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Algebraic Methods for Permutations with Prescribed Patterns

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1. INTRODUCTION

Many problems which arise in a variety of disciplines may be expressed in terms of the enumeration of sequences, over a finite alphabet, which possess certain prescribed characteristics. Typical problems include the enumeration of non-self-intersecting paths on a rectangular lattice (a problem from crystal physics, Seymour and Welsh [16]), and the enumeration of sequences with no substrings in a prescribed set (Guibas and Odlyzko [10]). The variety of such problems is attributable to the observation that sequences often may be used as a device for encoding combinatorial structures. For this reason, a considerable amount of attention has been focused on the development of general methods for sequence enumeration. A number of approaches to an algebraic theory of sequence enumeration have been adopted by Cartier and Foata [3], Cori and Richard [6], Doubilet *et al.* [7], Foata and Schützenberger [8], Gessel [9], Jackson and Goulden [12], Spears *et al.* [17], Stanley [20], and several others.

Many of the problems which have been considered already belong to the class in which the recognisable characteristics are those which are expressible in terms of adjacent pairs of elements. Typically, this class has been treated by a collection of special methods. However, the class may in fact be treated more generally as follows. Let $\Pi = (\pi_1, \pi_2)$ be an arbitrary bipartition of $\mathcal{N}_n^2$, where $\mathcal{N}_n = \{1,..., n\}$, and let $\sigma = \sigma_1 \cdots \sigma_l \in \mathcal{N}_n^+$ be called a *π_1-path* if $(\sigma_i, \sigma_{i+1}) \in \pi_1$ for $i = 1,..., l-1$. Clearly, π_1-paths are recognisable in terms of pairs of adjacent elements in a sequence. The following is a general theorem for enumerating sequences with respect to maximal π_1-paths.

THEOREM 1.1. *Let* $F(x) = 1 + f_1x + f_2x^2 + \cdots$ *and* $G(x) = g_1x + g_2x^2 + \cdots$ *be generating functions in which f_i and g_i, for $i \geqslant 1$, are indeterminates marking nonterminal and terminal paths of length i, respectively. Let* $\boldsymbol{\gamma} = (\gamma_0, \gamma_1,...)$, *where γ_k is the generating function for π_1-paths of length $k \geqslant 0$, where $\gamma_0 = 1$. Then*

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(1) *the number of sequences in* $\mathscr{N}_n^+$ *with* i_j *occurrences of* j, *for* $j = 1,\dots, n$, *with* m_i *maximal non-terminal* π_1-*paths of length* i *for* $i \geqslant 1$, *and with a terminal maximal* π_1-*path of length* k *is* $[\mathbf{x}^{\mathbf{i}}\mathbf{f}^{\mathbf{m}}g_k](GF^{-1} \circ \gamma)(F^{-1} \circ \gamma)^{-1}$, *where* $\mathbf{i} = (i_1,\dots, i_n)$ *and* $\mathbf{m} = (m_1,\dots)$.

(2) *Moreover, when* $\Pi = (<, \geqslant)$, *the number of permutations on* $\mathscr{N}_n$ *with* j *inversions, with* m_i *maximal non-terminal increasing paths of length* i *for* $i \geqslant 1$, *and with a terminal maximal increasing path of length* k *is* $[\mathbf{f}^{\mathbf{m}}g_k q^j(x^n/n!_q)]\ \Phi(1, x/1!_q, x^2/2!_q,\dots)$, *where* $\Phi(\gamma) = (GF^{-1} \circ \gamma)(F^{-1} \circ \gamma)^{-1}$ *and* $k!_q = \prod_{i=1}^k (1 - q^i)$.

The first part of this result was given by Jackson and Aleliunas [11], and the second follows from Gessel's inversion homomorphism [9]. The following is an example of the use of this theorem.

EXAMPLE 1.2. The number of permutations on $\mathscr{N}_n$ with k inversions and no increasing paths of length p is

$$\left[q^k \frac{x^n}{n!_q}\right]\left\{\sum_{j=0}^{\infty}\left(\frac{x^{jp}}{(jp)!_q} - \frac{x^{jp+1}}{(jp+1)!_q}\right)\right\}^{-1}.$$

Proof. Since the terminal maximal path is not distinguished we set $G(x) = F(x) - 1$. Accordingly, from Theorem 1.1, we have $1 + \Phi = (F^{-1} \circ \gamma)^{-1}$, where Φ is the required generating function. Since paths of length greater than $p-1$ do not occur then $F(x) = 1 + x + x^2 + \cdots + x^{p-1}$ so $F^{-1} \circ \gamma = \{(1-x)(1-x^p)^{-1}\} \circ \gamma = \sum_{j=0}^{\infty} (\gamma_{jp} - \gamma_{jp+1})$. But for increasing paths $\pi_1 = <$, and the result follows from Theorem 1.1.2. ∎

The purpose of this paper is to consider generalisations in which restrictions may be placed on elements which may or may not be adjacent. An extreme example, but one which will not be considered further here, is the enumeration of plane partitions with given shape. The configurations which correspond to permutations in this case are Young's tableaux. Both plane partitions and Young's tableaux have been considered elsewhere (see, for example, Stanley [18, 19]). They may be treated by the techniques considered here. However, this is beyond the scope of the present paper.

Less extreme cases, however, may be treated. Although the problems may appear to be artificial, they are of interest enumeratively because they capture, in a concise way, the combinatorial characteristic which defeats the classical methods of enumeration except in special cases. A particular instance in the class we shall consider is the enumeration of the set of permutations $\sigma = \sigma_1 \cdots \sigma_{4K+1}$ such that

(i) $\sigma_{4k+1} \geqslant \sigma_{4k+2} < \sigma_{4k+3} \geqslant \sigma_{4k+4} < \sigma_{4k+5}$,

(ii) $\sigma_{4k+1} < \sigma_{4k+3} \geqslant \sigma_{4k+5}$,

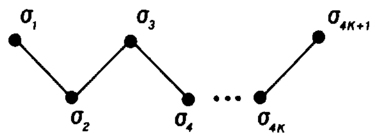

FIG. 1. The graphical representation of the pattern for permutations satisfying condition (i).

where $0 \leqslant k < K$. We adopt the convention that the edge with label $<$ between σ_i and σ_j is represented by

while an edge with label $\geqslant$ between σ_i and σ_j is represented by

with the understanding that the edges are directed from left to right. If σ is such that $(\sigma_i, \sigma_{i+1}) \in \pi_{k_i}$ for $i = 1, \ldots, K$, where $\pi_{k_1} \cdots \pi_{k_K} \in \Pi^*$, then we say that σ has *pattern* $\pi_{k_1} \cdots \pi_{k_K}$. Permutations which satisfy (i) alone have a pattern which may be represented graphically. This is given in Fig. 1.

These permutations are contained in the set of alternating permutations (André [1]) and we note that they may be enumerated by Theorem 1.1 with $G(x) = F(x) - 1$ and $F(x) = 1 + x^2$.

Permutations which satisfy conditions (i) and (ii) have a pattern which may be represented graphically. This is done in Fig. 2.

We say that the pattern given in Fig. 2 is obtained by the operation of *triangling* (denoted by ∇) the edges of the pattern given in Fig. 1. The patterns obtained in the closure of Π^* with respect to ∇ are called *T-graphs.* In this paper we consider the enumeration of sets of sequences whose patterns belong to an arbitrary prescribed set of T-graphs. Clearly the enumeration of plane partitions is excluded. This may be seen from Fig. 3, giving the pattern for a plane partition of shape $(4, 3^2, 2)$.

There is no pattern in Π^* from which the pattern in Fig. 3 is derivable by triangling, and consequently the pattern for plane partitions of shape $(4, 3^2, 2)$ is not a T-graph. Baxter sequences, treated by Chung, *et al.* [4] and by Mallows [13], are similarly excluded.

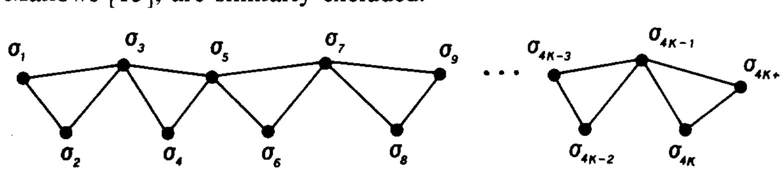

FIG. 2. The graphical representation of the pattern for permutations satisfying conditions (i) and (ii).

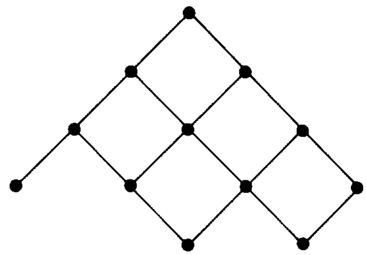

FIG. 3. The pattern for a plane partition of shape $(4, 3^2, 2)$.

Section 2 contains the main enumerative theorems. A certain power series in non-commutative indeterminates may be obtained as the solution to a system of linear equations. This series may be transformed to obtain the ordinary generating function for the enumeration of sequences whose patterns belong to a prescribed set of T-graphs, or to obtain the Eulerian generating function for the enumeration, with respect to inversions, of permutations whose patterns belong to a prescribed set of T-graphs. The system is given in Theorem 2.11. In Section 3 we demonstrate the use of the material of Section 2 in detail, and apply it to a number of non-trivial enumeration problems. Finally, in Section 4, we note that the generating function for the enumeration of permutations whose patterns belong to a prescribed set of T-graphs may be expressed as the solution to a system of matrix Riccati equations.

The following notational apparatus is used throughout. If $\mathbf{i} = (i_1, ..., i_n)$ and $\mathbf{x} = (x_1, ..., x_n)$ then $\mathbf{x}^{\mathbf{i}}$ denotes $x_1^{i_1} \cdots x_n^{i_n}$. Moreover, if $f(\mathbf{x})$ is a power series in $\mathbf{x}$ then $[\mathbf{x}^{\mathbf{i}}]f(\mathbf{x})$ denotes the coefficient of $\mathbf{x}^{\mathbf{i}}$ in $f(\mathbf{x})$. If $g(x) = g_0 + g_1 x + g_2 x + \cdots$, where x is an indeterminate, and if $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, ...)$ then $g \circ \boldsymbol{\gamma} = g_0 \gamma_0 + g_1 \gamma_1 + \cdots$, the umbral composition of g and $\boldsymbol{\gamma}$. If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{k}$ is a column vector with n components then $[\mathbf{A} : \mathbf{k}]_p$, $1 \leqslant p \leqslant n$, denotes the matrix obtained from $\mathbf{A}$ by replacing column p by $\mathbf{k}$. The $n \times n$ identity matrix is denoted by $\mathbf{e}$. A number of matrices have rows and columns indexed from zero, instead of one. Attention is not drawn further to this distinction since it is clear from the particular context. $\mathbf{J}$ is the $n \times n$ matrix, each of whose elements is equal to one, $\mathbf{X} = \operatorname{diag}(x_1, ..., x_n)$ and $\boldsymbol{\omega}$ denotes the matrix $\mathbf{XJ}$. Finally, $[\mathbf{A}]_{ij}$ denotes the (i, j)-element of A, and ε denotes the empty sequence.

