then (2.10) equals

$$\frac{1}{(2\pi i)^k}\oint\cdots\oint q\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\frac{\exp(\sum_{j=1}^{k}x_jz_j)}{\prod_{j=1}^{k}z_j^{2k}}\,dz_1\dots dz_k. \tag{2.13}$$

Pulling the differential operator outside the integral (Leibniz's rule) we conclude that (2.13) equals

$$q\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\frac{1}{(2\pi i)^k}\oint\cdots\oint\frac{\exp(\sum_{j=1}^{k}x_jz_j)}{\prod_{j=1}^{k}z_j^{2k}}\,dz_1\dots dz_k. \tag{2.14}$$

The integrand in (2.14) can be written as a product of integrals in one variable,

$$q\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\frac{1}{(2\pi i)^k}\prod_{j=1}^{k}\oint\frac{\exp(x_jz_j)}{z_j^{2k}}\,dz_j. \tag{2.15}$$

Each integral in the above product can be evaluated by expanding $\exp(x_jz_j)=\sum_{n=0}^{\infty}(x_jz_j)^n/n!$. The coefficient of z_j^{2k-1} is $x_j^{2k-1}/(2k-1)!$, and thus (2.15) equals

$$q\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\prod_{i=1}^{k}\frac{x_i^{2k-1}}{(2k-1)!}. \tag{2.16}$$

We have turned our computation of $c_\pm(0,k)$ into the question of determining the result of applying $q(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k})$ to $\prod_{i=1}^{k}\frac{x_i^{2k-1}}{(2k-1)!}$, and finding the value of the resulting polynomial at $(1,\dots,1)$. This calculation is done in Lemma 2.4. The proof of Lemma 2.4 uses Lemmas 2.2, and 2.3.

Lemma 2.2, a variant of Lemma 2.1 in [CFKRS2], gives a formula for applying the differential operator $\Delta(\frac{\partial^2}{\partial x_1^2},\dots,\frac{\partial^2}{\partial x_k^2})$ to a product of functions.

Lemma 2.2.

$$\Delta\left(\frac{\partial^2}{\partial x_1^2},\dots,\frac{\partial^2}{\partial x_k^2}\right)\prod_{i=1}^{k}f(x_i)=\left|f^{(2j-2)}(x_i)\right|_{k\times k}. \tag{2.17}$$

Proof. Write

$$\Delta\left(\frac{\partial^2}{\partial x_1^2},\dots,\frac{\partial^2}{\partial x_k^2}\right)=\left|\frac{\partial^{2j-2}}{x_i^{(2j-2)}}\right|_{k\times k}. \tag{2.18}$$

Applying this to $\prod_{i=1}^{k} f(x_i)$, and noticing that x_I only appears in the i-th row of the determinant, we can move $f(x_i)$ into that row. $\square$

Lemma 2.3 gives a formula for applying a product of differentials to a determinant of functions.

Lemma 2.3. *Let $f_1(x), \dots, f_k(x)$ be smooth functions of one variable. Then*

$$\frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_k}}{\partial x_k^{n_k}} \begin{vmatrix} f_1(x_1) & \dots\dots & f_k(x_1) \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ f_1(x_k) & \dots\dots & f_k(x_k) \end{vmatrix} = \begin{vmatrix} f_1^{(n_1)}(x_1) & \dots\dots & f_k^{(n_1)}(x_1) \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ f_1^{(n_k)}(x_k) & \dots\dots & f_k^{(n_k)}(x_k) \end{vmatrix}. \tag{2.19}$$

Proof. It is easy to see if we first look at a simple case, say $\frac{\partial}{\partial x_1}$ applied to the determinant on the left hand side of (2.19). $\square$

Lemma 2.4. *Let $q(z_1, \dots, z_k) = \Delta(z_1, \dots, z_k)\Delta(z_1^2, \dots, z_k^2)$, then*

$$q\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \prod_{i=1}^{k} \frac{x_j^{2k-1}}{(2k-1)!} \tag{2.20}$$

evaluated at $(x_1, \dots, x_k) = (1, \dots, 1)$ is

$$(-1)^{\frac{k(k-1)}{2}} \times k! \left(\prod_{j=0}^{k-1} \frac{(2j)!}{(k+j)!} \right) 2^{\frac{k(k-1)}{2}}. \tag{2.21}$$

Proof. To prove the lemma, we relate the value of (2.20) evaluated at $(x_1, \dots, x_k) = (1, \dots, 1)$ to a determinant of a matrix whose entries are binomial coefficients. We then use an identity for binomial coefficients to rewrite the determinant as a product of two determinants, and evaluate each of them separately.

Applying Lemma 2.2, we can deduce that

$$\Delta\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \Delta\left(\frac{\partial^2}{\partial^2 x_1^2}, \dots, \frac{\partial^2}{\partial x_k^2}\right) \prod_{j=1}^{k} f(x_j) \tag{2.22}$$

equals

$$\Delta\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right) \left| f^{(2(j-1))}(x_i) \right|_{k \times k}. \tag{2.23}$$

Expanding the Vandermonde determinant of partial differential operators, we obtain

$$\sum_{\mu \in S_k} \operatorname{sgn}(\mu) \frac{\partial^{\mu_1 - 1}}{\partial x_1^{\mu_1 - 1}} \cdots \frac{\partial^{\mu_k - 1}}{\partial x_k^{\mu_k - 1}} \begin{vmatrix} f(x_1) & f^{(2)}(x_1) & \cdots & f^{(2(k-1))}(x_1) \\ f(x_2) & f^{(2)}(x_2) & \cdots & f^{(2(k-1))}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_k) & f^{(2)}(x_k) & \cdots & f^{(2(k-1))}(x_k) \end{vmatrix}, \tag{2.24}$$

where $\mu_1, \dots, \mu_k$ is the image of the permutation μ of $1, \dots, k$. Applying Lemma 2.3, we can see that (2.24) equals

$$\sum_{\mu \in S_k} \operatorname{sgn}(\mu) \begin{vmatrix} f^{(\mu_1-1)}(x_1) & f^{(\mu_1+1)}(x_1) & \cdots & f^{(\mu_1-1+2(k-1))}(x_1) \\ f^{(\mu_2-1)}(x_2) & f^{(\mu_2+1)}(x_2) & \cdots & f^{(\mu_2-1+2(k-1))}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(\mu_k-1)}(x_k) & f^{(\mu_k+1)}(x_k) & \cdots & f^{(\mu_k-1+2(k-1))}(x_k) \end{vmatrix}. \tag{2.25}$$

Let $f(x) = \frac{x^{2k-1}}{(2k-1)!}$. Expression (2.25) evaluated at $(x_1, \dots, x_k) = (1, \dots, 1)$ is

$$\sum_{\mu \in S_n} \operatorname{sgn}(\mu) \begin{vmatrix} \frac{1}{(2k-\mu_1)!} & \frac{1}{(2k-\mu_1-2)!} & \cdots & \frac{1}{(-\mu_1+2)!} \\ \frac{1}{(2k-\mu_2)!} & \frac{1}{(2k-\mu_2-2)!} & \cdots & \frac{1}{(-\mu_2+2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(2k-\mu_k)!} & \frac{1}{(2k-\mu_k-2)!} & \cdots & \frac{1}{(-\mu_k+2)!} \end{vmatrix}. \tag{2.26}$$

Rearranging the rows to cancel the effect of μ (this introduces another $\operatorname{sgn}(\mu)$ in front of the determinant) and evaluating at $(x_1, \dots, x_k) = (1, \dots, 1)$, we get (2.26) equals

$$k! \begin{vmatrix} \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdots & \frac{1}{0!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k!} & \frac{1}{(k-1)!} & \cdots & 0 \end{vmatrix}. \tag{2.27}$$

We can convert the determinant (2.27) into a determinant of matrices whose entries are binomial coefficients. Multiplying the j-th column by $\frac{1}{(2(j-1))!}$ and the i-th row by $(2k-i)!$, we see that (2.27) equals

$$k! \frac{0!2! \cdots (2k-2)!}{(2k-1)!(2k-2)! \cdots k!} \begin{vmatrix} \binom{2k-1}{0} & \binom{2k-1}{2} & \cdots & \binom{2k-1}{2k-2} \\ \binom{2k-2}{0} & \binom{2k-2}{2} & \cdots & \binom{2k-2}{2k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k}{0} & \binom{k}{2} & \cdots & \binom{k}{2k-2} \end{vmatrix}. \tag{2.28}$$

The determinant in (2.28) is

$$\left| \binom{2k-i}{2j-2} \right|_{k \times k}. \tag{2.29}$$

In Section 4 we study this determinant. From (1.23) and Corollary 1.3, the determinant of this matrix equals $(-2)^{\binom{k}{2}}$. The extra $(-1)^{\binom{k}{2}}$ here comes about from the fact that the $D_0(k)$ in (1.23) has its columns reversed from the above determinant. □

Applying Lemma 2.4 to (2.16), we find that the leading term is:

$$\frac{a_k}{k!} \left(\frac{x}{2} \right)^{\frac{k(k+1)}{2}} \left(k! \frac{0!2! \cdots (2k-2)!}{(2k-1)! \cdots k!} \right) 2^{\frac{k(k-1)}{2}} = \frac{a_k}{2^k} \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j)!} x^{k(k+1)/2}. \tag{2.30}$$

Hence the coefficient of the leading term is

$$\frac{a_k}{2^k}\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!}. \tag{2.31}$$

This proves Proposition 2.1.

