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Expository Paper

Sequence enumeration

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1. Introduction

The purpose of this paper is to present a method for treating uniformly a large number of enumeration problems involving sequences over a finite alphabet subject to general restrictions. Many of these problems are classical in origin, including the Ménage problem (Lucas [16]), the alternating permutation problem (Netto [19], André [1, 2]), the derangement problem (Montmort [18]), and more recently, the Simon Newcomb problem (Riordan [21]) and the Smirnov problem (Smirnov, Saramanov, Zaharov [23]). Renewed interest in sequence enumeration has been shown in the recent literature.

In a given situation, we obtain a regular expression which generates all permissible sequences. The generating function associated with the problem is obtained by a simple transformation of this expression. This process of transformation has a direct combinatorial basis and involves the insertion of indeterminates to record the appropriate combinatorial information. Thus, we consider all sequences in a class and choose those with the required property by extracting the coefficient of the appropriate power of the indeterminate. Additive and multiplicative properties of these generating functions are given, which allow us to induce a linear system of equations for the required function. The solution, obtained by Cramer's Rule, consists of generating functions for elementary objects. These in turn are easily found by combinatorial means.

Our concern in this paper is to present a general method, together with a few examples which illustrate the ways in which the method is used. The reference which is cited for each result is, as far as it is possible to discover, a reference to the first solution, usually by other means. Several proofs are omitted. These are accessible in the literature and concern algebraic devices used in the theory.

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However, the proofs for combinatorial theorems which use our theory are included. A more complete collection of examples of sequence enumeration using the techniques given here may be found in [8, 9, 11, 12, 13, 14, 15]. Other treatments in this area may be found in Cartier and Foata [5], Foata and Schützenberger [6], Gessel [7], Jackson and Aleliunas [10], Reilly [20], Stanley [24], and others.

We adopt the following notational apparatus. The set $\{1, 2, \ldots, n\}$ for $n \geq 1$ is denoted by $\mathcal{N}_n$. If $\mathbf{i} = (i_1, \ldots, i_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$ then $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_n^{i_n}$. If $f(\mathbf{x})$ is a power series in $\mathbf{x}$ then $[\mathbf{x}^{\mathbf{i}}]f(\mathbf{x})$ denotes the coefficient of $\mathbf{x}^{\mathbf{i}}$ in $f(\mathbf{x})$. Let $g(x) = g_0 + g_1 x + g_2 x^2 + \cdots$, where x is an indeterminate, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$. Then $g \circ \boldsymbol{\gamma} = g_0 + g_1\gamma_1 + g_2\gamma_2 + \cdots$ is the *umbral composition* of g and $\boldsymbol{\gamma}$.

Let $n!_q$ denote $\prod_{i=1}^{n} \frac{1-q^i}{1-q}$, and $I(\sigma)$ denote the number of inversions in $\sigma = \sigma_1 \cdots \sigma_k$, where an *inversion* is a pair (σ_i, σ_j) with $\sigma_i > \sigma_j$ and $i < j$.

If $\mathbf{A}$ is an $n \times n$ matrix with elements a_{ij} for $i, j = 1, \ldots, n$, we write $[\mathbf{A}]_{ij} = a_{ij}$. Moreover $|\mathbf{A}|$ and $\|a_{ij}\|_{n \times n}$ denote the determinant of $\mathbf{A}$. If $\mathbf{b}$ is a column vector with n components then $[\mathbf{A} : \mathbf{b}]_i$ where $i = 1, \ldots, n$, denotes the matrix obtained from $\mathbf{A}$ by replacing column i of $\mathbf{A}$ by $\mathbf{b}$.

$\mathcal{H}^*$ denotes the sequence monoid on $\mathcal{H} = \{h_1, h_2, \ldots\}$, the free monoid with concatenation (denoted by juxtaposition). Let $\mathcal{H}^+ = \mathcal{H}^* \backslash \{\varepsilon\}$ where ε is the empty sequence.

2. The maximal decomposition theorem

Let $\{\pi_1, \pi_2\}$ be a bipartition of $\mathcal{N}_n^2$. A *π_1-path* of *length* r is a sequence $\sigma = \sigma_1 \cdots \sigma_r$ such that $(\sigma_i, \sigma_{i+1}) \in \pi_1$ for $i = 1, \ldots, r-1$. If $r = 1, \sigma$ is always a π_1-path. The *type* of a sequence σ is $\tau(\sigma) = (i_1, \ldots, i_n)$ where j occurs i_j times in σ. The generating function for π_1-paths of length k is $\gamma_k(\pi_1) = \sum_\sigma \mathbf{x}^{\tau(\sigma)}$, where the summation extends over all π_1-paths σ of length k. A substring ρ of a sequence is a *maximal* π_1-path if it is a π_1-path which is not properly contained in another. For example, the sequence 13213545 contains 4 maximal *strictly increasing paths* (i.e. π_1-paths where π_1 is the set of all (i, j) with $i < j$), namely 13, 2, 135 and 45.