2. THE MAIN THEOREMS

If α is a directed path with edge labels in $\Pi = (\pi_1, \pi_2)$ which, when listed serially from origin to terminus are $\pi_{k_1}, ..., \pi_{k_m}$, then α is denoted by

$\pi_{k_1} \cdots \pi_{k_m}$. If $\overrightarrow{uv}$ is an isolated directed edge with origin u, terminus v and edge label $\pi \in \Pi$, then ∇ denotes the operation of connecting u to v by a directed path $\pi_2^r \pi_1^s$, where $r, s \geqslant 1$. We call the resulting graph the T-graph obtained by *triangling* π with $\pi_2^r \pi_1^s$. The closure of α under ∇ is denoted by α^∇. This is the set of all T-graphs on α, and the latter is called the *spine* of the T-graphs in this set. Let $\mathscr{T}$ denote the set, $(\Pi^*)^\nabla$, of all T-graphs, where Π^* is the set of all directed paths with edge labels in Π.

If $A \in \mathscr{T}$, then A has a unique directed Hamiltonian path, and suppose now that the m vertices of A are labeled so that this path is $\overrightarrow{v_1 \cdots v_m}$. Let $i_1, \ldots, i_m \in \mathscr{N}_n$ and suppose that i_j is assigned to v_j, for $1 \leqslant j \leqslant m$, in such a way that if $\overrightarrow{v_k v_{k'}}$ is in the edge set of A and has label π then $(i_k, i_{k'}) \in \pi$. Then $\sigma = i_1 \cdots i_m \in \mathscr{N}_n^+$ is called a sequence with *length* $|\sigma| = m$ over $\mathscr{N}_n$ with *pattern* A. The length, $|A|$, of A is m.

For convenience in representing the elements of $\mathscr{T}$ graphically, we represent a directed edge with label π_1 by an edge drawn from bottom left to top right, and a directed edge with label π_2 by an edge drawn from top left to bottom right, with the convention that the edges so drawn are directed from left to right. Figure 4 gives a T-graph with spine $\pi_1^2 \pi_2^2$, of length 19, represented by this graphical convention.

The unique Hamiltonian path in this pattern is $\overrightarrow{v_1 \cdots v_{19}}$. If $\Pi = (<, \geqslant)$ then the sequence 4 1 12 9 5 2 6 10 19 18 15 7 3 8 13 11 14 16 17 has the pattern whose T-graph is given in Fig. 4. For example, $(i_3, i_9) = (12, 19) \in \pi_1$ since $12 < 19$, and $\overrightarrow{v_3 v_9}$ has label π_1 according to the above convention. Further use of this T-graph is made in Section 3.

DEFINITION 2.1. (1) Let $\sigma = \sigma_1 \cdots \sigma_m \in \mathscr{N}_n^+$. An *inversion* in σ is a pair (i,j) with $1 \leqslant i < j \leqslant m$ such that $\sigma_i > \sigma_j$. The number of inversions in σ is denoted by $I(\sigma)$.

(2) Let $\alpha \subseteq \mathscr{N}_n$, $\beta = \mathscr{N}_n - \alpha$ and let $\sigma = \sigma' \sigma'' \in \mathscr{S}_n$ (the symmetric group on n symbols) be such that σ' is a permutation on α and σ'' is a

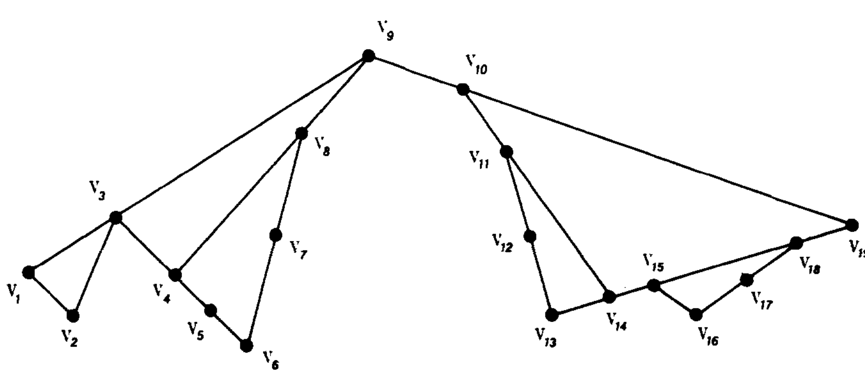

FIG. 4. A T-graph with spine $\pi_1^2 \pi_2^2$.

permutation on β. Then $I(\alpha,\beta)$ denotes the number of inversions (i,j) in σ with $i\in\alpha$ and $j\in\beta$, for any such α.

We note that $I(\alpha,\beta)$ is constant for all $\sigma=\sigma'\sigma''$ such that σ' is a permutation on α and σ'' is a permutation on β. Accordingly, $I(\alpha,\beta)$ is well-defined.

The following proposition is well-known, but its proof is included since the methods employed are used later in Proposition 2.7. The first part is attributed to Rodrigues [15], while the second may be found in Gessel [9].

PROPOSITION 2.2. *Let $p_k(n)$ be the number of permutations on $\mathscr{N}_n$ with k inversions. Then*

$$(1)\quad \sum_{k\geqslant 0} p_k(n)\,q^k=(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}),$$

where q is an indeterminate.

$$(2)\quad \sum_{\substack{\alpha\subset\mathscr{N}_n\\|\alpha|=m}} q^{I(\alpha,\beta)}=\binom{n}{m}_q,\qquad \textit{where}\quad \binom{n}{m}_q=n!_q(m!_q(n-m)!_q)^{-1}$$

(*the q-binomial, or Gaussian, coefficient*).

Proof. (1) Let $f_n(q)=\Sigma_{\sigma\in\mathscr{S}_n} q^{I(\sigma)}$. We obtain a recurrence equation for $f_n(q)$ as follows. Each element of σ_{n+1} may be constructed uniquely from an element σ of $\mathscr{S}_n$ by inserting the number $n+1$ into any one of the $n+1$ gaps preceding or following elements in σ. We index these gaps sequentially from right to left, beginning from zero. The insertion of the number $n+1$ into gap i in any $\sigma\in\mathscr{S}_n$ contributes an additional (disjoint) set of i inversions to the set of inversions of σ. These contributions are enumerated by $1+q+\cdots+q^n$ independently of σ. Thus $f_{n+1}(q)=(1+q+\cdots+q^n)f_n(q)$ and $f_1(q)=1$. The result follows.

(2) Let $\alpha\subseteq\mathscr{N}_n$, $\beta=\mathscr{N}_n-\alpha$ and $\sigma=\sigma'\sigma''\in\mathscr{S}_n$, where σ' is a permutation on α and σ'' is a permutation on β. Then clearly $I(\sigma)=I(\sigma')+I(\sigma'')+I(\alpha,\beta)$, because the three sets of inversions are disjoint. Let $\mathscr{P}(\alpha)$ denote the set of permutations on α. Accordingly

$$\sum_{\sigma\in\mathscr{S}_n} q^{I(\sigma)}=\sum_{\substack{\alpha\subseteq\mathscr{N}_n\\|\alpha|=m}} q^{I(\alpha,\beta)}\sum_{\substack{\sigma'\in\mathscr{P}(\alpha)\\\sigma''\in\mathscr{P}(\beta)}} q^{I(\sigma')+I(\sigma'')}$$

$$=\sum_{\substack{\alpha\subseteq\mathscr{N}_n\\|\alpha|=m}} q^{I(\alpha,\beta)}\sum_{\substack{\sigma'\in\mathscr{S}_m\\\sigma''\in\mathscr{S}_{n-m}}} q^{I(\sigma')+I(\sigma'')}$$

since $I(\sigma')$ and $I(\sigma'')$ are invariant under monotonic functions of the elements of α and β, respectively. Thus from (1) we have

$$n!_q = (n-m)!_q m!_q \sum_{\substack{\alpha \subseteq \mathcal{N}_n \\ |\alpha| = m}} q^{I(\alpha,\beta)}$$

and the result follows. ∎

We now consider the generating function for sequences with prescribed patterns. Let W denote the graph consisting of a directed edge with label $\pi_1 \cup \pi_2$. If $A, B \in \mathscr{E} \cup \{W\}$ then AB denotes the configuration obtained by identifying the right-most vertex of A with the left-most vertex of B. Let $\tilde{\mathscr{E}} = (\mathscr{E} \cup \{W\})^*$. If $\mathscr{A} \subseteq \tilde{\mathscr{E}}$ then $\langle A \rangle$ denotes the set of all sequences in $\mathcal{N}_n^*$ with pattern in $\mathscr{A}$. If $\sigma \in \mathcal{N}_n^+$ and $\sigma = \sigma_1 \cdots \sigma_k$ then $\rho(\sigma)$ denotes $x_{\sigma_1} \cdots x_{\sigma_k}$, where $x_1, \ldots, x_n$ are non-commutative indeterminates.

DEFINITION 2.3. Let $A \in \tilde{\mathscr{E}}$. The *incidence matrix*, $\mathscr{I}(A)$, of A is the $n \times n$ matrix such that $[\mathscr{I}(A)]_{ij} = \sum_{\sigma \in \langle A \rangle} x_{\sigma_1} \cdots x_{\sigma_{l-1}}$, where $|A| = l$.

We note that there is a $[1:1]$ correspondence between incidence matrices and the associated patterns, and accordingly we write $A = \mathscr{I}^{-1}(\mathbf{a})$ when $\mathbf{a} = \mathscr{I}(A)$.

PROPOSITION 2.4. *If $A, B \in \tilde{\mathscr{E}}$ then $\mathscr{I}(AB) = \mathscr{I}(A)\,\mathscr{I}(B)$.*

Proof. If $A, B \in \tilde{\mathscr{E}}$ then $AB \in \tilde{\mathscr{E}}$ so AB has a unique directed Hamiltonian path. The result then follows from Definition 2.3.