2.2. Further lower order terms

In this section we consider a general term occurring in the sum of integrals (2.6). Let λ be a partition. We shall calculate

$$\begin{aligned}&\frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{a_k}{(2\pi i)^k}b_\lambda^{\pm}(k)\oint\cdots\oint m_\lambda(z_1,\dots,z_k)\\&\quad\times\frac{\Delta(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)}{2^k\prod_{j=1}^k z_j^{2k}}\exp\left(\frac{x}{2}\sum_{j=1}^k z_j\right)dz_1\dots dz_k.\end{aligned} \tag{2.32}$$

Modifying the approach of the previous section to incorporate the extra monomial $m_\lambda(z_1,\dots,z_k)$, we define

$$q_\lambda(z_1,\dots,z_k)=m_\lambda(z_1,\dots,z_k)\Delta(z_1,\dots,z_k)\Delta\bigl(z_1^2,\dots,z_k^2\bigr). \tag{2.33}$$

Following the same steps as in the evaluation of the leading term, expression (2.32) becomes

$$\frac{(-1)^{\frac{k(k-1)}{2}}a_k b_\lambda^{\pm}(k)}{k!}\left(\frac{x}{2}\right)^{\frac{k(k+1)}{2}-|\lambda|}\left(q_\lambda\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\prod_{j=1}^k\frac{x_j^{2k-1}}{(2k-1)!}\right)_{\text{evaluated at } x_j=1}. \tag{2.34}$$

This section is devoted to calculating (2.34).

Let $f(x)=x^{2k-1}/(2k-1)!$. Let $|\lambda|=\sum_i\lambda_i$, and length $l(\lambda)$. Thus, $l(\lambda)$ is the number of non-zero elements of the partition λ, i.e. $\lambda_j=0$ for $j>l(\lambda)$. Let $m_j(\lambda)$ be the number of j's in the partition λ, so that $|\lambda|=m_1(\lambda)+2m_2(\lambda)+3m_3(\lambda)+\cdots$. There are $\binom{k}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda),m_2(\lambda),\dots}$ monomials in $m_\lambda(x_1,\dots,x_k)$ [S, 7.8]. Here $\binom{l(\lambda)}{m_1(\lambda),m_2(\lambda),\dots}$ is the multinomial coefficient. Since we are working with symmetric functions, it is enough to compute (2.32), i.e. (2.34), for one monomial of $m_\lambda(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k})$. Therefore,

$$q_\lambda\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}\right)\prod_{j=1}^k f(x_j)\bigg|_{\text{evaluated at } x_j=1} \tag{2.35}$$

equals

$$\binom{k}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda),m_2(\lambda),\dots}\frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1}\dots\partial x_{l(\lambda)}^{\lambda_{l(\lambda)}}}\Delta\left(\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_k}\right)\Delta\left(\frac{\partial^2}{\partial x_1^2}\cdots\frac{\partial^2}{\partial x_k^2}\right)\prod_{j=1}^k f(x_j) \tag{2.36}$$

evaluated at $(x_1,\dots,x_k)=(1,\dots,1)$. We already have the expression for the effect of Vandermonde determinant operators in (2.25). Therefore by Lemma 2.3, the expression (2.36) equals

$$\binom{k}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda),m_2(\lambda),\dots}\frac{\partial^{l}}{\partial x_1^{\lambda_1}\dots\partial x_{l(\lambda)}^{\lambda_{l(\lambda)}}}\sum_{\mu\in S_k}\operatorname{sgn}(\mu)\det\bigl(f^{(\mu_i-1)+2(j-1)}(x_i)\bigr). \tag{2.37}$$

The expression (2.37) is equal to

$$\binom{k}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} \sum_{\mu \in S_k} \operatorname{sgn}(\mu) \det\bigl(f^{(\mu_i - 1 + 2(j-1) + \lambda_i)}(1)\bigr). \tag{2.38}$$

In each summand of (2.38), rearrange the rows so as to reverse the effect of μ. We get

$$\binom{k}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} \sum_{\nu \in S_k} \det\bigl(f^{(i - 1 + 2(j-1) + \lambda_{\nu_i})}(1)\bigr). \tag{2.39}$$

Here ν is μ^{-1}. The expression (2.39) is

$$\binom{k}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} \sum_{\nu \in S_k} \det\left(\frac{1}{(2k - 1 - (i-1) - 2(j-1) - \lambda_{\nu_i})!}\right)_{ij}. \tag{2.40}$$

Each determinant inside the sum is of the form

$$\det\left(\frac{1}{(2k - 1 - (i-1) - 2(j-1) - d_i)!}\right)_{ij}, \tag{2.41}$$

and $\sum d_i = |\lambda|$. In Proposition 2.5, we determine a necessary condition for the determinant (2.41) to be non-zero. This condition will imply that a large portion of terms in (2.40) are zero.

Proposition 2.5. *Consider the determinant*

$$\det\left(\frac{1}{(2k - 1 - (i-1) - 2(j-1) - d_i)!}\right)_{ij}. \tag{2.42}$$

Assume that $\sum_{i=1}^{k} d_i = |\lambda|$, with $d_i \in \mathbb{Z}_{\geqslant 0}$. The determinant (2.42) is zero if any of $d_1, \ldots, d_{k-|\lambda|}$ is non-zero.

Proof. Let u be a number between 1 and k such that d_u is non-zero. The u-th row in the matrix is

$$\left(\frac{1}{(2k - 1 - (u-1) - 2(j-1) - d_u)!}\right)_{1 \leqslant j \leqslant k}. \tag{2.43}$$

Now look at the row which is d_u rows below the row u in the matrix (2.42). Let this be row v where $v = u + d_u$. Row v,

$$\left(\frac{1}{(2k - 1 - (v-1) - 2(j-1) - d_v)!}\right)_{1 \leqslant j \leqslant k}, \tag{2.44}$$

is identical to row u if d_v is zero. We have a necessary condition for the matrix to have a non-zero determinant; for every u such that $d_u \neq 0$, either d_{u+d_u} is also non-zero or $u + d_u > k$. We look at this cascading process, and see that if we start at a row above the row $k - |\lambda|$, that is if $d_u \neq 0$ for some $u \leqslant k - |\lambda|$, then we cannot go down beyond row k since all d_i add to $|\lambda|$. Hence we will have two identical rows. We can then conclude that we obtain non-zero determinants in (2.42) only when $d_u = 0$ for $1 \leqslant u \leqslant k - |\lambda|$. $\square$

The above proposition tells us that, in (2.46) all the action takes place in the last $|\lambda|$ rows or lower. Thus, let $\mathbf{u} = (u_k, \ldots, u_1)$ be a permutation of $(\lambda_1, \ldots, \lambda_k)$. Notice that we have reversed the order of the subscripts on the u's, starting at k and ending at 1. Applying the above proposition, we shall assume $u_i = 0$ for $i > |\lambda|$. Note, however, that some of the u_i's, with $i \leqslant |\lambda|$ can also equal 0. For a given permutation $\mathbf{u}$, let $i(\mathbf{u})$ be the smallest positive integer such that $u_i = 0$ for all $i > i(\mathbf{u})$. Thus, $i(\mathbf{u}) \leqslant |\lambda|$.

Next, any two permutations that have identical non-zero u_i's, i.e. that move around the 0's, produce the same determinant. For any given way of selecting where the non-zero λ_i's go, there are $(k - l(\lambda))!$ ways to move around the remaining zero-valued λ_i's. Furthermore, permuting identical non-zero λ_i's also produces the same determinant. For a given permutation, there are $m_1(\lambda)!m_2(\lambda)!\ldots$ ways to move around the identical non-zero λ_i's. Using the fact that

$$\binom{k}{l(\lambda)}\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots}\big(k - l(\lambda)\big)!m_1(\lambda)!m_2(\lambda)!\ldots = k!, \tag{2.45}$$

and taking into account the above two paragraphs, expression (2.40) can thus be written as

$$k! \sum_{\mathbf{u}}{}' \left| \begin{array}{cccc} \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdots & \frac{1}{0!} \\ \vdots & \vdots & & \vdots \\ \frac{1}{(k+i(\mathbf{u}))!} & \frac{1}{(k-i(\mathbf{u})-2)!} & & \vdots \\ \hline \frac{1}{(k+i(\mathbf{u})-1-u_{i(\mathbf{u})})!} & \frac{1}{(k+i(\mathbf{u})-3-u_{i(\mathbf{u})})!} & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{(k-u_1)!} & \frac{1}{(k-u_1-2)!} & \cdots & 0 \end{array} \right|. \tag{2.46}$$

There are $k - i(\mathbf{u})$ rows above the horizontal dashed line and $i(\mathbf{u})$ rows below the dotted line. The sum is over *distinct* permutations $(u_k, \ldots, u_1)$ of $(\lambda_1, \ldots, \lambda_k)$, satisfying $u_i = 0$ for $i > |\lambda|$. Note that, in order for a given permutation $\mathbf{u}$ to appear in the sum, we require that $k \geqslant i(\mathbf{u})$.

We may also reduce the number of terms in the sum by excluding matrices where two or more rows of the matrix are identical. The $'$ on the sum indicates that such terms have been excluded from the sum.