Many questions in sequence enumeration may be rephrased in terms of restrictions on sequence type, and restrictions on maximal π_1-path lengths for some bipartition $\{\pi_1, \pi_2\}$ of $\mathcal{N}_n^2$. The following theorem expresses the generating function for sequences on $\mathcal{N}_n$ with respect to type and maximal π_1-path lengths in terms of the π_1-path generating functions $\gamma_k(\pi_1)$, $k \geq 0$.

THEOREM 2.1 (Jackson and Aleliunas [10], Gessel [7]). *Let* $F(x) = 1 + f_1 x + f_2 x^2 + \cdots$ *where* $f_1, f_2, \ldots$ *are indeterminates. Then the number of sequences on* $\mathcal{N}_n$ *of type* $\mathbf{i}$, *and with* k_j *occurrences of maximal* π_1*-paths of length*

$j, j \geq 1$ is

(1) $[\mathbf{x}^{\mathbf{i}}\mathbf{f}^{\mathbf{k}}](F^{-1} \circ \boldsymbol{\gamma})^{-1}$

where $\mathbf{f} = (f_1, f_2, \ldots)$, $\mathbf{k} = (k_1, k_2, \ldots)$ and $\boldsymbol{\gamma} = (\gamma_1(\pi_1), \gamma_2(\pi_1), \ldots)$. Moreover, the number of permutations on $\mathcal{N}_n$ with m inversions and k_j maximal increasing paths of length j, $j \geq 1$ is

(2) $\left[\mathbf{f}^{\mathbf{k}} q^m \dfrac{x^n}{n!_q}\right](F^{-1} \circ \boldsymbol{\eta}_q)^{-1}$

where

$$\boldsymbol{\eta}_q = \left(\frac{x}{1!_q}, \frac{x^2}{2!_q}, \ldots\right).$$

The proof of this theorem is given in Section 4. For the remainder of this section we consider examples of the use of this theorem in the enumeration of sequences with respect to a variety of characteristics. Each application requires a specialization of $F(x)$ and a choice of π_1. The first set of examples demonstrates how the same $F(x)$ allows us to solve a number of distinct problems through appropriate choice of π_1. Since a permutation on $\mathcal{N}_n$ is a sequence of type $(1, 1, \ldots, 1)$, the following device for the extraction of the coefficient of $x_1 x_2 \cdots x_n$ from a power series in $\gamma_k(\pi_1)$, $k \geq 1$, for three choices of π_1, allows us also to solve the permutation version of any sequence problem associated with the given $F(x)$ and any of the three special choices for π_1.

LEMMA 2.2 ([11]). *Let $\Phi(\gamma_1, \gamma_2, \ldots)$ be a power series in $\gamma_1(\pi_1), \gamma_2(\pi_1), \ldots$. Then $[x_1 \cdots x_n]\Phi(\gamma_1, \gamma_2, \ldots)$ is equal to*

(1) $\left[\dfrac{x^n}{n!}\right]\Phi\left(\dfrac{x}{1!}, \dfrac{x^2}{2!}, \ldots\right)$ *if π_1 is the set of rises ((i, j) with $i < j$).*

(2) $[x^n] \sum_{k \geq 0} k![y^k]\Phi(xy, x^2y, \ldots)$ *if π_1 is the set of successions ($(i, i+1) \in \mathcal{N}_n^2$).*

(3) $[x^n] \sum_{k \geq 1} (k-1)![y^k]x\dfrac{\partial}{\partial x}\Phi(xy, x^2y, \ldots)$ *if π_1 is the set of *-successions ($(i, 1 + (i \bmod n)) \in \mathcal{N}_n^2$).*

EXAMPLE 2.3 ([8]). We consider first the enumeration of sequences with respect to type and number of occurrences of π_1 (a (*not necessarily maximal*) π_1-path of length 2), recorded by the indeterminate u. A maximal π_1-path of length k contains $k-1$ occurrences of π_1, and occurrences of π_1 may only be found internal to π_1-paths. Accordingly we may set $f_k = u^{k-1}$, $k \geq 1$ in Theorem 2.1(1), so

$$F(x) = 1 + x + ux^2 + u^2x^3 + \cdots = (1-(u-1)x)(1-ux)^{-1}$$

and there are $[u^j\mathbf{x}^{\mathbf{i}}]\{1-\sum_{k\geq 1}(u-1)^{k-1}\gamma_k(\pi_1)\}^{-1}$ sequences of type $\mathbf{i}$ with j occurrences of elements in π_1.

EXAMPLE 2.4 (Stanley [24]. It follows from Theorem 2.1(2) and the above argument that there are

$$\left[u^j q^m \frac{x^n}{n!_q}\right]\left\{1-\sum_{k\geq 1}(u-1)^{k-1}\frac{x^k}{k!_q}\right\}^{-1}$$

permutations on $\mathcal{N}_n$ with m inversions and j rises.

EXAMPLE 2.5. If π_1 is the set of rises on $\mathcal{N}_n$ then we immediately obtain

$$\gamma_k(\pi_1) = [z^k]\prod_{i=1}^{n}(1+zx_i).$$

Thus from Example 2.3, there are

$$[u^j\mathbf{x}^{\mathbf{i}}](u-1)\left\{u-\prod_{k=1}^{n}(1+(u-1)x_k)\right\}^{-1}$$

sequences of type $\mathbf{i}$ over $\mathcal{N}_n$ with j rises.