The following power series is central to the subsequent development and may be specialised to yield the generating functions for the enumeration of sequences and permutations whose patterns belong to a prescribed set of T-graphs.

DEFINITION 2.5. (1) $y_1, \ldots, y_t$ are indeterminates and

$$\mathscr{V} = \left\{ \sum_{i > 0} m_i \mathbf{a}_i \mid \mathscr{I}^{-1}(\mathbf{a}_i) \in \tilde{\mathscr{E}}, m_i \in Q[\mathbf{y}] \right\}.$$

(2) If $\mathbf{u} = \sum_{i>0} m_i \mathbf{a}_i \in \mathscr{V}$ then $\Psi(\mathbf{u}) = \sum_{i>0} m_i \sum_{\sigma \in \langle \mathscr{I}^{-1}(\mathbf{a}_i) \rangle} \rho(\sigma)$.

The following proposition gives an important property of Ψ.

PROPOSITION 2.6. *Let* $\mathbf{u}$, $\mathbf{v} \in \mathscr{V}$. *Then* $\Psi(\mathbf{u}\boldsymbol{\omega}\mathbf{v}) = \Psi(\mathbf{u})\,\Psi(\mathbf{v})$, *where* $\boldsymbol{\omega} = \mathscr{I}(W)$.

Proof. Clearly $\Psi(\mathbf{u}) = \operatorname{tr} \mathbf{u}\boldsymbol{\omega}$. But $\operatorname{tr} \mathbf{a}_i \boldsymbol{\omega} \mathbf{a}_j \boldsymbol{\omega} = (\operatorname{tr} \mathbf{a}_i \boldsymbol{\omega})(\operatorname{tr} \mathbf{a}_j \boldsymbol{\omega})$ and the result follows. ∎

Two further generating functions are needed. These are concerned with the enumeration of sequences and the enumeration of permutations.

PROPOSITION 2.7. *If* $\mathbf{u} \in \mathscr{V}$ *and* $\mathbf{u} = \sum_{i\geqslant 0} m_i \mathbf{a}_i$ *let* $\Psi_1(\mathbf{u}) = \sum_{i\geqslant 0} m_i \sum_{\sigma\in\langle\mathscr{T}^{-1}(\mathbf{a}_i)\rangle} \mathbf{x}^{\tau(\sigma)}$ *and*

$$\Psi_2(\mathbf{u}) = \sum_{i\geqslant 0} m_i \frac{x^{|\mathscr{T}^{-1}(\mathbf{a}_i)|}}{|\mathscr{T}^{-1}(\mathbf{a}_i)|!_q} \sum_{\sigma\in\langle\langle\mathscr{T}^{-1}(\mathbf{a}_i)\rangle\rangle} q^{I(\sigma)},$$

where $\langle\langle A\rangle\rangle = \langle A\rangle \cap \mathscr{S}_{|A|}$, *the set of permutations with pattern* $A \in \tilde{\mathscr{E}}$, *and* $\tau(\sigma) = (i_1,..., i_n)$, *the type of* σ, *where* i_j *is the number of occurrences of* j *in* σ *for* $1 \leqslant j \leqslant n$. *Let* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathscr{V}$ *and* $\Pi = (<, \geqslant)$. *Then the following holds for* $i = 1, 2$:

(1) *If* $\Psi(\mathbf{u}_1) + \Psi(\mathbf{u}_2) = \Psi(\mathbf{u}_3)$ *then* $\Psi_i(\mathbf{u}_1) + \Psi_i(\mathbf{u}_2) = \Psi_i(\mathbf{u}_3)$.

(2) *If* $\Psi(\mathbf{u}_1)\, \Psi(\mathbf{u}_2) = \Psi(\mathbf{u}_3)$ *then* $\Psi_i(\mathbf{u}_1)\, \Psi_i(\mathbf{u}_2) = \Psi_i(\mathbf{u}_3)$.

Proof. Let $\mathbf{u}_j = \sum_{i\geqslant 0} m_i^{(j)} \mathbf{a}_i$ for $j = 1, 2, 3$ and let $\mathscr{T}^{-1}(\mathbf{a}_i) = A_i$ for $i \geqslant 0$.

(1) $[i = 2]$. Now $\Psi(\mathbf{u}_1) + \Psi(\mathbf{u}_2) = \Psi(\mathbf{u}_3)$ so retaining only the contributions of permutations we have

$$\sum_{i\geqslant 0} m_i^{(1)} \sum_{\sigma\in\langle\langle A_i\rangle\rangle} \rho(\sigma) + \sum_{i\geqslant 0} m_i^{(2)} \sum_{\sigma\in\langle\langle A_i\rangle\rangle} \rho(\sigma) = \sum_{i\geqslant 0} m_i^{(3)} \sum_{\sigma\in\langle\langle A_i\rangle\rangle} \rho(\sigma),$$

whence

$$\sum_{i\geqslant 0} m_i^{(1)} \sum_{\sigma\in\langle\langle A_i\rangle\rangle} \frac{q^{I(\sigma)}}{|A_i|!_q} + \sum_{i\geqslant 0} m_i^{(2)} \sum_{\sigma\in\langle\langle A_i\rangle\rangle} \frac{q^{I(\sigma)}}{|A_i|!_q}$$
$$= \sum_{i\geqslant 0} m_i^{(3)} \sum_{\sigma\in\langle\langle A_i\rangle\rangle} \frac{q^{I(\sigma)}}{|A_i|!_q}$$

and the result follows. The case $i = 1$ is treated similarly.

(2) $[i = 2]$. Now $\Psi(u_1)\, \Psi(u_2) = \Psi(u_3)$ so retaining the contribution of permutations we have

$$\sum_{i\geqslant 0} m_i^{(3)} \sum_{\sigma\in\langle A_i\rangle} \rho(\sigma) = \sum_{i,j\geqslant 0} m_i^{(1)} m_j^{(2)} \sum_{\sigma\in\langle A_i W A_j\rangle} \rho(\sigma)$$

by Propositions 2.6 and 2.4, whence

$$\Psi_2(\mathbf{u}_3) = \sum_{i\geqslant 0} m_i^{(3)} \sum_{\sigma\in\langle\langle A_i\rangle\rangle} \frac{q^{I(\sigma)}}{|A_i|!_q}$$
$$= \sum_{i,j\geqslant 0} m_i^{(1)} m_j^{(2)} \sum_{\sigma\in\langle\langle A_i W A_j\rangle\rangle} \frac{q^{I(\sigma)}}{|A_i W A_j|!_q}.$$

We now construct each element of $\langle\!\langle A_i W A_j \rangle\!\rangle$ by concatenating an element of $\mathscr{P}(\alpha)$ of shape A_i with an element of $\mathscr{P}(\beta)$ of shape A_j, for all choices of $\alpha \subseteq \mathscr{N}_{|A_i W A_j|}$ and $\beta = \mathscr{N}_{|A_i W A_j|} - \alpha$, where $|\alpha| = |A_i|$ and, of course, $|\beta| = |A_j|$.

Accordingly

$$\sum_{\sigma \in \langle\!\langle A_i W A_j \rangle\!\rangle} \frac{q^{I(\sigma)}}{|A_i W A_j|!_q} = \sum_{\substack{\alpha \subseteq \mathscr{N}_{|A_i W A_j|} \\ |\alpha| = |A_i|}} \sum_{\substack{\sigma' \in \mathscr{P}(\alpha) \cap \langle A_i \rangle \\ \sigma'' \in \mathscr{P}(\beta) \cap \langle A_j \rangle}} \frac{q^{I(\sigma') + I(\sigma'') + I(\alpha, \beta)}}{|A_i W A_j|!_q}$$

But there exists a bijective map $o_\alpha : \mathscr{P}(\alpha) \to \mathscr{S}_{|A_i|} : (a_{j_1}, \ldots, a_{j_{|A_i|}}) = (\iota_\alpha(j_1), \ldots, \iota_\alpha(j_{|A_i|}))$, where $\alpha = (\alpha_1, \ldots, \alpha_{|A_i|})$ with $\alpha_1 < \cdots < \alpha_{|A_i|}$ and $\iota_\alpha(\alpha_k) = k$ for $k = 1, \ldots, |A_i|$. Moreover, o_α preserves the number of inversions since $\Pi = (<, \geqslant)$. Thus

$$\sum_{\sigma \in \langle\!\langle A_i W A_j \rangle\!\rangle} \frac{q^{I(\sigma)}}{|A_i W A_j|!_q} = \sum_{\substack{\alpha \subseteq \mathscr{N}_{|A_i W A_j|} \\ |\alpha| = |A_i|}} \frac{q^{I(\alpha, \beta)}}{|A_i W A_j|!_q} \sum_{\substack{\sigma' \in \mathscr{S}_{|A_i|} \cap \langle A_i \rangle \\ \sigma'' \in \mathscr{S}_{|A_j|} \cap \langle A_j \rangle}} q^{I(\sigma') + I(\sigma'')}$$

so by Proposition 2.2.2 we have

$$\sum_{\sigma \in \langle\!\langle A_i W A_j \rangle\!\rangle} \frac{q^{I(\sigma)}}{|A_i W A_j|!_q} = \sum_{\sigma' \in \langle\!\langle A_i \rangle\!\rangle} \frac{q^{I(\sigma')}}{|A_i|!_q} \sum_{\sigma'' \in \langle\!\langle A_j \rangle\!\rangle} \frac{q^{I(\sigma'')}}{|A_j|!_q},$$

whence

$$\Psi_2(\mathbf{u}_3) = \sum_{i, j \geqslant 0} m_i^{(1)} m_j^{(2)} \sum_{\sigma' \in \langle\!\langle A_i \rangle\!\rangle} \frac{q^{I(\sigma')}}{|A_i|!_q} \sum_{\sigma'' \in \langle\!\langle A_j \rangle\!\rangle} \frac{q^{I(\sigma'')}}{|A_j|!_q}$$

and the result follows. The case $i = 1$ follows similarly. ∎

We now obtain the required generating function for sequences and permutations as solutions to systems of linear equations. This is done by representing the T-graphs equationally. It will be seen that certain T-graphs, namely, the *headed* ones, may be treated indirectly and accordingly the equational representation of the T-graphs is used to decompose the T-graphs into headed T-graphs. Certain preliminary results are needed.