Now consider one specific term in the sum (2.46). As in the calculation of the leading coefficient, multiply its i-th row by $(2k - i)!$ and its j-th column by $1/(2(j - 1))!$. This enables us to write the determinant in a term of (2.46) as a product of a known quantity and a determinant of binomial coefficients,

$$\begin{aligned} &\frac{\prod_{j=1}^{k}(2(j-1))!}{\prod_{i=1}^{k}(2k-i)!} \times \left| \begin{array}{cccc} \binom{2k-1}{0} & \binom{2k-1}{2} & \cdots\cdots & \binom{2k-1}{2k-2} \\ \binom{2k-2}{0} & \binom{2k-2}{2} & \cdots\cdots & \binom{2k-2}{2k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k+i(\mathbf{u})}{0} & \binom{k+i(\mathbf{u})}{2} & \ddots & \binom{k+i(\mathbf{u})}{2k-2} \\ \binom{k+i(\mathbf{u})-1-u_{i(\mathbf{u})}}{0} & \binom{k+i(\mathbf{u})-1-u_{i(\mathbf{u})}}{2} & \cdots\cdots & \binom{k+i(\mathbf{u})-1-u_{i(\mathbf{u})}}{2k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k-u_1}{0} & \binom{k-u_1}{2} & \cdots\cdots & \binom{k-u_1}{2k-2} \end{array} \right| \\ &\quad \times \big(k + i(\mathbf{u}) - 1\big)_{u_{i(\mathbf{u})}} \big(k + i(\mathbf{u}) - 2\big)_{u_{i(\mathbf{u})-1}} \cdots (k)_{u_1}. \end{aligned} \tag{2.47}$$

Here $(x)_n$ is the falling factorial $x(x - 1)\ldots(x - n + 1)$. The last factor, the product of falling factorials, is a polynomial of degree $|\lambda|$ in k. Expressions (2.46) and (2.47) for the lower terms are the analogue of (2.28) for the leading coefficient. The difference is the presence of $u_1, \ldots, u_{i(\mathbf{u})}$ in the determinant and the appearance of the product of falling factorials. The latter are accounted for by the fact that the $(2k-i)!$ is not entirely cancelled by the numerator of the binomial coefficients in the last $i(\mathbf{u})$ rows.

We study the above determinant in the next section. To apply the formulas of that section, we require $u_1 \geqslant u_2 \geqslant \cdots$, which does not typically hold for the terms in the sum of (2.46). However, by swapping adjacent rows, we can arrange that these inequalities hold. More precisely, say that $u_m < u_{m+1}$. We can assume that, in fact, $u_m + 2 \leqslant u_{m+1}$ since if $u_m + 1 = u_{m+1}$ then the m-th and $m+1$-st rows from the bottom of the matrix coincide, and such terms are excluded from (2.46) since the determinant in such cases is 0.

Consider what happens when we swap the m-th and $m+1$-st rows from the bottom. The binomial coefficient $\binom{k+m-1-u_m}{2j-2}$ gets switched with $\binom{k+m-u_{m+1}}{2j-2}$ at a cost of a sign change to the determinant. On the other hand, the new determinant is of the same form, but with $\mathbf{u}$ replaced by $\mathbf{u}'$, where $u'_j = u_j$ for all j, except for $u'_m = u_{m+1} - 1$ and $u'_{m+1} = u_m + 1$. Thus we have reversed the inequality, i.e. $u'_m \geqslant u'_{m+1}$. Notice also that this swapping also satisfies $\sum u'_j = \sum u_j = |\lambda|$.

Therefore, continuing in this fashion, any given determinant in the sum in (2.47) is equal, up to a power of -1, to the same kind of determinant but with $\mathbf{u}$ replaced by, say, $\alpha(\mathbf{u})$, where α is a partition of $|\lambda|$, i.e. with $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant 0$. Let the power of -1 introduced by the row swaps that take $\mathbf{u}$ to $\alpha(\mathbf{u})$ be denoted by $n(\mathbf{a})$. Thus, a given determinant in (2.47) is equal, on performing the row swaps, to

$$(-1)^{n(\mathbf{u})+\binom{k}{2}} D_{\alpha(\mathbf{u})}(k), \tag{2.48}$$

where D is the determinant defined in (1.23). The extra $(-1)^{\binom{k}{2}}$ arises because the columns of D in (1.23) are in the reverse order from the determinants in (2.47).

Therefore, returning to Eqs. (2.6), (2.34), and (2.32), we have, on simplifying, that the coefficient $c_\pm(r,k)$ of $x^{\frac{k(k+1)}{2}-r}$ in $Q_\pm(k,x)$ can be expressed as:

$$\frac{a_k}{2^{\frac{k(k+1)}{2}-r}} \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j)!} \\ \times \sum_{|\lambda|=r} b_\lambda^\pm(k) \sum_{\mathbf{u}}{}' (-1)^{n(\mathbf{u})} D_{\alpha(\mathbf{u})}(k) \times \big(k+i(\mathbf{u})-1\big)_{u_{i(\mathbf{u})}} \big(k+i(\mathbf{u})-2\big)_{u_{i(\mathbf{u})-1}} \cdots (k)_{u_1}. \tag{2.49}$$

In Theorem 1.2 and Corollary 1.3 we show that, for $k \geqslant \max(l(\alpha), \alpha_1)$,

$$D_\alpha(k) = 2^{\binom{k}{2}-r} P_\alpha(k), \tag{2.50}$$

where $P_\alpha(k)$ is a polynomial in k of degree $|\alpha|$. Theorem 1.2 also gives a formula for determining the polynomials P. Hence

$$c_\pm(r,k) = \left(\frac{a_k}{2^k} \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j)!}\right) \sum_{|\lambda|=r} b_\lambda^\pm(k) N_\lambda(k), \tag{2.51}$$

where

$$N_\lambda(k) = \sum_{\mathbf{u}}{}' (-1)^{n(\mathbf{u})} P_{\alpha(\mathbf{u})}(k) \times \big(k+i(\mathbf{u})-1\big)_{u_{i(\mathbf{u})}} \big(k+i(\mathbf{u})-2\big)_{u_{i(\mathbf{u})-1}} \cdots (k)_{u_1}. \tag{2.52}$$

The sum is over distinct permutations $\mathbf{u} = (u_k, \dots, u_1)$ formed from the partition λ by appending 0's if necessary. Furthermore, we have restricted to permutations such that $i(\mathbf{u}) \leqslant |\lambda|$, and have also excluded $\mathbf{u}$ where the corresponding matrix has any identical rows. Finally, for a given $\mathbf{u}$ to appear in the sum we have assumed that $i(\mathbf{u}) \leqslant k$, and to apply (2.50), we also required that $k \geqslant \max(l(\alpha(\mathbf{u})), \alpha(\mathbf{u})_1)$.

We show that the latter assumption can be removed. First note that $i(\mathbf{u}) = l(\alpha(\mathbf{u}))$, because our swapping procedure that replaces a given $\mathbf{u}$ with $\mathbf{u}'$ has $i(\mathbf{u}') = i(\mathbf{u})$. Next, Corollary 1.4 tells us that $P_\alpha(k)$ vanishes for $\alpha_1 \leqslant k \leqslant l(\alpha) - 1$. Furthermore,

$$\bigl(k + i(\mathbf{u}) - 1\bigr)_{u_{i(\mathbf{u})}} \bigl(k + i(\mathbf{u}) - 2\bigr)_{u_{i(\mathbf{u})-1}} \cdots (k)_{u_1} \tag{2.53}$$

vanishes if $0 \leqslant k < \alpha_1$ as can be seen by examining the factor associated to α_1: let u_j be the term that, under our swapping procedure, gets swapped down to α_1. The corresponding falling factorial is $(k + j - 1)_{u_j}$. But $\alpha_1 = u_j - (j - 1)$, because u_j gets moved down $j - 1$ rows to the bottom row. Therefore,

$$(k + j - 1)_{u_j} = (k + j - 1)_{\alpha_1 + j - 1} = (k + j - 1)(k + j - 2) \dots (k - \alpha_1 + 1) \tag{2.54}$$

which is divisible by

$$k(k - 1) \dots (k - \alpha_1 + 1). \tag{2.55}$$

Thus, we have shown that

$$P_{\alpha(\mathbf{u})}(k) \times \bigl(k + i(\mathbf{u}) - 1\bigr)_{u_{i(\mathbf{u})}} \bigl(k + i(\mathbf{u}) - 2\bigr)_{u_{i(\mathbf{u})-1}} \cdots (k)_{u_1} \tag{2.56}$$

vanishes for $0 \leqslant k < \max(l(\alpha(\mathbf{u})), \alpha(\mathbf{u})_1)$. We can, therefore, ignore, in (2.52), the condition that $k \geqslant \max(l(\alpha(\mathbf{u})), \alpha(\mathbf{u})_1)$, since, in including terms with $k < \max(l(\alpha(\mathbf{u})), \alpha(\mathbf{u})_1)$, the corresponding summand in (2.52) vanishes.

Hence, $N_\lambda(k)$ is given by a sum over a *fixed*, i.e. depending only on λ but not on k, number of terms $\mathbf{u}$. Each term is a polynomial of degree $2|\lambda|$ in k, thus $N_\lambda(k)$ is a polynomial in k of degree $\leqslant 2|\lambda|$.

This completes the proof of Theorem 1.1.