This is the *Simon Newcomb problem* [21].

EXAMPLE 2.6 (Carlitz [3]). If π_1 is the set of levels on $\mathcal{N}_n$ $((i,i)\in\mathcal{N}_n^2)$, then $\gamma_k(\pi_1) = x_1^k + \cdots + x_n^k = s_k$, a power sum symmetric function in $x_1, \ldots, x_n$. Thus, from Example 2.3, there are $[u^j\mathbf{x}^{\mathbf{i}}]\{1-\sum_{k\geq 1}(u-1)^{k-1}s_k\}^{-1}$ sequences on $\mathcal{N}_n$ of type $\mathbf{i}$ with j levels.

It follows that there are $[\mathbf{x}^{\mathbf{i}}]\{1-s_1+s_2-s_3+\cdots\}^{-1}$ sequences of type $\mathbf{i}$ with no levels, so that adjacent elements in the sequence are distinct. This is called the *Smirnov problem* [23].

EXAMPLE 2.7. From Example 2.3 and Lemma 2.2 there are

$$\left[u^j \frac{x^n}{n!}\right](u-1)\{u-e^{(u-1)x}\}^{-1}$$

permutations on $\mathcal{N}_n$ with j rises,

$$[u^j x^n] \sum_{k\geq 0} k!\, x^k \{1-(u-1)x\}^{-k}$$

permutations on $\mathcal{N}_n$ with j successions, and

$$[u^j x^n] \sum_{k\geq 1} k!\, x^k \{1-(u-1)x\}^{-(k+1)}$$

permutations on $\mathcal{N}_n$ with j *-successions.

The first of these results is an *Eulerian number* [21] and may, of course, also be obtained from Example 2.4 by specializing to the case $q=1$. The latter are given in [12]. Tanny [25] has found these two numbers in another form.

Of the many other specializations of $F(x)$, we give one involving the enumeration of sequences with respect to π_1-paths (not necessarily maximal) of length p, $p\geq 1$.

EXAMPLE 2.8 (Jackson and Aleliunas [10]). The only π_1-paths of length p in a sequence are internal to maximal π_1-paths of length $k\geq p$. Each of these maximal paths contains $k-p+1$ π_1-paths of length p and thus, if π_1-paths of length p are recorded by the indeterminate u, we may set $f_k=u^{k-p+1}$, $k\geq p$ and $f_k=1$, $k<p$ in Theorem 2.1. Accordingly

$$\begin{aligned} F(x) &= 1+x+x^2+\cdots+x^{p-1}+ux^p+u^2x^{p+1}+\cdots \\ &= (1-ux+(u-1)x^p)(1-ux)^{-1}(1-x)^{-1} \end{aligned}$$

and from Theorem 2.1(1) there are $[u^j\mathbf{x^i}](\{1-x)(1-ux)\{1-ux+(u-1)x^p\}^{-1}\}\circ \boldsymbol{\gamma}(\pi_1))^{-1}$ sequences on $\mathcal{N}_n$ of type $\mathbf{i}$ with j π_1-paths of length p.

EXAMPLE 2.9. When $p=3$, we may conclude from Example 2.8 that there are

$$[u^j\mathbf{x^i}]\left\{\sum_{k\geq 0}(u-1)^k \sum_{l=0}\binom{k}{l}\{\gamma_{k+l}(\pi_1)-u\gamma_{k+l+1}(\pi_1)\}\right\}^{-1}$$

sequences of type $\mathbf{i}$ with j π_1-paths of length 3.

EXAMPLE 2.10. It follows from Theorem 2.1(2), and the above argument, that there are

$$\left[u^j q^m \frac{x^n}{n!_q}\right]\left(\sum_{k\geq 0}(u-1)^k \sum_{l=0}^{k}\binom{k}{l}\left\{\frac{x^{k+l}}{(k+l)!_q}-u\frac{x^{k+l+1}}{(k+l+1)!_q}\right\}\right)^{-1}$$

permutations on $\mathcal{N}_n$ with m inversions and j increasing paths of length 3.

3. The algebra of patterns

In this section we introduce the general theory which enables us to enumerate sequences on $\mathcal{N}_n$ with various restrictions placed on strings of adjacent elements.

Let $\Pi=\{\pi_1, \pi_2\}$ be a bipartition of $\mathcal{N}_n^2$, and let $\mathcal{N}_n^2$ be denoted by ω. Let $\sigma=\sigma_1\cdots\sigma_l$ be a non-empty sequence over $\mathcal{N}_n$ and let $\mu=\pi_{m_1}\cdots\pi_{m_{l-1}}$ be a sequence over the alphabet $\{\pi_0, \pi_1, \pi_2\}$ where $\pi_0=\omega$. We say that the sequence σ has *pattern* μ if $(\sigma_j, \sigma_{j+1})\in\pi_{m_j}$ for $j=1,\ldots,l-1$. We let $\langle\mu\rangle$ denote the set of all sequences over $\mathcal{N}_n$ whose pattern is μ. The subset of these sequences which are permutations on $\mathcal{N}_k$, for some k, is denoted by $\langle\langle\mu\rangle\rangle$.