If $A \in \pi^\nabla$, where $\pi \in \Pi$ then A is called *elementary*. In particular, if A is the T-graph obtained by triangling π with $\pi_2^r \pi_1^s$ and if $B \in A^\nabla$ then B is called an elementary T-graph with *spine-type* (r, s). We call $(\pi_1^+)^\nabla$ and $(\pi_2^+)^\nabla$ the sets of *headed* and *non-headed* T-graphs. If $A \subseteq \tilde{\mathscr{E}}$ has the property that $|A| = |B|$ for all $A, B \in \mathscr{A}$ then $\mathscr{A}$ is said to be *homogeneous*. Similarly if $u \in \mathscr{V}$ is a linear combination of incidence matrices of headed T-graphs, then u is called *headed*. Moreover, u is called *homogeneous* if it is a linear combination of incidence matrices of T-graphs of the same length.

PROPOSITION 2.8. *Let $A \in (\pi_2)^\nabla$ and let $\nu(A)$ denote the element of $(\pi_1)^\nabla$ obtained from A by replacing the label on the spine of A by π_1. Let A have spine-type (r, s). Then there exist unique $B_1, \ldots, B_r \in (\pi_2)^\nabla$ and unique $C_1, \ldots, C_s \in (\pi_1)^\nabla$ such that $\mathscr{I}(A) + \mathscr{I}(\nu(A)) = \mathscr{I}(B_1) \cdots \mathscr{I}(B_r)\mathscr{I}(C_1) \cdots \mathscr{I}(C_s)$.*

Proof. $\langle A \rangle \cap \langle \nu(A) \rangle = \varnothing$ and $\langle A \rangle \cup \langle \nu(A) \rangle = \langle D \rangle$, where $D \in (\pi_2^r \pi_1^s)^\nabla$, since A has spine-type (r, s). Clearly, D may be written uniquely in the form $B_1 \cdots B_r C_1 \cdots C_s$, where $B_1, \ldots, B_r \in (\pi_2)^\nabla$ and $C_1, \ldots, C_s \in (\pi_1)^\nabla$. Thus, from Definition 2.3, we have $\mathscr{I}(A) + \mathscr{I}(\nu(A)) = \mathscr{I}(B_1 \cdots B_r C_1 \cdots C_s)$ and the result follows from Proposition 2.4. ∎

If $A \in (\pi_2)^\nabla$ and $B = \nu(A)$ then $\nu^{-1}(B)$ denotes A.

The following result gives the equational representation of the T-graphs.

COROLLARY 2.9. *Let $A \in \mathscr{E}$. Then*

$$\begin{aligned} &\mathscr{I}(A) = \boldsymbol{\omega}_0, && \\ &\mathbf{a}_i + \mathbf{b}_i = \boldsymbol{\omega}_i && \text{for } 1 \leqslant i \leqslant k, \\ &\mathbf{a} + \mathbf{b} = \boldsymbol{\omega}, && \text{where } \mathbf{a} = \mathscr{I}(\pi_1) \text{ and } \mathbf{b} = \mathscr{I}(\pi_2), \end{aligned}$$

where

(i) *$\mathbf{a}_i = \mathscr{I}(\nu(\mathscr{I}^{-1}(\mathbf{b}_i)))$ and $\mathbf{b}_i$ are incidence matrices for headed and non-headed elementary T-graphs, and are mutually distinct.*

(ii) *$\boldsymbol{\omega}_i$ is a product of elements from $\{\mathbf{a}, \mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_k\}$.*

(iii) *For each i, $1 \leqslant i \leqslant k$, either $\mathbf{a}_i$ or $\mathbf{b}_i$ occurs in at least one $\boldsymbol{\omega}_j$, where $j \neq i$, but not in $\boldsymbol{\omega}_i$.*

(iv) *The system of equations is unique, up to reordering of equations and relabeling of matrices.*

Proof. By repeated application of Proposition 2.8. ∎

The system of equations in Corollary 2.9 is called *the incidence system* for A.

LEMMA 2.10. *Let $H \in \mathscr{E}$ and let $\mathscr{I}(H) = \mathbf{h}$. Then $\mathbf{h}$ admits an expansion of the form $\mathbf{h} = \mathbf{p} - \sum_{j=1}^{s} \boldsymbol{l}_j \boldsymbol{\omega} \mathbf{r}_j$ for some $s \geqslant 0$ (empty sums are zero), where*

(1) *$\mathbf{p}, \boldsymbol{l}_1, \ldots, \boldsymbol{l}_s$ are headed and homogeneous (for a left-expansion of $\mathbf{h}$),*

(2) *$\mathbf{p}, \mathbf{r}_1, \ldots, \mathbf{r}_s$ are headed and homogeneous (for a right-expansion of $\mathbf{h}$).*

Proof. From Corollary 2.9, H has an incidence system. Suppose $\mathscr{I}(H) = \boldsymbol{\omega}_0$. This incidence system may be used to eliminate the incidence matrix of the left-most (resp. right-most) non-headed elementary T-graph in $\boldsymbol{\omega}_0$. This process of elimination may be applied successively to each term generated in this fashion until the left-most (resp. right-most) expansion is obtained. ∎

The following theorem gives the systems of linear equations for Ψ_1 and Ψ_2 defined in Proposition 2.7.

THEOREM 2.11 (The linear system). *Let H, C, $D \in \mathscr{E}$ and let $\mathbf{h} = \mathscr{I}(H)$, $\mathbf{c} = \mathscr{I}(C)$, $\mathbf{d} = \mathscr{I}(D)$, $\mathbf{r}_0 = \mathbf{c}$ and $\boldsymbol{l}_0 = \mathbf{0}$.*

(1) *Let $\mathbf{h} = \mathbf{p} - \sum_{j=1}^{t} \boldsymbol{l}_j \boldsymbol{\omega} \mathbf{r}_j$ be an expansion (left or right) of $\boldsymbol{h}$. Then for $s = 1, 2$ we have*

$$\Psi_s(\mathbf{c}(\mathbf{e}-\mathbf{k})^{-1}\mathbf{d}) = |[\mathbf{M}^{(s)} : \mathbf{k}^{(s)}]_0| \cdot |\mathbf{M}^{(s)}|^{-1},$$

where $\mathbf{M}^{(s)}$ is $(t+1) \times (t+1)$, $\mathbf{k}^{(s)} = (\mathbf{k}_0^{(s)}, \dots, \mathbf{k}_t^{(s)})^T$, $k_j^{(s)} = \Psi_s(\mathbf{r}_j(\mathbf{e}-\mathbf{p})^{-1}\mathbf{d})$ and

$$\begin{aligned}
[\mathbf{M}^{(s)}]_{ij} &= \delta_{ij} && \text{for } 0 \leqslant i \leqslant t,\ j = 0 \\
&= \delta_{ij} + \Psi_s(\mathbf{r}_i(\mathbf{e}-\mathbf{p})^{-1}\boldsymbol{l}_j) && \text{for } 0 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant t.
\end{aligned}$$

(2) *Moreover $[\mathbf{x}^{\mathbf{m}}]\, \Psi_1(\mathbf{c}(\mathbf{e}-\mathbf{h})^{-1}\mathbf{d})$ is the number of sequences in $\mathscr{N}_n^*$ with type $\mathbf{m}$ and pattern in CH^*D, and $[q^k(x^n/n!_q)]\, \Psi_2(\mathbf{c}(\mathbf{e}-\mathbf{h})^{-1}\mathbf{d})$ is the number of permutations over $\mathscr{N}_n$ with k inversions and pattern in CH^*D.*

Proof. (1) Now $H \in \mathscr{E}$ so, from Lemma 2.10, $\mathbf{h}$ has an expansion (left or right) of the form $\mathbf{h} = \mathbf{p} - \sum_{j=1}^{t} \boldsymbol{l}_j \boldsymbol{\omega} \mathbf{r}_j$. But $\mathbf{r}_i(\mathbf{e}-\mathbf{p})^{-1}\mathbf{d} = \mathbf{r}_i(\mathbf{e}-\mathbf{p})^{-1}(\mathbf{e}-\mathbf{h})(\mathbf{e}-\mathbf{h})^{-1}\mathbf{d}$ so $\mathbf{r}_i(\mathbf{e}-\mathbf{p})^{-1}\mathbf{d} = \mathbf{r}_i(\mathbf{e}-\mathbf{h})^{-1}\mathbf{d} + \sum_{j=0}^{t} \mathbf{r}_i(\mathbf{e}-\mathbf{p})^{-1}\boldsymbol{l}_j\boldsymbol{\omega}\mathbf{r}_j(\mathbf{e}-\mathbf{h})^{-1}\mathbf{d}$ for $0 \leqslant i \leqslant t$. Thus, from Proposition 2.6 we have

$$\Psi(\boldsymbol{r}_i(\mathbf{e}-\mathbf{p})^{-1}\mathbf{d}) = \sum_{j=0}^{t} \{\delta_{ij} + \Psi(\mathbf{r}_i(\mathbf{e}-\mathbf{p})^{-1}\boldsymbol{l}_j)\}\, \Psi(\mathbf{r}_j(\mathbf{e}-\mathbf{h})^{-1}\mathbf{d}).$$

Let $\boldsymbol{\xi}^{(s)} = (\xi_0^{(s)}, \dots, \xi_t^{(s)})^T$, where $\xi_j^{(s)} = \Psi_s(\mathbf{r}_j(\mathbf{e}-\mathbf{h})^{-1}\mathbf{d})$. Thus from Proposition 2.7 we have $\mathbf{M}^{(s)}\boldsymbol{\xi}^{(s)} = \mathbf{k}^{(s)}$. But $\xi_0^{(s)} = \Psi_s(\mathbf{c}(\mathbf{e}-\mathbf{h})^{-1}\mathbf{d})$ and the result follows by Cramer's rule.

(2) Immediate. ∎

The power series in $\mathbf{M}^{(s)}$ and $\mathbf{k}^{(s)}$ involve only headed T-graphs or T-graphs of bounded length which may or may not be headed. Both cases may be treated and, accordingly, Theorem 2.11 may be regarded as a means for transforming the generating function for a T-graph problem into an expression whose constituents are obtainable by other means. It will be shown in Theorem 2.18 that another system of linear equations may be obtained for headed T-graphs and for T-graphs of bounded length. The system is constructed combinatorially by considering the largest element in a permutation. The same device may be used to demonstrate that the generating function for permutations, whose patterns belong to a prescribed set of T-graphs, satisfies a matrix Riccati equation.

Attention is confined henceforth to the bipartition $(<, \geqslant)$ and to the enumeration of permutations. We now consider the properties of $\psi_{\mathbf{a}}^{(q)}(m)$, a combinatorial object, which is defined below. These properties are used to derive the system of linear equations given in Theorem 2.18.