As an example, we compute $N_{(2,1,1)}(k)$ using (2.52). In this case, the sum (2.52) is over the 12 distinct permutations of $(2, 1, 1, 0)$. We can truncate at 4 terms because $|\lambda| = 4$, and $i(\mathbf{u}) \leqslant |\lambda|$. Of these 12 permutations, only $(2, 1, 1, 0)$, $(0, 2, 1, 1)$, $(1, 0, 2, 1)$, and $(1, 1, 0, 2)$ give non-zero determinants. The sign $(-1)^{n(\mathbf{u})}$ is 1 for $(2, 1, 1, 0)$, and -1 for the rest. We have

$$\begin{aligned}
N_{(2,1,1)}(k) = {} & P_{(2,1,1)}(k)_2(k+1)_1(k+2)_1 - P_{(1,1,1,1)}(k+1)_2(k+2)_1(k+3)_1 \\
& - P_{(1,1,1,1)}(k)_1(k+2)_2(k+3)_1 - P_{(1,1,1,1)}(k)_1(k+1)_1(k+3)_2 \\
= {} & \frac{1}{8}(k-2)(k+3)\bigl(k^2 + k - 4\bigr) \\
& \times k(k-1)(k+1)(k+2) - \frac{1}{24}(k-3)(k-2)(k-1)(k+4) \\
& \times \bigl((k+1)k(k+2)(k+3) + k(k+2)(k+1)(k+3) + k(k+1)(k+3)(k+2)\bigr) \\
= {} & k(k-1)(k-2)(k+3)(k+2)(k+1).
\end{aligned} \tag{2.57}$$

Having shown that $N_\lambda(k)$ is a polynomial in k of degree $\leqslant 2|\lambda|$, we can determine it either using formula (2.52) and the formulas in Theorem 1.2 for the polynomials P, or else by evaluating (2.46) for $2|\lambda|+1$ values of k and applying polynomial interpolation. More specifically, we can work back from (2.46) to (2.32), and divide by $\frac{a_k}{2^k}\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j)!}$ to get the formula

$$N_\lambda(k) = \left(\frac{-1}{2}\right)^{k(k-1)/2} 2^{|\lambda|}\prod_{j=0}^{k-1}\frac{(k+j)!}{(2j)!}\sum_{\mathbf{u}}{}' \left|\begin{array}{cccc} \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdots & \frac{1}{1!} \\ \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdots & \frac{1}{0!} \\ \vdots & \vdots & & \vdots \\ \frac{1}{(k+i(\mathbf{u}))!} & \frac{1}{(k-i(\mathbf{u})-2)!} & & \vdots \\ \hdashline \frac{1}{(k+i(\mathbf{u})-1-u_{i(\mathbf{u})})!} & \frac{1}{(k+i(\mathbf{u})-3-u_{i(\mathbf{u})})!} & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{(k-u_1)!} & \frac{1}{(k-u_1-2)!} & \cdots & 0 \end{array}\right|. \tag{2.58}$$

This formula can be used for a specific choice of λ and several values of k to create a table of values of $N_\lambda(k)$ to which polynomial interpolation can be applied. Table 1 lists the polynomials $N_\lambda(k)$ for all $|\lambda|\leqslant 7$.

3. The coefficients $b_\lambda^{\pm}(k)$

In order to compute the multivariate Taylor expansion of (5.18), we consider the series expansion of its logarithm. We first examine the arithmetic product, and let

$$\log\bigl(A_k(z_1,\dots,z_k)/a_k\bigr) =: \sum_{r=1}^{\infty}\sum_{|\lambda|=r} B_\lambda(k)m_\lambda(z). \tag{3.1}$$

We start the sum at $r=1$ because the division by a_k makes the constant term 0. Now, the left hand side is symmetric in the z_i's, and we can find $B_\lambda(k)$ by applying

$$\frac{1}{\lambda_1!\lambda_2!\dots\lambda_l!}\frac{\partial^{\lambda_1}}{\partial z_1^{\lambda_1}}\frac{\partial^{\lambda_2}}{\partial z_2^{\lambda_2}}\cdots\frac{\partial^{\lambda_l}}{\partial z_l^{\lambda_l}}, \tag{3.2}$$

where $l=l(\lambda)$, and setting $z_1=\dots=z_k=0$. Since the partial derivatives do not involve $z_{l+1},\dots,z_k$ we can set these to 0 before the differentiation. Thus, by (1.10), $B_\lambda(k)$ is equal to (3.2) applied to

$$\begin{aligned} &-\log(a_k) + \sum_p \sum_{1\leqslant i\leqslant j\leqslant l}\log\left(1-\frac{1}{p^{1+z_i+z_j}}\right) + \sum_{1\leqslant i\leqslant l}(k-l)\log\left(1-\frac{1}{p^{1+z_i}}\right) \\ &\quad + \log\left(\frac{1}{2}\left(\prod_{j=1}^{l}\left(1-\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\left(1-\frac{1}{p^{\frac{1}{2}}}\right)^{l-k} + \prod_{j=1}^{l}\left(1+\frac{1}{p^{\frac{1}{2}+z_j}}\right)^{-1}\left(1+\frac{1}{p^{\frac{1}{2}}}\right)^{l-k}\right)+\frac{1}{p}\right) \\ &\quad - \log\left(1+\frac{1}{p}\right), \end{aligned} \tag{3.3}$$

evaluated at $z_1=\dots=z_l=0$. Likewise, we can find the coefficients of the expansions

$$-\frac{1}{2}\sum_{j=1}^{k}\log X(1/2+z_j,a) =: \sum_{r=1}^{\infty}\sum_{|\lambda|=r} f_\lambda^{\pm}(k)m_\lambda(z), \tag{3.4}$$

where $a=1$ for f^- and 0 for f^+, and of

$$\sum_{1\leqslant i\leqslant j\leqslant k}\log\bigl(\zeta(1+z_i+z_j)(z_i+z_j)\bigr)=:\sum_{r=1}^{\infty}\sum_{|\lambda|=r}g_\lambda(k)m_\lambda(z),\tag{3.5}$$

by applying (3.2), at $z_1=\cdots=z_l=0$, to

$$-\frac{1}{2}\sum_{j=1}^{l}\log X(1/2+z_j,a),\tag{3.6}$$

and to

$$\sum_{1\leqslant i\leqslant j\leqslant l}\log\bigl(\zeta(1+z_i+z_j)(z_i+z_j)\bigr)+\sum_{1\leqslant i\leqslant l}(k-l)\log\bigl(\zeta(1+z_i)z_i\bigr),\tag{3.7}$$

respectively.

Next, by composing the three series expansions (3.1), (3.4), (3.5) with the series for the exponential function, we can derive formulas for the coefficients $b_\lambda^{\pm}(k)$. Example formulas, for $b_{(1)}^{\pm}(k)$ and $b_{(1,1)}^{\pm}(k)$, are displayed in the Introduction.

To obtain numerical approximations to $b_\lambda^{\pm}(k)$ for specific choices of k and λ one needs to compute infinite sums over primes where the summand is a rational function of $p^{1/2}$ times $\log(p)^{|\lambda|}$. This can be achieved to high precision using Mobius inversion as described, in the context of the moment polynomials of the Riemann zeta function, in Section 4.1 of [CFKRS2]. In this fashion, and using (1.15), we computed the values of $c_\pm(r,k)$, for $r\leqslant 10$ and $k\leqslant 9$, given in Tables 3 and 4.

4. Determinant of a matrix of binomial coefficients

Proof of Theorem 1.2. We shall first prove (1.27), and use it to prove (1.26).

Proof of (1.27). For a k-tuple $(\alpha_1,\ldots,\alpha_k)$ and $x=(x_1,\ldots,x_k)$, let x^α denote the monomial $x_1^{\alpha_1}\ldots x_k^{\alpha_k}$. For a partition λ of length less than or equal to k, x^λ can be defined by appending zeros after the positive elements of λ to make it a k-tuple.

Reversing the rows of the matrix in (1.23), we see that

$$D_\lambda(k)=(-1)^{\binom{k}{2}}\det\left(\binom{k+i-1-\lambda_i}{2k-2j}\right)_{1\leqslant i,j\leqslant k}.\tag{4.1}$$

The (i,j)-th entry of the matrix in (4.1) can be written using the coefficient operator defined in (1.25). Let $x=(x_1,\ldots,x_k)$. Then

$$D_\lambda(k)=(-1)^{\binom{k}{2}}\det\bigl(\bigl[x_j^{2k-2j}\bigr](1+x_j)^{k+i-1-\lambda_i}\bigr)_{1\leqslant i,j\leqslant k}.\tag{4.2}$$

Noticing that column j only involves x_j, we can move all the $[x_j^{2k-2j}]$ in front of the determinant to get

$$\begin{aligned}&(-1)^{\binom{k}{2}}\bigl[x^{2\delta_k}\bigr]\det\bigl((1+x_j)^{k+i-1-\lambda_i}\bigr)_{1\leqslant i,j\leqslant k}\\&\quad=(-1)^{\binom{k}{2}}\bigl[x^{2\delta_k}\bigr]\det\bigl((1+x_j)^{-(k-i+\lambda_i)}\bigr)\prod_{l=1}^{k}(1+x_l)^{2k-1}.\end{aligned}\tag{4.3}$$

Table 3
The coefficients $c_-(r,k)$ of $Q_-(k)$.

r	$c_-(r,1)$	$c_-(r,2)$	$c_-(r,3)$
0	3.522211004995827732e−01	1.238375103096108452e−02	1.528376099282021425e−05
1	6.175500336140218316e−01	1.807468351186638511e−01	8.968276397996084726e−04
2		3.658991414081511628e−01	1.701420175947633562e−02
3		−1.398953902867718369e−01	1.093281830681910732e−01
4			1.358556940901993748e−01
5			−2.329509111366616925e−01
6			4.735303837788046866e−01
r	$c_-(r,4)$	$c_-(r,5)$	$c_-(r,6)$
0	3.158268332443340154e−10	6.712517611066278238e−17	1.036004645427003276e−25
1	5.062201340608140133e−08	2.341233253582258184e−14	6.796814066740219201e−23
2	3.252070477914552180e−06	3.571169234103129887e−12	2.037808336505920108e−20
3	1.065078255299183117e−04	3.127118490785452708e−10	3.698051408075659748e−18
4	1.865791348720969960e−03	1.734617312939144360e−08	4.534838798273249707e−16
5	1.658674128885722146e−02	6.342941105701246722e−07	3.972866885083416336e−14
6	5.985999910494527870e−02	1.541064437383931078e−05	2.563279107875100164e−12
7	5.231179842747744717e−03	2.441498848686470880e−04	1.237229229636910631e−10
8	−1.097356193524353096e−01	2.390928284573956911e−03	4.491515829566301398e−09
9	5.581253300381869842e−01	1.275610736275904766e−02	1.222154548508955419e−07
10	1.918594095122517496e−01	2.430382016767882944e−02	2.461203700713661380e−06
r	$c_-(r,7)$	$c_-(r,8)$	$c_-(r,9)$
0	8.864927187204894781e−37	3.372009502181036150e−50	4.727735796587526113e−66
1	9.894437508330137269e−34	5.951191608649093822e−47	1.248019487993274422e−62
2	5.176293026015439716e−31	5.002043249634522587e−44	1.585820955757896443e−59
3	1.686724585610585967e−28	2.664702289380503418e−41	1.291823649274241834e−56
4	3.837267516078630273e−26	1.010164553397544484e−38	7.580660624239738211e−54
5	6.474635477336820480e−24	2.900498887294046119e−36	3.413900516458523702e−51
6	8.402114103039537077e−22	6.555588245821587108e−34	1.227404779731471396e−48
7	8.581764459399681586e−20	1.196609980002393296e−31	3.618608212113140382e−46
8	7.002464589632248733e−18	1.795828629692653400e−29	8.916974338520402569e−44
9	4.607034349981096374e−16	2.244368542496810519e−27	1.862786263819570034e−41
10	2.455973970379903840e−14	2.357312576663548340e−25	3.334524507937658586e−39