The *incidence matrix*, $\mathcal{I}(\mu)$, for the pattern μ is the matrix whose (i, j)-element is

$$\sum_{\substack{\sigma_1,\ldots,\sigma_l\in\langle\mu\rangle\\ \sigma_1=i,\,\sigma_l=j}} x_{\sigma_1}x_{\sigma_2}\cdots x_{\sigma_{l-1}}.$$

The properties of the incidence matrix which are required in our theory are given in the following result.

PROPOSITION 3.1 ([11]). *Let* $\mu_1, \mu_2, \mu_3\in\{\pi_1, \pi_2, \omega\}^*$, *and let* $\langle\mu_1\rangle\cup\langle\mu_2\rangle=\langle\mu_3\rangle$ *and* $\langle\mu_1\rangle\cap\langle\mu_2\rangle=\emptyset$. *Then*

(1) $\mathcal{I}(\mu_1\mu_2)=\mathcal{I}(\mu_1)\mathcal{I}(\mu_2)$

(2) $\mathcal{I}(\mu_1)+\mathcal{I}(\mu_2)=\mathcal{I}(\mu_3)$

(3) $\mathcal{I}(\pi_1)+\mathcal{I}(\pi_2)=\mathbf{XJ}$ *where* $\mathbf{X}=\operatorname{diag}(x_1,\ldots,x_n)$ *and* $\mathbf{J}$ *is the* $n\times n$ *matrix of ones.*

In any problem involving sequence enumeration over $\mathcal{N}_n$ and a bipartition $\{\pi_1, \pi_2\}$ of $\mathcal{N}_n^2=\omega$, we use the notation $\mathcal{I}(\pi_1)=\mathbf{A}$, $\mathcal{I}(\pi_2)=\mathbf{B}$ and $\mathcal{I}(\omega)=\mathbf{XJ}=\mathbf{W}$, for convenience.

Let $\mathcal{R}$ denote the set of all matrices of the form $\sum_{i\geq 1} c_i\mathcal{I}(\mu_i)$ where

$$\{\mu_i \mid i\geq 1\}\subseteq\{\pi_1, \pi_2, \omega\}^*$$

and the c_i's are polynomials in the commutative indeterminates $y_1, y_2, \ldots$ with rational coefficients (i.e. $c_i \in \mathbb{Q}[\mathbf{y}]$, $i \geq 1$). A particular element of $\mathcal{R}$ represents the sum of the incidence matrices of permissible patterns associated with a given problem. The polynomials c_i are used to convey combinatorial information additional to the sequence type. The latter is recorded by the incidence matrices. We next associate a generating function with each element of $\mathcal{R}$ and establish the properties of such generating functions.

LEMMA 3.2. *Let*

$$\mathbf{U} = \sum_{i\geq 1} c_i \mathcal{I}(\mu_i) \in \mathcal{R}, \qquad \Psi_1(\mathbf{U}) = \sum_{i\geq 1} c_i \sum_{\sigma \in \langle \mu_i \rangle} \mathbf{x}^{\tau(\sigma)}$$

and

$$\Psi_2(\mathbf{U}) = \sum_{i\geq 1} c_i \sum_{\sigma \in \langle\langle \mu_i \rangle\rangle} \frac{x^{|\sigma|}}{|\sigma|\,!_q} q^{I(\sigma)}.$$

Then, if $\mathbf{V} \in \mathcal{R}$,

(1) $\Psi_i(\mathbf{U}+\mathbf{V}) = \Psi_i(\mathbf{U}) + \Psi_i(\mathbf{V})$ *for* $i = 1, 2$.

(2) $\Psi_i(c\mathbf{U}) = c\Psi_i(\mathbf{U})$ *for* $i = 1, 2$, *and* $c \in \mathbb{Q}[\mathbf{y}]$.

(3) $\Psi_i(\mathbf{UWV}) = \Psi_i(\mathbf{U})\Psi_i(\mathbf{V})$ *for* $i = 1$, *and for* $i = 2$ *if* π_1 *is the set of rises.*

(4) $\Psi_1(\mathcal{I}(\pi_1^{k-1})) = \gamma_k(\pi_1)$, *and* $\Psi_2(\mathcal{I}(\pi_1^{k-1})) = \dfrac{x^k}{k\,!_q}$ *where* π_1 *is the set of rises.*

(5) $\Psi_1(\mathbf{U}) = \operatorname{trace}(\mathbf{UW})$.

These properties are all immediate except for (3) with $i = 2$. A proof for the latter may be found in [14].

To exploit these results in the solution of a particular sequence enumeration problem, we must first determine $\mathbf{U} \in \mathcal{R}$ so that $\Psi_i(\mathbf{U})$ is the required generating function. This is in fact, often, a routine matter using direct combinatorial constructions and is illustrated in the examples. By Proposition 3.1(1), $\mathbf{U}$ is a power series in the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{W}$. We may, using Proposition 3.1(3), replace $\mathbf{B}$ by $\mathbf{W}-\mathbf{A}$, and then use Lemma 3.2(1–3) to express $\Psi_i(\mathbf{U})$ as sums and products of $\Psi_i(\mathbf{A}^{k-1})$ for some $k \geq 1$.