DEFINITION 2.12. (1) If $\mathbf{a} = \mathscr{I}(A)$, where $A \in \mathscr{E}$, then $\psi_{\mathbf{a}}^{(q)}(m) = \sum_{\sigma \in \langle\langle \pi_1^m \mathscr{I}^{-1}(\mathbf{a}) \rangle\rangle} q^{I(\sigma)}$.

(2) If $\mathbf{u} = \sum_{i \geqslant 0} m_i \mathbf{a}_i$, $\mathscr{I}^{-1}(\mathbf{a}_i) \in \mathscr{E}$ for $i \geqslant 0$ and $m_i \in Q[\mathbf{y}]$ then $\psi_{\mathbf{u}}^{(q)}(m) = \sum_{i \geqslant 0} m_i \psi_{\mathbf{a}_i}^{(q)}(m)$.

(3) The *edge-length*, $l_e(A)$, of A is $|A| - 1$. The *vertex-length*, $l_v(A)$, of A is $|A|$. Moreover, we write $l_e(\mathbf{a}) = l_e(A)$. If $\mathbf{u} \in \sum_{i \geqslant 0} m_i \mathbf{a}_i \in \mathscr{T}$ is homogeneous then $l_e(\mathbf{u}) = l_e(\mathbf{a}_i)$.

The combinatorial interpretation of $\psi_{\mathbf{a}}^{(q)}(m)$ is given below. The notation for edge-length and vertex-length is introduced to simplify certain expressions.

PROPOSITIONS 2.13. *Let $\mathscr{A} = \{A_i\}$ be a homogeneous set of headed T-graphs in which each T-graph has edge-length p. Let $\mathscr{B} = \{B_i\}$ be a set of T-graphs. If $\mathbf{u} = \sum_{i \geqslant 0} m_i \mathscr{I}(A_i)$ and $\mathbf{v} = \sum_{i \geqslant 0} m_i' \mathscr{I}(B_i)$, where m_i, $m_i' \in Q[\mathbf{y}]$ for $i \geqslant 0$, then $\psi_{\mathbf{uv}}^{(q)}(m) = \psi_{\mathbf{u}}^{(q)}(m)\, \psi_{\mathbf{v}}^{(q)}(m+p)$.*

Proof. Now $\psi_{\mathbf{uv}}^{(q)}(m) = \sum_{i,j \geqslant 0} m_i m_j' \psi_{\mathbf{a}_i \mathbf{b}_j}^{(q)}(m)$, where $\mathbf{a}_i = \mathscr{I}(A_i)$, $\mathbf{b}_i = \mathscr{I}(B_i)$ for $i \geqslant 0$. But each element of $\langle\langle \pi_1^m A_i B_j \rangle\rangle$ may be constructed uniquely from $\langle\langle \pi_1^{m+p} B_j \rangle\rangle$ by replacing the single string in $\langle\langle \pi_1^{m+p} \rangle\rangle$ by an element of $\langle\langle \pi_1^m A_i \rangle\rangle$. Thus

$$\psi_{\mathbf{a}_i \mathbf{b}_j}^{(q)}(m) = \sum_{\sigma \in \langle\langle \pi_1^m A_i B_j \rangle\rangle} q^{I(\sigma)} = \sum_{\sigma_1 \in \langle\langle \pi_1^{m+p} B_j \rangle\rangle} \sum_{\sigma_2 \in \langle\langle \pi_1^m A_i \rangle\rangle} q^{I(\sigma)}.$$

However, $I(\sigma_1) = I(\sigma') + I(\pi_1^{m+p}, B_j)$, where σ_1 is formed by identifying the right-hand element of σ' with the left-hand element of σ'', and where $\sigma' \in \langle\langle \pi_1^{m+p} \rangle\rangle$ and $\sigma'' \in \langle\langle B_j \rangle\rangle$. Accordingly $I(\sigma_1) = I(\sigma'') + I(\pi_1^m A_i, B_j)$. Thus $I(\sigma_1) + I(\sigma_2) = I(\sigma'') + I(\sigma_2) + I(\pi_1^m A_i, B_j) = I(\sigma)$, where $\sigma \in \langle\langle \pi_1^m A_i B_j \rangle\rangle$. Thus

$$\psi_{\mathbf{a}_i \mathbf{b}_j}^{(q)}(m) = \sum_{\sigma_1 \in \langle\langle \pi_1^{m+p} B_j \rangle\rangle} q^{I(\sigma_1)} \sum_{\sigma_2 \in \langle\langle \pi_1^m A_i \rangle\rangle} q^{I(\sigma_2)} = \psi_{\mathbf{a}_i}^{(q)}(m)\, \psi_{\mathbf{b}_j}^{(q)}(m+p),$$

whence

$$\psi_{\mathbf{uv}}^{(q)}(m) = \sum_{i,j\geq 0} m_i m_j' \psi_{\mathbf{a}_i}^{(q)}(m)\, \psi_{\mathbf{b}_j}^{(q)}(m+p) = \psi_{\mathbf{u}}^{(q)}(m)\, \psi_{\mathbf{u}}^{(q)}(m+p)$$

and this completes the proof. ∎

PROPOSITION 2.14. *Let* $\mathbf{c}$, $\mathbf{p}$, $\mathbf{d} \in \mathscr{V}$ *be homogeneous and let* $\mathbf{c}$, $\mathbf{p}$ *be headed. Then* $\Psi_2(\mathbf{c}(\mathbf{e}-\mathbf{p})^{-1}\mathbf{d}) = \sum_{k\geq 0} \mathbf{c}_k(x^{\lambda_k}/\lambda_k!_q)$, *where* $\lambda_k = l_e(\mathbf{c}) + kl_e(\mathbf{p}) + l_e(\mathbf{d}) + 1$ *and*

$$c_k = \psi_{\mathbf{c}}^{(q)}(0) \left\{ \sum_{i=0}^{k-1} \psi_{\mathbf{p}}^{(q)}(l_e(\mathbf{c}) + il_e(\mathbf{p})) \right\} \psi_{\mathbf{d}}^{(q)}(l_e(\mathbf{c}) + kl_e(\mathbf{p})).$$

Proof. From Proposition 2.7 we have

$$\Psi_2(\mathbf{c}(\mathbf{e}-\mathbf{p})^{-1}\mathbf{d}) = \sum_{k\geq 0} \psi_{\mathbf{cp}^k\mathbf{d}}^{(q)}(0) \frac{x^{\lambda_k}}{\lambda_k!_q}$$

and the result follows by repeated applications of Proposition 2.13. ∎

We now consider the evaluation of $\psi_{\mathbf{a}}^{(q)}(m)$, where $\mathscr{T}^{-1}(\mathbf{a}) = A$ is an elementary T-graph which may be headed of non-headed. For this purpose we consider each maximum of A in turn and delete it and its incident edges. As a result $\psi_{\mathbf{a}}^{(q)}(m)$ may be expressed as the solution of a system of linear equations whose coefficient matrix contains $\psi_{\mathbf{b}_i}^{(q)}(0)$, where $\mathscr{T}^{-1}(\mathbf{b}_i)$ is a strict subpattern of A. These objects may be evaluated directly.

DEFINITION 2.15. Let $A \in \mathscr{E}$. A *maximum* in A is a vertex in the spine of A which is incident with a single edge in the spine and is

(i) the origin of an edge in the spine with label π_2, or

(ii) the terminus of an edge in the spine with label π_1;

or is incident with two edges in the spine and

(iii) both (i) and (ii) hold.

Clearly each element of $\mathscr{E}$ has at least one maximum. Moreover, if a maximum, u, is deleted from A together with its incident edges then A decomposes into an ordered pair $(\lambda_u(A), \rho_u(A))$. The set of maxima of A is denoted by $\mu(A)$, and $\lambda(A)$ denotes $\lambda_w(A)$, where w is the right-most maximum of A.

PROPOSITION 2.16. *Let* $A \in \mathscr{E}$ *and* $\mathscr{T}(A) = \mathbf{a}$. *Then*

$$\psi_{\mathbf{a}}^{(q)}(m) = \sum_{u\in\mu(A)} \binom{m + l_e(A)}{m + l_v(\lambda_u(A))}_q \psi_{\mathscr{T}(\lambda_u(A))}^{(q)}(m)\, \psi_{\mathscr{T}(\rho_u(A))}^{(q)}(0)\, q^{l_v(\rho_u(A))}$$

(*where* $l_v(A)$ *and* $l_e(A)$ *are given in Definition* 2.12.3)

Proof. We construct each of the elements of $\langle\langle \pi_1^m A\rangle\rangle$ uniquely as follows. Let $|\pi_1^m A| = p$, and consider the position of p in $\sigma \in \langle\langle \pi_1^m A\rangle\rangle$. This position is any one of the maxima of A. Let $u \in \mu(A)$ and suppose that p is assigned to u. Let $\alpha \subseteq \mathscr{N}_{p-1}$ and $\beta = \mathscr{N}_{p-1} - \alpha$. Let $\sigma_1 \in \mathscr{P}(\alpha) \cap \langle \pi_1^m \lambda_u(A)\rangle$ and $\sigma_2 \in \mathscr{P}(\beta) \cap \langle \rho_u(A)\rangle$, so σ_1 is a permutation of the elements of α with pattern $\pi_1^m \lambda_u(A)$ and σ_2 is a permutation of the elements of β with pattern $\rho_u(A)$. Thus $\sigma_1 p \sigma_2 \in \langle\langle A\rangle\rangle$, and because the construction is $[1:1]$ we have

$$\psi_{\mathbf{a}}^{(q)}(m) = \sum_{u \in \mu(A)} \sum_{\alpha \subseteq \mathscr{N}_{p-1}} \sum_{\sigma_1 \in \mathscr{P}(\alpha) \cap \langle \pi_1^m \lambda_u(A)\rangle} \sum_{\sigma_2 \in \mathscr{P}(\beta) \cap \langle \rho_u(A)\rangle} q^{I(\sigma_1 p \sigma_2)}.$$