The determinant in (4.2) can be written in terms of a_{δ_k} and s_λ defined in (1.53) and (1.55),

$$D_\lambda(k) = (-1)^{\binom{k}{2}}\left[x^{2\delta_k}\right] a_{\lambda+\delta_k}\left(\frac{1}{1+x_1},\dots,\frac{1}{1+x_k}\right)\prod_{l=1}^{k}(1+x_l)^{2k-1}$$

$$= (-1)^{\binom{k}{2}}\left[x^{2\delta_k}\right] a_{\delta_k}\left(\frac{1}{1+x_1},\dots,\frac{1}{1+x_k}\right) s_\lambda\left(\frac{1}{1+x_1},\dots,\frac{1}{1+x_k}\right)\prod_{l=1}^{k}(1+x_l)^{2k-1}. \quad (4.4)$$

But (1.56) gives $a_{\delta_k}(x_1,\dots,x_k)$ explicitly. Hence

$$a_{\delta_k}\left(\frac{1}{1+x_1},\dots,\frac{1}{1+x_k}\right) = \prod_{1\leqslant i<j\leqslant k}\left(\frac{1}{1+x_i}-\frac{1}{1+x_j}\right) = \prod_{1\leqslant i<j\leqslant k}\frac{(1+x_j)-(1+x_i)}{(1+x_j)(1+x_i)}$$

$$= \frac{\prod_{1\leqslant i<j\leqslant k}(x_j-x_i)}{\prod_{j=1}^{k}(1+x_j)^{k-1}} = \frac{(-1)^{\binom{k}{2}} a_{\delta_k}(x)}{\prod_{j=1}^{k}(1+x_j)^{k-1}}. \quad (4.5)$$

Using (4.5) in (4.4), we have

$$D_\lambda(k) = \left[x^{2\delta_k}\right] a_{\delta_k}(x_1,\dots,x_k)\, s_\lambda\left(\frac{1}{1+x_1},\dots,\frac{1}{1+x_k}\right)\prod_{l=1}^{k}(1+x_l)^{k}. \quad (4.6)$$

Table 4
The coefficients $c_+(r,k)$ of $Q_+(r,k)$.

r	$c_+(r,1)$	$c_+(r,2)$	$c_+(r,3)$
0	3.522211004995827732e−01	1.238375103096108452e−02	1.528376099282021425e−05
1	−4.889851881547797041e−01	6.403273133040673915e−02	6.087355322740111135e−04
2		−4.030985462971436450e−01	5.189536257221761054e−03
3		8.784723252866324383e−01	−2.070416696161206729e−02
4			−4.836560144295628388e−02
5			6.305676273169569246e−01
6			−1.231149543676485214
r	$c_+(r,4)$	$c_+(r,5)$	$c_+(r,6)$
0	3.158268332443340154e−10	6.712517611066278238e−17	1.036004645427003276e−25
1	4.070002081481211197e−08	2.024913313371989448e−14	6.113326104276961713e−23
2	1.961035634727995841e−06	2.611003455556346309e−12	1.632224321325099403e−20
3	4.187933734218812260e−05	1.870888923760240058e−10	2.605311255686981285e−18
4	3.233832982317403053e−04	8.086250862410257040e−09	2.766415183453526818e−16
5	−7.264209058002128044e−04	2.126496335543600159e−07	2.056437432501927988e−14
6	−9.741303115420443803e−03	3.194157049041922835e−06	1.095709499896029594e−12
7	6.254058547607513341e−02	2.120198748289444789e−05	4.206172871179562219e−11
8	5.338039400180279170e−02	−3.390055513847315853e−05	1.149109718292255815e−09
9	−1.125787514381924481e+00	−7.750613901748660065e−04	2.154509460431619112e−08
10	2.125417457224375362	3.339978554290242568e−03	2.543371224701971233e−07
r	$c_+(r,7)$	$c_+(r,8)$	$c_+(r,9)$
0	8.864927187204894781e−37	3.372009502181036150e−50	4.727735796587526113e−66
1	9.114637784804059894e−34	5.569826318573164385e−47	1.181182697783246367e−62
2	4.370089613567423486e−31	4.368642207198861832e−44	1.417926553457661234e−59
3	1.297363094463138851e−28	2.164658555649376388e−41	1.089051480593133551e−56
4	2.670392092372496088e−26	7.604817314362535383e−39	6.012641112088390226e−54
5	4.043466811338890795e−24	2.015327809331532264e−36	2.541594397695401893e−51
6	4.663148139710778893e−22	4.184593239584908611e−34	8.555207141044511720e−49
7	4.183154331210266578e−20	6.980465161514108456e−32	2.354807833463352272e−46
8	2.954857264190019988e−18	9.516651650236242059e−30	5.400892227120418237e−44
9	1.652770327042906306e−16	1.073015400698217206e−27	1.046573394851932219e−41
10	7.319238365079051443e−15	1.008662233782716849e−25	1.731269798305270612e−39

We shall now express s_λ as a coefficient in a polynomial which is easier to work with. The dual Jacobi–Trudi identity, (1.59), gives

$$s_\lambda = \det(e_{\mu_i - i + j})_{1 \leqslant i,j \leqslant n}, \tag{4.7}$$

where $(\mu_1, \ldots, \mu_n)$ is the conjugate partition of λ, and $n = l(\mu)$.

Expanding the determinant, we get

$$s_\lambda = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} e_{\mu_i - i + \sigma(i)}(x). \tag{4.8}$$

From (1.46), we have $e_r = [t^r]E(t)$. We rewrite (4.8) using this notation. Let $t = (t_1, \ldots, t_n)$. Then

$$\begin{aligned} s_\lambda(x) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \big[t_1^{\mu_1 - 1 + \sigma(1)} \ldots t_n^{\mu_n - n + \sigma(n)}\big] \prod_{i=1}^{n} E(t_i) \\ &= \big[t_1^{\mu_1} \ldots t_n^{\mu_n}\big] \prod_{i=1}^{n} E(t_i) \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} t_i^{i - \sigma(i)} \right). \end{aligned}$$

Next, pull out $\prod t_i^i$ from the sum, and multiply and divide by $\prod t_i^n$, to get

$$\begin{aligned}\left[t_1^{\mu_1}\dots t_n^{\mu_n}\right]\prod_{i=1}^{n}E(t_i)\prod_{1\leqslant i<j\leqslant n}(t_i-t_j)\prod_{l=1}^{n}t_l^{l-n} &= \left[t^{\mu+\delta_n}\right]\prod_{i=1}^{n}E(t_i)\prod_{1\leqslant i<j\leqslant n}(t_i-t_j)\\ &= \left[t^{\mu+\delta_n}\right]a_{\delta_n}(t)\prod_{i=1}^{n}E(t_i).\end{aligned}\tag{4.9}$$

Here we have also used the Vandermonde determinant (up to $(-1)^{\binom{n}{2}}$):

$$\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}t_i^{n-\sigma(i)}=\prod_{1\leqslant i<j\leqslant n}(t_i-t_j).\tag{4.10}$$

We have expressed $s_\lambda(x)$ as a coefficient in a polynomial. Substituting (4.9) for s_λ in (4.6), and using the product form of $E(t)$, (1.46), we have

$$\begin{aligned}D_\lambda(k) &= \left[t^{\mu+\delta_n}x^{2\delta_k}\right]a_{\delta_k}(x)a_{\delta_n}(t)\left[\prod_{i=1}^{n}\left(\prod_{l=1}^{k}\left(1+\frac{t_i}{1+x_l}\right)\right)\right]\prod_{l=1}^{k}(1+x_l)^k\\ &= \left[t^{\mu+\delta_n}x^{2\delta_k}\right]a_{\delta_k}(x)a_{\delta_n}(t)\prod_{l=1}^{k}\left((1+x_l)^{k-n}\prod_{i=1}^{n}(1+x_l+t_i)\right)\\ &= \left[t^{\mu+\delta_n}x^{2\delta_k}\right]a_{\delta_k}(x)a_{\delta_n}(t)\prod_{l=1}^{k}\left((1+x_l)^{k-n}\prod_{i=1}^{n}\left(1+\frac{x_l}{1+t_i}\right)\right)\prod_{i=1}^{n}(1+t_i)^k.\end{aligned}\tag{4.11}$$