By Lemma 3.2(4), each of these terms may be obtained directly and the problem is solved. The procedure outlined above is impractical in most instances, but an indirect method using the same properties may be used to advantage, yielding a linear system of equations for a set of generating functions which contains the required generating function. This indirect argument is demonstrated for sequences with a fixed, arbitrary pattern in the next example, and in more general circumstances in the next section.

EXAMPLE 3.3 (Stanley [24]). We consider the enumeration of permutations on $\mathcal{N}_n$ with the fixed pattern $\pi_1^{\mu_1-1}\pi_2\pi_1^{\mu_2-1}\pi_2\cdots\pi_1^{\mu_m-1}$, with respect to inversions, where $\mu_1, \mu_2, \ldots, \mu_m$ are arbitrary positive integers with $\mu_1+\mu_2+\cdots+\mu_m=n$ and π_1 is the set of rises. Let $\xi_i=\Psi_2(\mathbf{A}^{\mu_i-1}\mathbf{B}\cdots\mathbf{A}^{\mu_m-1})$ and $a_i=\sum_{j=1}^{i}\mu_j$ for $i=1,\ldots,m$, with $a_0=0$. The required generating function, from Lemma 3.2, is thus ξ_1. Now by successively replacing the left-most occurrence of $\mathbf{B}$ by $\mathbf{W}-\mathbf{A}$, we have, from Proposition 3.1(3) and Lemma 3.2,

$$\xi_i=\sum_{j=i+1}^{m}(-1)^{j-i-1}\Psi_2(\mathbf{A}^{a_{j-1}-a_{i-1}-1})\xi_j+(-1)^{m-i}\Psi_2(\mathbf{A}^{a_m-a_{i-1}-1})$$

for $i=1,\ldots,m$. This is an $m\times m$ system of linear equations for $\xi_1,\ldots,\xi_m$. Since $\Psi_2(\mathbf{A}^{k-1})=\dfrac{x^k}{k!_q}$, from Lemma 3.2, we solve this linear system by Cramer's Rule to give

$$\xi_1=\left\|\frac{x^{a_j-a_{i-1}}}{(a_j-a_{i-1})!_q}\right\|_{m\times m}.$$

Accordingly, the number of permutations on $\mathcal{N}_n$ with fixed pattern $\pi_1^{\mu_1-1}\pi_2\pi_1^{\mu_2-1}\pi_2\cdots\pi_1^{\mu_m-1}$, for π_1 the set of rises, and k inversions is

$$\left[q^k\frac{x^n}{n!_q}\right]\xi_1=[q^k]\left\|\binom{n-a_{i-1}}{a_j-a_{i-1}}_q\right\|_{m\times m}.$$

By setting $q=1$ we obtain MacMahon's [17] result; that there are $\left\|\binom{n-a_{i-1}}{a_j-a_{i-1}}\right\|_{m\times m}$ permutations on $\mathcal{N}_n$ with the fixed pattern $\pi_1^{\mu_1-1}\pi_2\pi_1^{\mu_2-1}\pi_2\cdots\pi_1^{\mu_m-1}$, where π_1 is the set of rises and $a_i=\sum_{j=1}^{i}\mu_j$, $a_0=0$.

Stanley's result is more general than that of Example 3.3. He uses binomial posets to consider sets of r permutations simultaneously. The extension of our method to handle this situation is presented in [8]. It is based on the fact that the incidence matrices are r-fold tensor products of those used here, say

$$\mathbf{U}_1\otimes\cdots\otimes\mathbf{U}_r,$$

and that $\Psi_i(\mathbf{U}_1\otimes\cdots\otimes\mathbf{U}_r)=\prod_{j=1}^{r}\Psi_i(\mathbf{U}_j)$ for $i=1,2$.

4. The linear system

In many instances the generating function required as the solution to a problem has the form $\Psi_t(\mathbf{C}(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D})$. For example, the ordinary generating function for sequences with pattern in $\mu^* = \varepsilon \cup \mu \cup \mu^2 \cup \cdots$ for some non-empty pattern μ, is clearly $\Psi_1(\mathbf{I}+\mathbf{H}+\mathbf{H}^2+\cdots) = \Psi_1((\mathbf{I}-\mathbf{H})^{-1})$ where $\mathbf{H} = \mathscr{I}(\mu)$. The matrix $\mathbf{H}$ need not be a single incidence matrix, but can be an element in $\mathscr{R}$ with indeterminates which retain combinatorial information in coefficients. An example of this use of indeterminates is given in the proof of the maximal decomposition theorem, obtained as a corollary of the following theorem. This theorem allows us formally to express $\Psi_l(\mathbf{C}(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D})$ as a ratio of determinants whose elements are related generating functions. The theorem is given combinatorial meaning in later examples. The second part of the theorem gives a similar expression for trace log $(\mathbf{I}-\mathbf{H})^{-1}$, which appears in Section 7, as the form of the generating functions for circular sequences.

THEOREM 4.1 ([9], [14]). *Let* $\mathbf{Q}, \mathbf{H}, \mathbf{C}, \mathbf{D}, \mathbf{L}_i, \mathbf{R}_i \in \mathscr{R}$ *for* $i = 1, \ldots, s$ *and* $\mathbf{H} = \mathbf{Q} - \sum_{k=1}^{s} \mathbf{L}_k \mathbf{W} \mathbf{R}_k$. *Then*

(1) $\Psi_l(\mathbf{C}(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D}) = |[\mathbf{M}:\mathbf{d}]_0| \cdot |\mathbf{M}|^{-1}$ *for* $l = 1, 2$.