But $I(\sigma_1 p \sigma_2) = I(\sigma_1 \sigma_2) + l_v(\rho_u(A))$ since p is greater than every element in σ_2. Thus $I(\sigma_1 p \sigma_2) = I(\sigma_1) + I(\sigma_2) + I(\alpha, \beta) + l_v(\rho_u(A))$, whence

$$\begin{aligned}
\psi_{\mathbf{a}}^{(q)}(m) &= \sum_{u \in \mu(A)} q^{l_v(\rho_u(A)} \sum_{\alpha \subseteq \mathscr{N}_{p-1}} q^{(I(\alpha,\beta)} \\
&\quad \times \sum_{\sigma_1 \in \mathscr{S}_{|\pi_1^m \lambda_u(A)|} \cap \langle \pi_1^m \lambda_u(A)\rangle} q^{I(\sigma_1)} \sum_{\sigma_2 \in \mathscr{S}_{|\rho_u(A)|} \cap \langle \rho_u(A)\rangle} q^{I(\sigma_2)} \\
&= \sum_{u \in \mu(A)} q^{l_v(\rho_u(A))} \binom{p-1}{|\alpha|}_q \sum_{\sigma_1 \in \langle\langle \pi_1^m \lambda_u(A)\rangle\rangle} q^{I(\sigma_1)} \sum_{\sigma_1 \in \langle\langle \rho_u(A)\rangle\rangle} q^{I(\sigma_2)}
\end{aligned}$$

from Proposition 2.2.2

$$= \sum_{u \in \mu(A)} \binom{m + l_e(A)}{m + l_v(\lambda_u(A))}_q \psi_{\mathscr{T}(\lambda_u(A))}^{(q)}(m)\, \psi_{\mathscr{T}(\rho_u(A))}^{(q)}(0)\, q^{l_v(\rho_u(A))}$$

which completes the proof. ∎

Since $\lambda_u(A)$ and $\rho_u(A)$ are in $\mathscr{E}$, Proposition 2.16 may be applied to $\lambda_u(A)$ for each $u \in \mu(A)$ to obtain a system of linear equations for $\psi_{\mathbf{a}}^{(q)}(m)$. The following definition is needed.

DEFINITION 2.17. Let $U_i, V_i \in \mathscr{E}$ for $i = 1, ..., r$ and let $\mathbf{K}_m(U_1, ..., U_r; V_1, ..., V_r)$ be the $r \times r$ matrix defined by

$$\begin{aligned}
[\mathbf{K}_m]_{ij} &= 1 && \text{if } V_j = U_i \\
&= -\binom{m + l_e(V_i)}{m + l_v(V_j)}_q \psi_{\mathscr{T}(\rho_\sigma(U_i))}^{(q)}(0)\, q^{l_v(\rho_\sigma(U_i))} && \text{if there exists } \sigma \in \mu(V_i) \text{ such that } \lambda_\sigma(U_i) = V_j \\
&= 0 && \text{otherwise}
\end{aligned}$$

and $\psi_{\varepsilon}^{(q)}(m) = 1$.

The final result gives an explicit expression for $\psi_{\mathbf{a}}^{(q)}(m)$. This expression, in conjunction with Proposition 2.13, may be used to derive the matrix of the linear system given in Theorem 2.11.

THEOREM 2.18. *Let $A \in \mathscr{E}$ and let $A = \alpha_0$, $\lambda(\alpha_0) = \alpha_1, \ldots, \lambda(\alpha_{p-1}) = \alpha_p$, $\lambda(\alpha_p) = \varepsilon$, where none of $\alpha_0, \ldots, \alpha_{p-1}$ is ε. Then $\psi_{\mathbf{a}}^{(q)}(m) = -|\mathbf{K}_m(\alpha_0, \ldots, \alpha_{p-1}; \varepsilon, \alpha_1, \ldots, \alpha_{p-1})|$.*

Proof. Proposition 2.16 may be applied to $\alpha_0, \ldots, \alpha_{p-1}$ to give a set of p equations for $\psi_{\mathscr{I}(\alpha_i)}^{(q)}(m)$ for $i = 1, \ldots, p-1$. The coefficient matrix of this system of linear equations is upper triangular with unit diagonal, after reordering the equations if necessary. The result follows by Cramer's rule. ∎

3. AN APPLICATION

We now consider an application of the material of Section 2 to a specific enumeration problem. Let H be the T-graph given in Fig. 4. We determine the eulerian generating function for the number of permutations on $\mathscr{N}_n$ with pattern in H^*. Figure 5 displays a set of T-graphs associated with H.

Consider the T-graphs given in Fig. 5. By Corollary 2.9 the incidence system associated with H is

$$\begin{aligned}
\mathbf{h} = \mathscr{I}(H) &= \mathbf{a}_1 \mathbf{a}_2 \mathbf{b} \mathbf{b}_3, \\
\mathbf{a}_1 + \mathbf{b}_1 &= \mathbf{b}\mathbf{a}, \\
\mathbf{a}_2 + \mathbf{b}_2 &= \mathbf{b}\mathbf{a}_4 \mathbf{a}, \\
\mathbf{a}_3 + \mathbf{b}_3 &= \mathbf{b}\mathbf{b}_5 \mathbf{a}\mathbf{a}_6 \mathbf{a}, \\
\mathbf{a}_4 + \mathbf{b}_4 &= \mathbf{b}^2 \mathbf{a}^2, \\
\mathbf{a}_5 + \mathbf{b}_5 &= \mathbf{b}^2 \mathbf{a}, \\
\mathbf{a}_6 + \mathbf{b}_6 &= \mathbf{b}\mathbf{a}^2, \\
\mathbf{a} + \mathbf{b} &= \boldsymbol{\omega},
\end{aligned}$$

where $\mathbf{a}_i = \mathscr{I}(A_i)$, $\mathbf{b}_i = \mathscr{I}(\nu^{-1}(A_i))$ for $i = 1, \ldots, 6$, $\mathbf{a} = \mathscr{I}(\pi_1)$ and $\mathbf{b} = \mathscr{I}(\pi_2)$.

We now apply Theorem 2.11, for which the right expansion of $\mathbf{h}$ is required. Then eliminating the right-most occurrence of a $\mathbf{b}_i$ in $\mathbf{h}$ we have

$$\mathbf{h} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{b}^2 \mathbf{b}_5 \mathbf{a}\mathbf{a}_6 \mathbf{a} - \mathbf{a}_1 \mathbf{a}_2 \mathbf{b}\mathbf{a}_3.$$

By continuing this process we obtain $\mathbf{h} = \mathbf{p} - \sum_{i=1}^{t} \boldsymbol{l}_j \boldsymbol{\omega} \mathbf{r}_j$, where $t = 4$ and $\mathbf{p} = \mathbf{a}_1 \mathbf{a}_2 \mathbf{a}\{\mathbf{a}(\mathbf{a}^3 - \mathbf{a}_5)\mathbf{a}\mathbf{a}_6 \mathbf{a} + \mathbf{a}_3\}$, $\boldsymbol{l}_1 = \mathbf{a}_1 \mathbf{a}_2$, $\boldsymbol{l}_2 = \mathbf{a}_1 \mathbf{a}_2 \mathbf{b}$, $\boldsymbol{l}_3 = \mathbf{a}_1 \mathbf{a}_2 \mathbf{b}^2$, $\boldsymbol{l}_4 = \mathbf{a}_1 \mathbf{a}_2 \mathbf{b}^3$, $\mathbf{r}_1 = \mathbf{a}_3 + \mathbf{a}(\mathbf{a}^3 - \mathbf{a}_5)\mathbf{a}\mathbf{a}_6 \mathbf{a}$, $\mathbf{r}_2 = (\mathbf{a}_5 - \mathbf{a}^3)\mathbf{a}\mathbf{a}_6 \mathbf{a}$, $\mathbf{r}_3 = \mathbf{a}^3 \mathbf{a}_6 \mathbf{a}$, $\mathbf{r}_4 = -\mathbf{a}^2 \mathbf{a}_6 \mathbf{a}$. Moreover, by inspection, we have $l_e(\mathbf{p}) = 18$, $l_e(\boldsymbol{l}_i) = 7 + i$ and $l_e(\mathbf{r}_i) = 10 - i$ for $i = 1, \ldots, 4$. Also $l_e(\mathbf{r}_0) = 0$.

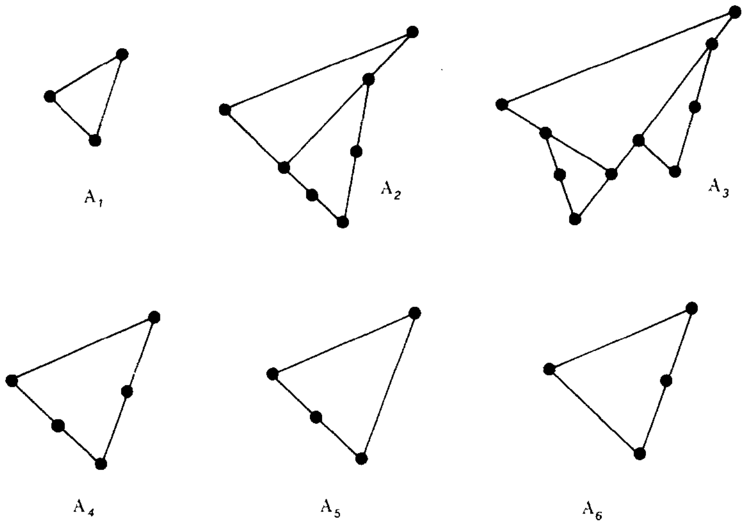

FIG. 5. A set of T-graphs associated with the T-graph of Fig. 4.