Applying the dual Cauchy identity (1.61) to the double product on the right hand side above gives

$$\prod_{l=1}^{k}\left((1+x_l)^{k-n}\prod_{i=1}^{n}\left(1+\frac{x_l}{1+t_i}\right)\right)=\sum_{\lambda}s_\lambda(x_1,\dots,x_k)s_{\lambda'}\left(1,\dots,1,\frac{1}{1+t_1},\dots,\frac{1}{1+t_n}\right).\tag{4.12}$$

The number of 1's in the second factor on the right hand side of (4.12) is $k-n$. Recall from (1.55) that $a_\delta s_\lambda=a_{\delta+\lambda}$. Hence (4.11) equals

$$\left[t^{\mu+\delta_n}\right]\left[x^{2\delta_k}\right]a_{\delta_n}(t)\prod_{i=1}^{n}(1+t_i)^k\sum_{\lambda}a_{\lambda+\delta_k}(x_1,\dots,x_k)s_{\lambda'}\left(1,\dots,1,\frac{1}{1+t_1},\dots,\frac{1}{1+t_n}\right).\tag{4.13}$$

The monomial $x^{2\delta_k}$ occurs in the sum in (4.13) only when $\lambda=\delta_k$. The coefficient of $x^{2\delta_k}$ in $a_{2\delta_k}(x_1,\dots,x_k)$ is 1. Simplifying (4.13), we have

$$D_\lambda(k)=\left[t^{\mu+\delta_n}\right]a_{\delta_n}(t)\prod_{i=1}^{n}(1+t_i)^k s_{\delta_k}\left(1,\dots,1,\frac{1}{1+t_1},\dots,\frac{1}{1+t_n}\right).\tag{4.14}$$

Note that we have used $\delta_k'=\delta_k$. Applying the formula s_{δ_k} in (1.57), we have

$$s_{\delta_k}\left(1,\dots,1,\frac{1}{1+t_1},\dots,\frac{1}{1+t_n}\right)=2^{\binom{k-n}{2}}\prod_{i=1}^{n}\left(1+\frac{1}{1+t_i}\right)^{k-n}\prod_{1\leqslant i<j\leqslant n}\left(\frac{1}{1+t_i}+\frac{1}{1+t_j}\right). \tag{4.15}$$

The $2^{\binom{k-n}{2}}$ comes from pairing, in applying (1.57), the $k-n$ 1's. The middle factor arises from matching each $1/(1+t_i)$ with $k-n$ 1's, and the last factor from matching all pairs of distinct $1/(1+t_i), 1/(1+t_j)$. Substituting (4.15) into (4.14), and collecting the powers of $(1+t_i)$ gives

$$D_\lambda(k)=\left[t^{\mu+\delta_n}\right]a_{\delta_n}(t)2^{\binom{k-n}{2}}\left(\prod_{i=1}^{n}(1+t_i)(2+t_i)^{k-n}\right)\prod_{1\leqslant i<j\leqslant n}(2+t_i+t_j). \tag{4.16}$$

Substituting $z_i=t_i/2$, and collecting powers of 2 (note that $\binom{k-n}{2}+(k-n)n+\binom{n}{2}=\binom{k}{2}$), we get

$$D_\lambda(k)=\left[z^{\mu+\delta_n}\right]a_{\delta_n}(z)2^{\binom{k}{2}+\binom{n}{2}-|\mu+\delta_n|}\prod_{i=1}^{n}(1+2z_i)(1+z_i)^{k-n}\prod_{1\leqslant i<j\leqslant n}(1+z_i+z_j). \tag{4.17}$$

Here we have also used $a_{\delta_n}(t)=a_{\delta_n}(z)2^{\binom{n}{2}}$. Since $|\delta_n|=\binom{n}{2}$, and $|\mu|=|\lambda|$, this proves (1.27). □

Proof of (1.26). We now use (1.27) to prove (1.26). As above, let $z=(z_1,\dots,z_n)$.

Since Schur symmetric functions form a $\mathbb{Z}$-basis for the ring of symmetric functions, the coefficient of s_γ of a symmetric function F is well defined. We denote this coefficient by $[s_\gamma]F$.

For a symmetric polynomial $F(z)$ in n-variables, and a partition γ with length at most n, we have

$$\left[s_\gamma(z)\right]F(z)=\left[z^{\gamma+\delta_n}\right]a_{\delta_n}(z)F(z). \tag{4.18}$$

This can be seen by writing $F(z)$ in terms of our Schur basis

$$F(z)=\sum_\gamma v_\gamma s_\gamma(z). \tag{4.19}$$

We wish to find the coefficient v_γ. Multiplying by $a_{\delta_n}(z)$ and using (1.55) gives

$$a_{\delta_n}(z)F(z)=\sum_\gamma v_\gamma a_{\gamma+\delta_n}(z). \tag{4.20}$$

Now, the monomials in $a_{\gamma+\delta_n}(z)$ are all distinct, and distinct from the monomials in $a_{\gamma'+\delta_n}$ for any different partition γ' of length at most n. Furthermore, $z^{\gamma+\delta_n}$ appears in $a_{\gamma+\delta_n}(z)$ with coefficient 1, coming from the main diagonal of (1.53). Thus, v_γ is equal to the coefficient of $z^{\gamma+\delta_n}$ in $a_{\delta_n}(z)F(z)$.

Therefore we can rewrite (1.27) as

$$D_\lambda(k)=2^{\binom{k}{2}-|\lambda|}\left[s_\mu(z)\right]\prod_{1\leqslant i<j\leqslant n}\left(\frac{1+z_i+z_j}{(1+z_i)(1+z_j)}\right)\prod_{i=1}^{n}(1+2z_i)(1+z_i)^{k-1}. \tag{4.21}$$

We shall work with the ring of symmetric functions Λ instead of the ring of symmetric polynomials in n variables Λ_n. The right hand side of (4.21) equals

$$2^{\binom{k}{2}-|\lambda|}\left[s_\mu(z)\right]\prod_{1\leqslant i<j}\left(\frac{1+z_i+z_j}{(1+z_i)(1+z_j)}\right)\prod_{i\geqslant 1}(1+2z_i)(1+z_i)^{k-1}. \tag{4.22}$$

Note that in (4.22), we are looking at elements in the ring of symmetric functions, Λ, i.e. as a product involving a countable number of variables $z_1, z_2, \dots,$ whereas in (4.21), we were considering the elements in the ring of symmetric polynomials in n variables, Λ_n.

Applying ω, and using (1.64) we obtain

$$D_\lambda(k) = 2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big]\omega\left(\prod_{1\leqslant i<j}\left(1-\frac{y_i y_j}{(1+y_i)(1+y_j)}\right)\prod_{i\geqslant 1}(1+2y_i)(1+y_i)^{k-1}\right). \quad (4.23)$$

We use the fact that $\exp(\log(1+u)) = 1+u$ to write the argument of ω as formal power series:

$$\begin{aligned} D_\lambda(k) &= 2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big]\omega\left(\exp\sum_{a\geqslant 1}\frac{1}{a}\left(-\sum_{1\leqslant i<j} y_i^a y_j^a (1+y_i)^{-a}(1+y_j)^{-a}\right.\right. \\ &\quad \left.\left. -(-2)^a\sum_{i\geqslant 1} y_i^a - (k-1)(-1)^a\sum_{i\geqslant 1} y_i^a\right)\right) \\ &= 2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big]\omega\left(\exp\sum_{a\geqslant 1}\frac{1}{a}\left(-\sum_{b,c\geqslant 0}\binom{-a}{b}\binom{-a}{c}\sum_{1\leqslant i<j} y_i^{a+b} y_j^{a+c}\right.\right. \\ &\quad \left.\left. -(-2)^a\sum_{i\geqslant 1} y_i^a - (k-1)(-1)^a\sum_{i\geqslant 1} y_i^a\right)\right). \end{aligned} \quad (4.24)$$

We can rewrite the argument of ω in (4.24) using power symmetric functions;

$$\begin{aligned} D_\lambda(k) &= 2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big]\omega\left(\exp\sum_{a\geqslant 0}\frac{1}{a}\left(-\sum_{b,c\geqslant 0}\binom{-a}{b}\binom{-a}{c}\frac{1}{2}(p_{a+b}p_{a+c} - p_{2a+b+c})\right.\right. \\ &\quad \left.\left. -(-2)^a p_a - (k-1)(-1)^a p_a\right)\right). \end{aligned} \quad (4.25)$$

In (1.64) we have seen that $\omega(p_a) = (-1)^{a-1}p_a$. This gives

$$\begin{aligned} D_\lambda(k) &= 2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big]\exp\sum_{a\geqslant 0}\frac{1}{a}\left(-\sum_{b,c\geqslant 0}\binom{-a}{b}\binom{-a}{c}\right. \quad (4.26) \\ &\quad \left. \times (-1)^{2a+b+c}\frac{1}{2}(p_{a+b}p_{a+c} + p_{2a+b+c}) + 2^a p_a + (k-1)p_a\right) \\ &= 2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big]\prod_{1\leqslant i\leqslant j}\left(1-\frac{y_i y_j}{(1-y_i)(1-y_j)}\right)\prod_{i\geqslant 1}(1-2y_i)^{-1}(1-y_i)^{-k+1}. \quad (4.27) \end{aligned}$$

If we isolate factors corresponding to $i = j$ in the first product, we are able to cancel some factors in the second product. Simplifying, we get

$$D_\lambda(k) = 2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big]\prod_{1\leqslant i<j}\left(1-\frac{y_i y_j}{(1-y_i)(1-y_j)}\right)\prod_{i\geqslant 1}(1-y_i)^{-k-1}. \quad (4.28)$$

To calculate the coefficient of s_λ in (4.28), we only have to look at the projection in any Λ_m such that m is greater than or equal to $l(\lambda)$. We can choose it to be equal to $l(\lambda)$ (also equals μ_1). Let $m = l(\lambda)$. Then

$$D_\lambda(k) = 2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big] \prod_{1\leqslant i<j\leqslant m}\left(1 - \frac{y_i y_j}{(1-y_i)(1-y_j)}\right)\prod_{i=1}^{m}(1-y_i)^{-k-1}, \tag{4.29}$$

which is equal to

$$2^{\binom{k}{2}-|\lambda|}\big[s_\lambda(y)\big] \prod_{1\leqslant i<j\leqslant m}(1-y_i-y_j)\prod_{i=1}^{m}(1-y_i)^{-k-m}. \tag{4.30}$$

Another application of (4.18) proves (1.26). □

Proof of Corollary 1.3. It is immediate from (1.26) or (1.27) that $P_\lambda(k)$ is a polynomial in k with integer values at integers of degree at most $|\lambda|$. We will show that it is in fact of degree $|\lambda|$ and determine its leading coefficient.