(2) $\text{trace}\log(\mathbf{I}-\mathbf{H})^{-1} = \log|\mathbf{M}|^{-1} + \text{trace}\log(\mathbf{I}-\mathbf{Q})^{-1}$ *for* $l = 1$.

where

$$[\mathbf{M}]_{ij} = \delta_{ij} + \Psi_l(\mathbf{R}_i(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{L}_j), \qquad 0 \le i, j \le s$$
$$d_i = \Psi_l(\mathbf{R}_i(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{D}) \quad \textit{for} \quad i = 0, 1, \ldots, s$$

and

$$\mathbf{L}_0 = \mathbf{0}, \mathbf{R}_0 = \mathbf{C}, \mathbf{d} = (d_0, d_1, \ldots, d_s)^T.$$

Proof. (1) We premultiply both sides of $\mathbf{I}-\mathbf{H} = \mathbf{I}-\mathbf{Q}+\sum_{k=1}^{s}\mathbf{L}_k\mathbf{W}\mathbf{R}_k$ by $\mathbf{T}(\mathbf{I}-\mathbf{Q})^{-1}$, for $\mathbf{T} \in \mathscr{R}$, postmultiply by $(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D}$, and apply Ψ_l, yielding

$$\Psi_l(\mathbf{T}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{D}) = \Psi_l\{\mathbf{T}(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D} + \sum_{k=1}^{s} \mathbf{T}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{L}_k\mathbf{W}\mathbf{R}_k(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D}\}.$$

Substituting $\mathbf{T} = \mathbf{C}, \mathbf{R}_1, \ldots, \mathbf{R}_s$ and using Lemma 3.2, we obtain an $(s+1)\times(s+1)$ system of linear equations for the generating functions $\xi_i = \Psi_l(\mathbf{R}_i(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D})$, for $i = 0, 1, \ldots, s$. The result follows by determining $\xi_0 = \Psi_l(\mathbf{C}(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D})$ by Cramer's Rule.

(2) We have $\mathbf{I}-\mathbf{H} = \mathbf{I}-\mathbf{Q}+\sum_{k=1}^{s}\mathbf{L}_k\mathbf{W}\mathbf{R}_k$ so that, taking determinants on both sides,

$$|\mathbf{I}-\mathbf{H}| = |\mathbf{I}-\mathbf{Q}| \cdot |\mathbf{I}+(\mathbf{I}-\mathbf{Q})^{-1}\sum_{k=1}^{s}\mathbf{L}_k\mathbf{W}\mathbf{R}_k|.$$

Thus, taking inverses and logs of both sides, we have

$$\log|\mathbf{I}-\mathbf{H}|^{-1} = \log|\mathbf{I}-\mathbf{Q}|^{-1} + \log|\mathbf{I}+\mathbf{E}\mathbf{F}^T|^{-1},$$

where $\mathbf{E}=[\mathbf{E}_1|\cdots|\mathbf{E}_s]$, $\mathbf{F}=[\mathbf{F}_1|\cdots|\mathbf{F}_s]$, in which $\mathbf{E}_i=(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{L}_i\mathbf{X}\mathbf{1}$, $\mathbf{F}_i=\mathbf{R}_i^T\mathbf{1}$, for $i=1,\ldots,s$, and $\mathbf{1}$ is the column vector of n ones. Thus

$$[\mathbf{F}^T\mathbf{E}]_{ij}=\operatorname{trace}\mathbf{R}_i(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{L}_j\mathbf{W}=\Psi_1(\mathbf{R}_i(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{L}_j),$$

from Lemma 3.2. But $|\mathbf{I}+\mathbf{E}\mathbf{F}^T|=|\mathbf{I}+\mathbf{F}^T\mathbf{E}|$ since $\mathbf{E}$ and $\mathbf{F}$ have the same dimensions, and $|\mathbf{I}+\mathbf{F}^T\mathbf{E}|=|\mathbf{M}|$ from the above and since the only non-zero element in the first column of $\mathbf{M}$ is $[\mathbf{M}]_{00}=1$. The result follows from the identity $e^{\operatorname{trace}\mathbf{A}}=|e^{\mathbf{A}}|$.

The above result is clearly useful only when generating functions of the form $\Psi_l(\mathbf{R}_i(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{L}_j)$ are easier to determine than $\Psi_l(\mathbf{C}(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D})$. This is true when $\mathbf{Q}$ is expressed in terms of $\mathbf{A}$ alone, and thus our strategy is to use $\mathbf{B}=\mathbf{W}-\mathbf{A}$ to express $\mathbf{H}$ in the form $\mathbf{Q}-\sum_{i=1}^{s}\mathbf{L}_i\mathbf{W}\mathbf{R}_i$, where $\mathbf{Q}$ and $\mathbf{L}_i$, $i=1,\ldots,s$ involve $\mathbf{A}$ alone. Using this strategy for $s=1$, we prove the maximal decomposition theorem (Theorem 2.1) below.