We now consider the evaluation of $\mathbf{M}^{(2)}$ and $\mathbf{k}^{(2)}$. From Theorem 2.11 and Proposition 2.14, with $\mathbf{c} = \mathbf{r}_0 = \mathbf{d} = \mathbf{e}$, we have

$$
\begin{aligned}
[\mathbf{M}^{(2)}]_{ij} &= \delta_{ij} \qquad \text{for} \quad 0 \leqslant i \leqslant 4, \quad j = 0 \\
&= \delta_{ij} + \sum_{k \geqslant 0} c_k(i, j) \frac{x^{l_e(\mathbf{r}_i) + l_e(\boldsymbol{l}_j) + 18k + 1}}{(l_e(\mathbf{r}_i) + l_e(\boldsymbol{l}_j) + 18k + 1)!_q} \\
&\qquad \text{for} \quad 0 \leqslant i \leqslant 4, \quad 1 \leqslant j \leqslant 4
\end{aligned}
$$

and

$$
k_i^{(2)} = \sum_{k \geqslant 0} d_k(i) \frac{x^{l_e(\mathbf{r}_i) + 18k + 1}}{(l_e(\mathbf{r}_i) + 18k + 1)!_q} \qquad \text{for} \quad 0 \leqslant i \leqslant 4,
$$

where

$$
\begin{aligned}
d_k(i) &= \psi_{\mathbf{r}_i}^{(q)}(0) \prod_{l=0}^{k-1} \psi_{\mathbf{p}}^{(q)}(l_e(\mathbf{r}_i) + 18l) && \text{for} \quad 0 \leqslant i \leqslant 4, \\
c_k(i, j) &= \psi_{\boldsymbol{l}_j}^{(q)}(l_e(\mathbf{r}_i) + 18k)\, d_k(i) && \text{for} \quad 0 \leqslant i \leqslant 4, \quad 1 \leqslant j \leqslant 4.
\end{aligned}
$$

Moreover, by repeated application of Proposition 2.13 we have, letting $\psi_{\mathbf{a}_i}^{(q)}(m) = g_i(m)$,

$$
\psi_{\mathbf{p}}^{(q)}(m) = g_1(m)\, g_2(m+2)\{g_3(m+9) + (1 - g_5(m+10) g_6(m+14))\} \qquad \text{for} \quad m \geqslant 0,
$$

$$\psi_{r_0}^{(q)}(0)=1, \qquad \psi_{r_1}^{(q)}(0)=g_3(0)+(1-g_5(1))\,g_6(5),$$
$$\psi_{r_2}^{(q)}(0)=(g_5(0)-1)\,g_6(4), \qquad \psi_{r_3}^{(q)}(0)=g_6(3), \qquad \psi_{r_4}^{(q)}(0)=-\,g_6(2).$$

Also $\psi_{l_1}^{(q)}(m)=g_1(m)\,g_2(m+2)$ for $m\geqslant 0$, and from Proposition 2.16, we have

$$\psi_{l_i}^{(q)}(m)=q^{\binom{i}{2}}\binom{m+7+i}{i-1}_q \psi_{l_1}^{(q)}(m) \qquad \text{for} \quad i=2,3,4 \text{ and } m\geqslant 0.$$

Now $g_i(m)$, for $i=1,\ldots,6$, may be obtained directly from Theorem 2.18 and Proposition 2.16. We obtain the following expressions for them.

$$g_1(m)=\begin{vmatrix} 0 & -1 \\ q\binom{m+1}{1}_q & 1 \end{vmatrix}=q\binom{m+1}{1}_q,$$

$$g_2(m)=\begin{vmatrix} 0 & -1 & 0 & 0 \\ q^8\binom{3}{1}_q\binom{m+5}{5}_q & 1 & -1 & 0 \\ q^7\binom{3}{1}_q\binom{m+4}{4}_q & 0 & 1 & -1 \\ q^6\binom{m+3}{3}_q & 0 & 0 & 1 \end{vmatrix}$$
$$=q^7\binom{3}{1}_q\left\{\binom{m+4}{4}_q+q\binom{m+5}{5}_q\right\}+q^6\binom{m+3}{3}_q,$$

$$g_3(m)=\begin{vmatrix} 0 & -1 & 0 & 0 & 0 \\ q^8P_1(q)\binom{m+8}{8}_q & 1 & -1 & 0 & 0 \\ q^7P_2(q)\binom{m+7}{7}_q & 0 & 1 & -1 & -q^2\binom{m+7}{2}_q \\ q^6P_3(q)\binom{m+6}{6}_q & 0 & 0 & 1 & -q\binom{m+6}{1}_q \\ q^8\binom{2}{1}_q\binom{m+4}{4}_q & 0 & 0 & 0 & 1 \end{vmatrix}$$

where

$$P_3(q)=q^5\binom{2}{1}_q\binom{5}{1}_q+q^7\binom{2}{1}_q\binom{4}{1}_q+q^{10}\binom{3}{1}_q,$$

$$P_2(q)=P_3(q)+q^6\binom{3}{1}_q\binom{6}{3}_q+q^9\binom{5}{2}_q,$$

$$P_1(q)=P_2(q)+q^8\binom{3}{1}_q\binom{6}{2}_q-q^8\binom{6}{1}_q+q^{13}\binom{5}{2}_q,$$

$$g_5(m)=\begin{vmatrix} 0 & -1 \\ q^3\binom{m+2}{2}_q & 1 \end{vmatrix}=q^3\binom{m+2}{2}_q,$$

$$g_6(m)=\begin{vmatrix} 0 & -1 & 0 \\ q^2\binom{m+2}{2}_q & 1 & -1 \\ q\binom{m+1}{1}_q & 0 & 1 \end{vmatrix}=q^2\binom{m+2}{2}_q+q\binom{m+1}{1}_q.$$

This completes the determination of $\mathbf{M}^{(2)}$ and $\mathbf{k}^{(2)}$. The required generating function is, from Theorem 2.11,

$$|[\mathbf{M}^{(2)}:\mathbf{k}^{(2)}]_0|\cdot|\mathbf{M}^{(2)}|^{-1}.$$

EXAMPLE 3.1. We next consider a simpler example which has been treated previously by Carlitz [2] in the case $q=1$. The example concerns the enumeration of alternating permutations of $\mathcal{N}_n$ with i inversions and m rises between successive maxima. Let A_1 be the T-graph given previously in Fig. 5. The incidence system for the problem is

$$\mathbf{h}=z\mathbf{a}_1+\mathbf{b}_1,$$

$$\mathbf{a}_1+\mathbf{b}_1=\mathbf{ba}$$

and the required number is $[z^m q^i (x^n/n!_q)]\ \Psi_2((\mathbf{e}-\mathbf{h})^{-1})$. The right expansion of $\mathbf{e}-\mathbf{h}$ is $\mathbf{e}-\mathbf{h}=\mathbf{e}-\mathbf{p}-\boldsymbol{l}\omega\mathbf{r}$, where $\mathbf{p}=(z-1)\,\mathbf{a}_1-\mathbf{a}^2$, $\boldsymbol{l}=\mathbf{e}$ and $\mathbf{r}=\mathbf{a}$. Thus, by Theorem 2.11, we have

$$\Psi_2((\mathbf{e}-\mathbf{h})^{-1})=\frac{\Psi_2((\mathbf{e}-\mathbf{p})^{-1})}{1-\Psi_2(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1})}.$$

The numerator and denominator may be obtained immediately from Proposition 2.14, and the required generating function is $F_0(x)\{1-F_1(x)\}^{-1}$, where

$$F_l(x)=\sum_{k=0}^{\infty}\frac{x^{2k+l+1}}{(2k+l+1)!_q}\prod_{j=0}^{k-1}\left\{(z-1)\,q\binom{2j+l+1}{1}_q-1\right\}$$

since $\psi_{\mathbf{a}_1}^{(q)}(j)=q\binom{j+1}{1}_q$ and $\psi_{\mathbf{a}_2}^{(q)}(j)=1$.

The material of Section 2 may be applied more generally as the following example demonstrates.

EXAMPLE 3.2. We consider the enumeration of permutations of odd lengths, with respect to the number of rises (between adjacent elements) and the number of rises between elements in adjacent odd positions.

Let $\mathbf{m}_i=\mathscr{I}(M_i)$ for $i=1,\ldots,6$, where the M_i are given in Fig. 6. Then the required generating function is $\Psi_1((\mathbf{e}-\mathbf{h})^{-1})$, where the incidence system is

$$\begin{aligned}
&\mathbf{h}=ru\mathbf{m}_1+r^2u\mathbf{m}_2+ru\mathbf{m}_3+r\mathbf{m}_4+\mathbf{m}_5+r\mathbf{m}_6,\\
&\mathbf{m}_1+\mathbf{m}_4=\mathbf{ba},\\
&\mathbf{m}_3+\mathbf{m}_6=\mathbf{ab},\\
&\mathbf{m}_2=\mathbf{a}^2,\\
&\mathbf{m}_5=\mathbf{b}^2,
\end{aligned}$$

in which r marks rises, and u marks rises between elements in adjacent odd positions. To see this, it suffices to note that if $\sigma=\sigma_1\cdots\sigma_{2k+1}$ then $\sigma_{2j-1}\sigma_{2j}\sigma_{2j+1}$ has pattern M_i for some i with $1\leqslant i\leqslant 6$, and $1\leqslant j\leqslant k$. We note also that M_1 represents a rise and a rise between adjacent odd positions,

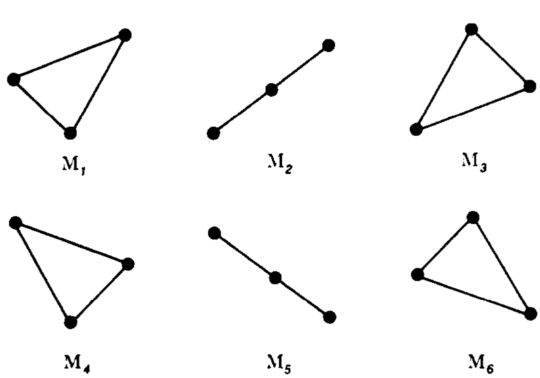

FIGURE 6

M_2 represents two rises and a rise between adjacent odd positions, and so on for $M_3, \ldots, M_6$. Accordingly the indeterminates ru, $r^2u, \ldots, r$ are associated with $M_1, M_2, \ldots, M_6$.