From (1.26), the highest power of k occurs when we pick as many powers of y_i as possible from the last product. This happens when none of the y_i are picked from $(1-y_i-y_j)$. Note that

$$(1-y)^{-k-m} = 1 + (k+m)y + \frac{(k+m)(k+m+1)}{2!}y^2 + \cdots = \sum_{j=0}^{\infty}(k+m+j-1)_j y^j/j!. \tag{4.31}$$

The coefficient of the highest power of k that appears in the j-th term of this Taylor series is $1/j!$. Thus, the coefficient of $k^{|\lambda|}$ in $P_\lambda(k)$ equals

$$\begin{aligned}
&\big[y_1^{\lambda_1+m-1}\ldots y_m^{\lambda_m}\big]\Bigg(\prod_{1\leqslant i<j\leqslant m}(y_i-y_j)\Bigg)\exp(y_1+\cdots+y_m)\\
&\quad= \big[y_1^{\lambda_1+m-1}\ldots y_m^{\lambda_m}\big]\sum_{\sigma\in S_m}\operatorname{sgn}(\sigma)\Bigg(\prod_{i=1}^{m} y_i^{m-\sigma(i)}e^{y_i}\Bigg)\\
&\quad= \sum_{\sigma\in S_m}\operatorname{sgn}(\sigma)\frac{1}{(\lambda_i-i+\sigma(i))!} = \det\big(1/(\lambda_i-i+j)!\big)_{m\times m}\\
&\quad= \frac{\prod_{1\leqslant i<j\leqslant m}(\lambda_i-\lambda_j-i+j)}{\prod_{1\leqslant i\leqslant m}(\lambda_i+m-i)!} = \frac{\chi^\lambda(1)}{|\lambda|!},
\end{aligned} \tag{4.32}$$

where $\chi^\lambda(1)$ is the degree of the irreducible representation of $S_{|\lambda|}$ indexed by λ. See Example 6 in Chapter I.7 of [M] for the last two equalities. □

Proof of Corollary 1.4. We use Eq. (1.27) which gives a formula for $P_\lambda(k)$. As part of the process of identifying the coefficient of $z_1^{\mu_1+n-1}\ldots z_n^{\mu_n}$ in that formula, we focus on the coefficient of $z_1^{\mu_1+n-1}$. Now, $\mu_1 = l(\lambda)$, and $n = \lambda_1$, hence $\mu_1 + n - 1 = l(\lambda) + \lambda_1 - 1$. When we expand (1.27), some of the powers of z_1 come from the factor $(1+z_1)^{k-\lambda_1}$, and the rest from

$$\prod_{1<j\leqslant\lambda_1}(z_1-z_j)(1+z_1+z_j)(1+2z_1). \tag{4.33}$$

Consider the terms arising from taking a z_1^j from the above. Notice that (4.33) is a polynomial in z_1 of degree $2\lambda_1 - 1$, and thus $0 \leqslant j \leqslant 2\lambda_1 - 1$. The remaining $l(\lambda) + \lambda_1 - 1 - j$ powers of z_1 come from expanding $(1 + z_1)^{k-\lambda_1}$ using the binomial theorem, so that the term associated with a particular choice of j is divisible by

$$\binom{k - \lambda_1}{l(\lambda) + \lambda_1 - 1 - j} = \frac{(k - \lambda_1)(k - \lambda_1 - 1)\dots(k - 2\lambda_1 - l(\lambda) + 2 + j)}{(l(\lambda) + \lambda_1 - 1 - j)!}. \tag{4.34}$$

For all $0 \leqslant j \leqslant 2\lambda_1 - 1$, this is divisible by

$$(k - \lambda_1)(k - \lambda_1 - 1)\dots\bigl(k - l(\lambda) + 1\bigr). \tag{4.35}$$

The coefficient of $z_1^{l(\lambda)+\lambda_1-1} = z_1^{\mu_1+n-1}$ in the expression in (1.27) is therefore divisible by (4.35). Thus, so is the coefficient of $z_1^{\mu_1+n-1}\dots z_n^{\mu_n}$.

The same analysis applied to (1.26), and using (4.31) gives that $P_\lambda(k)$ is divisible, for $l(\lambda) \leqslant \lambda_1$, by

$$\bigl(k + l(\lambda)\bigr)\dots(k + \lambda_1 - 1)(k + \lambda_1). \qquad \square \tag{4.36}$$

5. Family of quadratic twists of elliptic curve L-functions

Here we modify our techniques to the family of L-functions associated to the quadratic twists of an elliptic curve over $\mathbb{Q}$. To keep things explicit, we focus on the elliptic curve of conductor 11:

$$E_{11a} : y^2 + y = x^3 - x. \tag{5.1}$$

The L-function of E_{11a} is given by an Euler product of the form

$$L_{11}(s) = \frac{1}{1 - 11^{-s-1/2}} \prod_{p \neq 11} \frac{1}{1 - a(p)p^{-s-1/2} + p^{-2s}}, \tag{5.2}$$

which can be expanded into the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{1/2+s}}. \tag{5.3}$$

The Dirichlet series above is absolutely convergent in $\Re s > 1$. The coefficients $a(n)$ can be obtained from the Fourier expansion of the cusp form of weight two and level 11 given by

$$\sum_{n=1}^{\infty} a(n)q^n = q \prod_{n=1}^{\infty} \bigl(1 - q^n\bigr)^2 \bigl(1 - q^{11n}\bigr)^2, \tag{5.4}$$

or, alternatively, by counting points on E_{11a} over the fields F_p, p prime.

The function $L_{11}(s)$ has analytic continuation to all of $\mathbb{C}$ and satisfies the functional equation

$$L_{11}(s) = X(s)L_{11}(1 - s), \tag{5.5}$$

where

$$X(s) = \frac{\Gamma(3/2 - s)}{\Gamma(s + 1/2)} \left(\frac{2\pi}{11^{1/2}}\right)^{2s-1}. \tag{5.6}$$

The L-function associated to a quadratic twist of E_{11a} has a Dirichlet series of the form

$$L_{11}(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{1/2+s}} \chi_d(n), \tag{5.7}$$

where d is a fundamental discriminant which we further assume satisfies $(d, 11) = 1$. $L_{11}(s, \chi_d)$ satisfies the functional equation

$$L_{11}(s, \chi_d) = \chi_d(-11)|d|^{1-2s} X(s) L_{11}(1-s, \chi_d). \tag{5.8}$$

When considering the moments of $L_{11}(1/2, \chi_d)$ we should restrict $L(s, \chi_d)$ to have an even functional equation, i.e. $\chi_d(-11) = 1$, otherwise $L(1/2, \chi_d)$ is trivially equal to 0. In [CFKRS], d was also restricted to being negative since it allowed them to exploit a theorem of Kohnen and Zagier [KZ] to easily gather numerical data for $L_{11}(1/2, \chi_d)$ with which to check their conjecture.

When $d < 0$, $\chi_d(-1) = -1$, hence, in order to have an even functional equation, we require $\chi_d(11) = -1$, i.e. $d = 2, 6, 7, 8, 10 \bmod 11$. CFKRS conjectured, see Section 5.3 of [CFKRS], the asymptotic expansion:

$$\sum_{\substack{d \in S_-(X) \\ d = 2,6,7,8,10 \bmod 11}} L_{11}(1/2, \chi_d)^k \sim \frac{15}{11\pi^2} \frac{1}{X} \int_1^X \Upsilon_k(\log t)\, dt. \tag{5.9}$$

The extra factor of 5/11 on the right hand side, compared to (1.5), reflects the fact that the sum on the left is over 5 out of 11 possible residue classes mod 11. Here, Υ_k is the polynomial of degree $k(k-1)/2$ given by the k-fold residue

$$\Upsilon_k(x) = \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{R_{11}(z_1, \ldots, z_k) \Delta(z_1^2, \ldots, z_k^2)^2}{\prod_{j=1}^k z_j^{2k-1}} e^{x \sum_{j=1}^k z_j}\, dz_1 \ldots dz_k, \tag{5.10}$$

where

$$R_{11}(z_1, \ldots, z_k) = A_k(z_1, \ldots, z_k) \prod_{j=1}^k X(1/2 + z_j)^{-1/2} \prod_{1 \leqslant i < j \leqslant k} \zeta(1 + z_i + z_j), \tag{5.11}$$

and, overloading the notation of Section 1.1, A_k is the Euler product which is absolutely convergent for $\sum_{j=1}^k |z_j| < \frac{1}{2}$,

$$A_k(z_1, \ldots, z_k) = \prod_p R_{11,p}(z_1, \ldots, z_k) \prod_{1 \leqslant i < j \leqslant k} \left(1 - \frac{1}{p^{1+z_i+z_j}}\right) \tag{5.12}$$

with, for $p \neq 11$,

$$R_{11,p} = \left(1 + \frac{1}{p}\right)^{-1} \left(\frac{1}{p} + \frac{1}{2}\left(\prod_{j=1}^k \frac{1}{1 - a(p)p^{-1-z_j} + p^{-1-2z_j}} + \prod_{j=1}^k \frac{1}{1 + a(p)p^{-1-z_j} + p^{-1-2z_j}}\right)\right) \tag{5.13}$$

and

$$R_{11,11} = \prod_{j=1}^{k} \frac{1}{1 + 11^{-1-z_j}}. \tag{5.14}$$

Note that, although here we are working with the specific elliptic curve E_{11a}, CFKRS' recipe provides a similar conjecture for the quadratic twists of any elliptic curve over $\mathbb{Q}$. For many examples, see the paper [CPRW]. The only difference is in the conductor, in the local factors of A_k for the primes dividing the conductor, and in the allowed residue classes (and modulus) for d.