Proof of Theorem 2.1. For any bipartition $\{\pi_1,\pi_2\}$ of $\mathcal{N}_n^2$, each sequence in $\mathcal{N}^+$ has, as its pattern, a unique element of

$$\bigcup_{k\geq 0}\{(\varepsilon\,\dot{\cup}\,\pi_1\,\dot{\cup}\,\pi_1^2\,\dot{\cup}\cdots)\pi_2\}^k(\varepsilon\,\dot{\cup}\,\pi_1\,\dot{\cup}\,\pi_1^2\,\dot{\cup}\cdots).$$

Furthermore, the elements ε, π_1, $\pi_1^2,\ldots$ are separated from other occurrences of π_1 by π_2's, so that they represent the occurrence of maximal π_1-paths of lengths $1,2,3,\ldots$ in all sequences with the given pattern. Thus, from Proposition 3.1 and Lemma 3.2, the required generating functions are $\Psi_1(\mathbf{U})$ and $\Psi_2(\mathbf{U})$ where

$$\mathbf{U}=\sum_{k\geq 0}\{(f_1\mathbf{I}+f_2\mathbf{A}+f_3\mathbf{A}^2+\cdots)\mathbf{B}\}^k(f_1\mathbf{I}+f_2\mathbf{A}+f_3\mathbf{A}^2+\cdots)$$

and f_j records the occurrences of a maximal π_1-path of length j. Thus $\mathbf{U}=(\mathbf{I}-\mathbf{H})^{-1}\mathbf{D}$ where $\mathbf{H}=f(\mathbf{A})\mathbf{B}$, $\mathbf{D}=f(\mathbf{A})$ and $f(x)=f_1+f_2x+f_3x^2+\cdots$. Now $\mathbf{H}=-f(\mathbf{A})\mathbf{A}+f(\mathbf{A})\mathbf{W}$ from Proposition 3.1 and we may apply Theorem 4.1 with $s=1$, $\mathbf{Q}=-f(\mathbf{A})\mathbf{A}$, $\mathbf{L}_1=-f(\mathbf{A})$, $\mathbf{R}_1=\mathbf{I}$, $\mathbf{C}=\mathbf{I}$, $\mathbf{D}=f(\mathbf{A})$, $\mathbf{H}=f(\mathbf{A})\mathbf{B}$, so that $\mathbf{I}-\mathbf{Q}=F(\mathbf{A})$. The result follows from Lemma 3.2(4).

The final set of examples concerns the enumeration of sequences with pattern in $(\pi_1^p\pi_2^q)^*$. From Theorem 4.1, the solution will in general be a ratio of determinants. The case $p=q=1$ for permutations with π_1 equal to the set of rises is André's [1] generating function, $\sec x+\tan x$. We consider the case $p=q=2$ below. The general solution is given in [8].

EXAMPLE 4.2 ([11]). The generating function for sequences on $\mathcal{N}_n$ with pattern in the set $(\pi_1^2\pi_2^2)^*\pi_1^k$, $k=0,1$, with respect to type is, from Lemma 3.2,

$$\Psi_1\left(\sum_{i\geq 0}(\mathbf{A}^2\mathbf{B}^2)^i\mathbf{A}^k\right)=\Psi_1((\mathbf{I}-\mathbf{A}^2\mathbf{B}^2)^{-1}\mathbf{A}^k).$$

We expand $\mathbf{A}^2\mathbf{B}^2$ by casting out $\mathbf{B}$'s occurring to the left of $\mathbf{W}$'s, using $\mathbf{B}=\mathbf{W}-\mathbf{A}$. Thus $\mathbf{I}-\mathbf{A}^2\mathbf{B}^2=\mathbf{I}-\mathbf{A}^4-\mathbf{A}^2\mathbf{W}\mathbf{B}+\mathbf{A}^3\mathbf{W}$. In the notation of Theorem 4.1 we have $\mathbf{L}_0=\mathbf{0}$, $\mathbf{L}_1=-\mathbf{A}^2$, $\mathbf{L}_2=\mathbf{A}^3$, $\mathbf{R}_0=\mathbf{C}=\mathbf{I}$, $\mathbf{R}_1=\mathbf{B}$, $\mathbf{R}_2=\mathbf{I}$, $\mathbf{D}=\mathbf{A}^k$, $\mathbf{H}=\mathbf{A}^2\mathbf{B}^2$ and $\mathbf{Q}=\mathbf{A}^4$. Thus $\Psi_1((\mathbf{I}-\mathbf{Q})^{-1}\mathbf{A}^r)=G_{r+1}$ where $G_{r+1}=x^{r+1}(1-x^4)^{-1}\circ\boldsymbol{\gamma}$. Accordingly $|\mathbf{M}|=G_0^2-G_1G_3$, $|[\mathbf{M}:\mathbf{d}]_0|=G_0G_{k+1}-G_3G_{k+2}$ and the number of sequences of type $\mathbf{i}$ and shape in $(\pi_1^2\pi_2^2)^*\pi_1^k$ is

$$[\mathbf{x}^{\mathbf{i}}](G_0G_{k+1}-G_3G_{k+2})(G_0^2-G_1G_3)^{-1}$$

where $G_j=\sum_{l\geq 0}\gamma_{4l+j}(\pi_1)$.