Now $\mathbf{h} = r(u-1)(\mathbf{m}_1 + r\mathbf{m}_2 + \mathbf{m}_3) + (r\mathbf{a} + \mathbf{b})^2 = \mathbf{p} - \mathbf{l}_1 \boldsymbol{\omega} \mathbf{r}_1 - \mathbf{l}_2 \boldsymbol{\omega} \mathbf{r}_2$,

where

$$\mathbf{p} = r(u-1)(\mathbf{m}_1 + r\mathbf{m}_2 + \mathbf{m}_3) + (r-1)^2\mathbf{a}^2,$$

$$\mathbf{l}_1 = \mathbf{e}, \qquad \mathbf{l}_2 = r\mathbf{a} + \mathbf{b}, \qquad \mathbf{r}_1 = -(r-1)\mathbf{a}, \qquad \mathbf{r}_2 = -\mathbf{e}.$$

Thus, from Theorem 2.11 we have

$$\Psi_1((\mathbf{e}-\mathbf{h})^{-1})$$

$$= \frac{-\begin{vmatrix} 1-(r-1)\,\Psi_1(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1}) & -(r-1)\,\Psi_1(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1} \\ -\Psi_1((\mathbf{e}-\mathbf{p})^{-1}) & -\Psi_1((\mathbf{e}-\mathbf{p})^{-1}) \end{vmatrix}}{\begin{vmatrix} 1-(r-1)\,\Psi_1(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1}) & -(r-1)\,\Psi_1(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1}(r\mathbf{a}+\mathbf{b})) \\ -\Psi_1((\mathbf{e}-\mathbf{p})^{-1}) & 1-\Psi_1((\mathbf{e}-\mathbf{p})^{-1}(r\mathbf{a}+\mathbf{b})) \end{vmatrix}}$$

$$= \Psi_1((\mathbf{e}-\mathbf{p})^{-1}) \times \begin{vmatrix} 1-(r-1)\,\Psi_1(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1}) & -\gamma_1-(r-1)^2\Psi_1(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1}\mathbf{a}) \\ -\Psi_1((\mathbf{e}-\mathbf{p})^{-1}) & 1-(r-1)\,\Psi_1((\mathbf{e}-\mathbf{p})^{-1}\mathbf{a}) \end{vmatrix}^{-1}$$

by replacing $\mathbf{b}$ by $\boldsymbol{\omega} - \mathbf{a}$ and by using elementary row and column operations. Now $M_1, \ldots, M_6$, are not T-graphs, and in particular $\mathbf{m}_1 + r\mathbf{m}_2 + \mathbf{m}_3$ is not headed, so Theorem 2.18 is not applicable. Instead, we observe that $\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 = \gamma_1 \mathbf{a}$ by inspection. Moreover, $\mathbf{m}_2 = \mathbf{a}^2$ so $\mathbf{p} = r(u-1)\,\gamma_1 \mathbf{a} + (r-1)(ru-1)\,\mathbf{a}^2$. Now for $s, t \geqslant 0$ we have

$$\Psi_1(\mathbf{a}^s(\mathbf{e}-(z_1\mathbf{a}+z_2\mathbf{a}^2))^{-1}\mathbf{a}^t) = \sum_{k \geqslant 0} \sum_{i=0}^{k} \binom{k}{i} z_1^i z_2^{k-i} \gamma_{2k+s+t+1-i}$$

so

$$\Psi_1(\mathbf{a}^s(\mathbf{e}-\mathbf{p})^{-1}\mathbf{a}^t)$$

$$= \sum_{k \geqslant 0} \sum_{i=0}^{k} \binom{k}{i} (r(u-1))^i((r-1)(ru-1))^{k-i} \gamma_1^i \gamma_{2k+s+t+1-i}.$$

Let

$$f(l) = (r-1)^l \sum_{l \geqslant 0} \sum_{i=0}^{k} \binom{k}{i} (r(u-1))^i((r-1)(ru-1))^{k-i} \frac{x^{2k+l+1}}{(2k+l+1-i)!}.$$

Thus the number of permutations in $\mathcal{N}_n$ of odd length, with s rises and t rises between elements in adjacent odd positions is

$$[r^s u^t (x^n/n!)]\Phi, \qquad \text{where} \quad \Phi = f(0)\{(1-f(1))^2 - f(0)(x+f(2))\}^{-1}.$$

4. Concluding Comments

The generating function for the number of permutations whose patterns belong to a prescribed set of T-graphs may also be obtained from the solution to a matrix Riccati equation, whose general form is $\mathbf{Y}' = \mathbf{YAY} + \mathbf{BY} + \mathbf{YC} + \mathbf{D}$, where $\mathbf{Y} = [y_{ij}]_{p \times p}$, y_{ij} are functions of x, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are $p \times p$ matrices which may depend on x. We demonstrate the principle briefly with the patterns given in Fig. 2. Accordingly, let H, A_1 and B_1 be the patterns given in Fig. 7.

Let y_{00}, y_{01}, y_{10}, y_{11} be the exponential generating functions for the numbers of permutations with pattern in H^*, H^*B_1, A_1H^* and $A_1H^*B_1$, respectively. Now $\langle H^* \rangle = \langle HH^* \rangle \cup \langle \varepsilon \rangle$, where $\langle \varepsilon \rangle = \mathscr{N}_n$. Thus, deleting the first occurrence of a maximum from the right in H^* we have $\langle H^* \rangle \cong \langle H^*B_1 \rangle \times \langle A_1H^*B_1 \rangle \cup \{\varepsilon\}$ since the deletion of the maximum in H yields B_1WA_1. Accordingly, from the generating function reformulation of Proposition 2.16 we have $y'_{00} = y_{01}y_{10} + 1$. Similarly $\langle H^*B_1 \rangle = \langle HH^*B_1 \rangle \cup \langle B_1 \rangle \cong \langle H^*B_1 \rangle \times \langle A_1H^*B_1 \rangle \cup \langle \varepsilon \rangle$, whence $y'_{01} = y_{01}y_{11} + x$. Also $\langle A_1H^* \rangle = \langle A_1HH^* \rangle \cup \langle A_1 \rangle \cong \langle A_1H^*B_1 \rangle \times \langle A_1H^*B_1 \rangle \cup \langle \varepsilon \rangle$ so $y'_{10} = y_{11}y_{10} + x$. Finally, $\langle A_1H^*B_1 \rangle = \langle A_1HH^*B_1 \rangle \cup \langle A_1B_1 \rangle \cong \langle A_1H^*B_1 \rangle \times \langle A_1H^*B_1 \rangle \cup \langle \varepsilon \rangle \times \langle \varepsilon \rangle$ so $y'_{11} = y^2_{11} + x^2$. Let

$$\mathbf{Y} = \begin{bmatrix} y_{00} & y_{01} \\ y_{10} & y_{11} \end{bmatrix}.$$

Thus $\mathbf{Y}$ satisfies the matrix Riccati equation $\mathbf{Y}' = \mathbf{YPY} + \mathbf{Q}$, where $\mathbf{P} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\mathbf{Q} = \left[\begin{smallmatrix} 1 & x \\ x & x^2 \end{smallmatrix}\right]$. This equation may be linearised as follows (see, for example, Reid [14]). Let $\mathbf{U}$, $\mathbf{V}$ be 2×2 matrices such that $\mathbf{UY} = \mathbf{V}$, where $\mathbf{U}$ is non-singular. But $\mathbf{Y}(0) = \mathbf{0}$ so $\mathbf{V}(0) = \mathbf{0}$ and we may set $\mathbf{U}(0) = \mathbf{I}$. Now $\mathbf{V}' = \mathbf{U}'\mathbf{Y} + \mathbf{U}\mathbf{Y}' = \mathbf{U}'\mathbf{Y} + \mathbf{VPY} + \mathbf{UQ}$ so $\mathbf{V}' - \mathbf{UQ} - (\mathbf{U}' + \mathbf{VP})\mathbf{Y} = \mathbf{0}$. Let $\mathbf{V}' - \mathbf{UQ} = \mathbf{0}$ and $\mathbf{U}' + \mathbf{VP} = \mathbf{0}$. Accordingly, we have linearised the matrix Riccati equation. It follows that $\mathbf{U}'' = -\mathbf{V}'\mathbf{P} = -\mathbf{UQP}$ so $\mathbf{U}'' + \mathbf{UQP} = \mathbf{0}$. This may be solved for $\mathbf{U}$ in power series by the method of Frobenius, from which $\mathbf{V}$ may be determined by means of $\mathbf{V}' = \mathbf{UQ}$. But $y_{00} = [\mathbf{U}^{-1}\mathbf{V}]_{00} = |[\mathbf{U} : \mathbf{k}]_0| \cdot |\mathbf{U}|^{-1}$, where $\mathbf{k}$ is the 0th column of $\mathbf{V}$, from which y_{00} may be obtained.

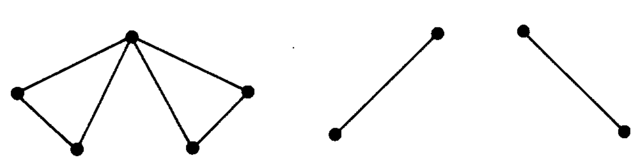

Fig. 7. Subpatterns of the T-graph given in Fig. 2.

In general, it appears that the dimension of the linear system of equations obtained by means of Theorem 2.11 is smaller than the dimension of the matrix obtained on the corresponding matrix Riccati formulation. Indeed, we have been unable to complete the hand calculations needed for deriving the matrix Riccati equations for the pattern given in Fig. 4. However, for a subclass of such problems the generating functions may be obtained by a direct application of matrix Riccati equations. This is done elsewhere [5].

We conclude with the derivation of y_{00} by the methods in Section 2. Instead of using Theorem 2.11, we proceed from first principles. Let $\mathbf{h} = \mathscr{I}(H)$, $\mathbf{a}_1 = \mathscr{I}(A_1)$, $\mathbf{b}_1 = \mathscr{I}(B_1)$, $\mathbf{a} = \mathscr{I}(\pi_1)$ and $\mathbf{b} = \mathscr{I}(\pi_2)$. The incidence system for $\mathbf{h}$ is, by Corollary 2.9,

$$\mathbf{h} = \mathbf{a}_1 \mathbf{b}_1,$$

$$\mathbf{a}_1 + \mathbf{b}_1 = \mathbf{ba},$$

$$\mathbf{a} + \mathbf{b} = \boldsymbol{\omega}.$$

Thus, from Theorem 2.11, the required generating functions is, for $q = 1$,

$$y_{00} = \Psi_2(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1})\{1 - \Psi_2(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1}\mathbf{a}_1)\}^{-1}$$

since $\mathbf{e} - \mathbf{h} = \mathbf{e} - \mathbf{p} - \mathbf{a}_1 \boldsymbol{\omega} \mathbf{a}$, where $\mathbf{p} = -\mathbf{a}_1^2 - \mathbf{a}_1 \mathbf{a}^2$. Now $\psi_{\mathbf{a}_1^2}(m) = \psi_{\mathbf{a}_1}(m)\, \psi_{\mathbf{a}_1}(m+2) = (m+1)(m+3)$ and $\psi_{\mathbf{a}_1\mathbf{a}^2}(m) = m+1$. Thus, from Proposition 2.14 we have

$$\Psi_2(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1}\mathbf{a}_1) = 1 + \sum_{k \geqslant 1} (-1)^k (4k-2) \left\{ \prod_{j=0}^{k-2} (4j+2)(4j+5) \right\} \frac{x^{4k}}{(4k)!}$$

and

$$\Psi_2(\mathbf{a}(\mathbf{e}-\mathbf{p})^{-1}) = \sum_{k \geqslant 0} (-1)^k \left\{ \prod_{j=0}^{k-1} (4j+2)(4j+5) \right\} \frac{x^{4k+2}}{(4k+2)!}$$

which completes the determination of the generating function.

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