Next, $\Upsilon_k(x)$ is a polynomial of degree of $k(k-1)/2$ given by the k-fold residue (5.10). The degree works out smaller compared to $Q_{\pm}(k,x)$ because the product of zetas in (5.11) involves fewer zetas, i.e. the product over $i < j$ has $\binom{k}{2}$ factors. Therefore, we can write

$$\Upsilon_k(x) = c_0(k)x^{k(k-1)/2} + c_1(k)x^{k(k-1)/2-1} + \cdots + c_{k(k-1)/2}(k). \tag{5.15}$$

Also note that the exponential in (5.10) has an x rather than $x/2$. This will impact the powers of 2 that enter into our formulas for the coefficients $c_r(k)$.

To address the poles coming from the zeta-product $\prod \zeta(1+z_i+z_j)$ we absorb some of the factors of $\Delta(z_1^2,\dots,z_k^2) = \prod_{1\leqslant i<j\leqslant k}(z_j - z_i)(z_j + z_i)$. Thus,

$$\begin{aligned} \Upsilon_k(x) = {} & \frac{(-1)^{k(k-1)/2}2^k}{k!} \frac{1}{(2\pi i)^k} \oint \cdots \oint A_k(z_1,\dots,z_k) \prod_{j=1}^{k} X(1/2+z_j)^{-1/2} \\ & \times \prod_{1\leqslant i<j\leqslant k} \zeta(1+z_i+z_j)(z_i+z_j) \frac{\Delta(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)}{\prod_{j=1}^{k} z_j^{2k-1}} \exp\left(x\sum_{j=1}^{k} z_j\right) dz_1 \dots dz_k. \end{aligned} \tag{5.16}$$

We overload notation again and set

$$a_k := A_k(0,\dots,0) \tag{5.17}$$

and expand

$$\frac{1}{a_k} A_k(z_1,\dots,z_k) \prod_{j=1}^{k} X(1/2+z_j)^{-1/2} \prod_{1\leqslant i<j\leqslant k} \zeta(1+z_i+z_j)(z_i+z_j) =: \sum_{j=0}^{\infty} \sum_{|\lambda|=j} b_\lambda(k) m_\lambda(z), \tag{5.18}$$

where, as before, $m_\lambda(z)$ is the monomial symmetric function for the partition λ. The left hand side above is holomorphic in a neighbourhood of $z_1 = \cdots = z_k = 0$, because the poles from the zeta-product $\prod \zeta(1+z_i+z_j)$ are cancelled by the product $\prod(z_i+z_j)$. We normalize by a_k so that the first coefficient is 1.

So (5.16) becomes

$$\begin{aligned} \Upsilon_k(x) = {} & \frac{(-1)^{k(k-1)/2}2^k}{k!} \frac{a_k}{(2\pi i)^k} \oint \cdots \oint \sum_{j=0}^{\infty} \sum_{|\lambda|=j} b_\lambda(k) m_\lambda(z) \\ & \times \frac{\Delta(z_1,\dots,z_k)\Delta(z_1^2,\dots,z_k^2)}{\prod_{j=1}^{k} z_j^{2k-1}} \exp\left(x\sum_{j=1}^{k} z_j\right) dz_1 \dots dz_k. \end{aligned} \tag{5.19}$$

Comparing to Eq. (2.6) we notice three differences: the extra 2^k in front of the integral, the $2k-1$ powers of each z_j, rather than $2k$ powers, in the denominator, and the x rather than $x/2$ in the exponential. The first two differences are accounted for by the fact that the product over zetas in (2.3) includes $i=j$, and this introduces, from (2.2), an extra $2z_j$, for each j. Therefore, proceeding as in Section 2.2, we get:

$$c_r(k) = \frac{(-1)^{k(k-1)/2}2^k}{k!}\frac{a_k}{(2\pi i)^k}\oint\cdots\oint\sum_{|\lambda|=r} b_\lambda(k)m_\lambda(z) \times \frac{\Delta(z_1,\ldots,z_k)\Delta(z_1^2,\ldots,z_k^2)}{\prod_{j=1}^k z_j^{2k-1}}\exp\left(\sum_{j=1}^k z_j\right)dz_1\ldots dz_k. \tag{5.20}$$

Analogously to (2.49), we have

$$c_r(k) = 2^k a_k\prod_{j=0}^{k-1}\frac{(2j)!}{(k+j-1)!} \times \sum_{|\lambda|=r} b_\lambda(k)\sum_{\mathbf{u}}{}' (-1)^{n(\mathbf{u})}E_{\alpha(\mathbf{u})}(k)\times \big(k+i(\mathbf{u})-2\big)_{u_{i(\mathbf{u})}}\big(k+i(\mathbf{u})-3\big)_{u_{i(\mathbf{u})-1}}\cdots(k-1)_{u_1}, \tag{5.21}$$

where, for a partition α,

$$E_\alpha(k) = \det\left(\binom{2k-i-1-\alpha_{k-i+1}}{2k-2j}\right)_{1\leqslant i,j\leqslant k}. \tag{5.22}$$

The equation for $c_r(k)$ differs from (2.49) in the power of 2 that appears, and also some of the factorials have an extra -1 in them. The latter comes from the one missing z_j in the denominator of (5.20) compared to (2.3).

Notice that $E_\alpha(k)$ is very similar to $D_\alpha(k)$. The only difference is the extra -1 in the binomial coefficient. We can relate the two determinants by taking advantage of the entries in the first column of the matrix for $E_\alpha(k)$, which are all 0 except for the 1, 1 entry. Assume, for now, that $k \geqslant \max(l(\alpha)+1, \alpha_1)$ (so, in particular, $\alpha_k = 0$). Expanding along the first column, and then reindexing i, j with $i+1, j+1$:

$$E_\alpha(k) = \det\left(\binom{2k-i-1-\alpha_{k-i+1}}{2k-2j}\right)_{2\leqslant i,j\leqslant k} \tag{5.23}$$

$$= \det\left(\binom{2(k-1)-i-\alpha_{(k-1)-i+1}}{2(k-1)-2j}\right)_{1\leqslant i,j\leqslant k-1} \tag{5.24}$$

$$= D_\alpha(k-1) = 2^{\binom{k-1}{2}-|\alpha|}\times P_\alpha(k-1). \tag{5.25}$$

Also note, while we have assumed that $k > l(\alpha)$, Corollary 1.4 tells us that $P_\alpha(k-1)$, and hence the right hand side of (5.23), vanishes for $\alpha_1+1 \leqslant k \leqslant l(\alpha)$. Furthermore, by the same method as was used around (2.55),

$$\big(k+i(\mathbf{u})-2\big)_{u_{i(\mathbf{u})}}\big(k+i(\mathbf{u})-3\big)_{u_{i(\mathbf{u})-1}}\cdots(k-1)_{u_1} \tag{5.26}$$

is divisible by $(k-1)\ldots(k-\alpha_1)$, and thus vanishes for $1 \leqslant k \leqslant \alpha_1$. Therefore, writing

$$c_r(k) = 2^{k+\binom{k-1}{2}-r} a_k \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j-1)!}$$
$$\times \sum_{|\lambda|=r} b_\lambda(k) \sum_{\mathbf{u}}{}' (-1)^{n(\mathbf{u})} P_{\alpha(\mathbf{u})}(k-1) \times \big(k+i(\mathbf{u})-2\big)_{u_{i(\mathbf{u})}} \big(k+i(\mathbf{u})-3\big)_{u_{i(\mathbf{u})-1}} \cdots (k-1)_{u_1}, \tag{5.27}$$

we can replace the requirement that $k \geqslant \max(l(\alpha)+1, \alpha_1)$ with $k > 0$. Finally, using (2.52), we get, for $k > 0$,

$$c_r(k) = 2^{k+\binom{k-1}{2}-r} a_k \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j-1)!} \sum_{|\lambda|=r} b_\lambda(k) N_\lambda(k-1). \tag{5.28}$$

For example, the $r = 0$ term equals

$$c_0(k) = 2^{k+\binom{k-1}{2}} a_k \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j-1)!}. \tag{5.29}$$

One can verify inductively that this matches the leading term as described in (1.5.26) of [CFKRS]:

$$2^{k+\binom{k-1}{2}} \prod_{j=0}^{k-1} \frac{(2j)!}{(k+j-1)!} = 2^{(k+1)k/2} \prod_{j=0}^{k-1} \frac{j!}{(2j)!}. \tag{5.30}$$

One should also pay attention here that the Taylor coefficients $b_\lambda(k)$, and also a_k, depend on the underlying elliptic curve E_{11a} and its $a(p)$'s. While we can derive similar formulas for $b_\lambda(k)$ as for quadratic Dirichlet L-functions (see the examples (1.21), (1.22)), in order to accelerate their numerical evaluation we would need to use the symmetric power L-functions associated to the L-function $L_{11}(s)$.

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