EXAMPLE 4.3. From Lemma 3.2 and Theorem 4.1, using a similar argument to that of Example 4.2, we find that the number of permutations on $\mathcal{N}_n$ with m inversions and pattern in $(\pi_1^2\pi_2^2)^*\pi_1^k$, where π_1 is the set of rises, is

$$\left[q^m\frac{x^n}{n!_q}\right](G_0G_{k+1}-G_3G_{k+2})(G_0^2-G_1G_3)^{-1}$$

where

$$G_r=\sum_{l\geq 0}\frac{x^{4l+r}}{(4l+r)!_q}.$$

EXAMPLE 4.4 ([11]). Setting $q=1$ in Example 4.3, or applying Lemma 2.2(1) to Example 4.2, we obtain the number of permutations on $\mathcal{N}_n$ with pattern in $(\pi_1^2\pi_2^2)^*\pi_1^k$, π_1 the sets of rises, as

$$\left[\frac{x^n}{n!}\right]\frac{\tan x+\tanh x}{1+\sec x\ \mathrm{sech}\ x}\quad\text{for}\quad k=0$$

and

$$\left[\frac{x^n}{n!}\right]\frac{\tan x\tanh x}{1+\sec x\ \mathrm{sech}\ x}\quad\text{for}\quad k=1.$$

The case $k=0$ has been considered in Carlitz and Scoville [4], where a different form of solution has been given.

5. Extension to a tripartition

In the previous sections we have given some examples of the application of our method to the enumeration of sequences over a bipartition of $\mathcal{N}_n^2$. The method relied on our being able to replace $\mathcal{I}(\pi_2)$ by $\mathbf{W}-\mathcal{I}(\pi_1)$, where $\mathbf{W}$ allows a special multiplicative property (Lemma 3.2(3)). Thus the solution is expressed in terms of generating functions for $\mathcal{I}(\pi_1)$ only. In considering enumeration over a partition $\{\pi_1, \ldots, \pi_p\}$ of $\mathcal{N}_n^2$ with $p>2$ blocks, it is evident that we can use the same properties for Ψ_1. However, since we now have $\mathcal{I}(\pi_1)+\cdots+\mathcal{I}(\pi_p)=\mathbf{W}$, we may replace $\mathcal{I}(\pi_p)$ (say) by $\mathbf{W}-\mathcal{I}(\pi_1)-\cdots-\mathcal{I}(\pi_{p-1})$ and thus express the solution totally in terms of $\mathcal{I}(\pi_1), \ldots, \mathcal{I}(\pi_{p-1})$. In general, this is not a particularly useful reduction, but in certain instances when $p=3$ we can exploit the combinatorial interpretation of the reduced problem to give a solution, at least for the permutation case. Such an instance is given in the following example.

EXAMPLE 5.1 (Roselle [22]). We wish to determine the number, $c(n, t, u)$, of permutations on $\mathcal{N}_n$ with t successions and u rises. If $\{\pi_1, \pi_2\}$ is the bipartition of $\mathcal{N}_n^2$ with π_1 the set of rises, and π_3 is the set of successions then, since $\pi_3 \subseteq \pi_1$, $(\pi_1-\pi_3, \pi_2, \pi_3)$ is a tripartition of $\mathcal{N}_n^2$. Now each sequence in $\mathcal{N}_n^+$ has, as its pattern, a unique element of $\{\pi_1-\pi_3, \pi_2, \pi_3\}^*$. Furthermore, the occurrence of $\pi_1-\pi_3$ in the pattern produces one rise and no succession in the sequence, the occurrence of π_3 produces one rise and one succession, and the occurrence of π_2 produces neither. If rises and successions are recorded by the indeterminates r and s, the required number is given by $[r^u s^t x_1 \cdots x_n]\Phi$ where

$$\Phi = \Psi_1(\{\mathbf{I}-(r(\mathbf{A}-\mathbf{E})+\mathbf{B}+rs\mathbf{E})\}^{-1})$$

and $\mathbf{E}=\mathcal{I}(\pi_3)$. Thus $\Phi=\Psi_1((\mathbf{I}-\mathbf{H})^{-1})$, and substituting $\mathbf{B}=\mathbf{W}-\mathbf{A}$, we obtain $\mathbf{H}=\mathbf{Q}+\mathbf{W}$ where $\mathbf{Q}=r(s-1)\mathbf{E}+(r-1)\mathbf{A}$. Accordingly, from Theorem 4.1 we obtain

$$1+\Phi=(1-\Psi_1(\{\mathbf{I}-r(s-1)\mathbf{E}-(r-1)\mathbf{A}\}^{-1}))^{-1}.$$

Let $r(s-1)=v$ and $r-1=y$. Thus

$$\begin{aligned}\Psi_1((I-v\mathbf{E}-y\mathbf{A})^{-1}) &= \Psi_1(y^{-1}\{\mathbf{I}-y(\mathbf{I}-v\mathbf{E})^{-1}\mathbf{A}\}^{-1}y(\mathbf{I}-v\mathbf{E})^{-1})\\ &= y^{-1}\sum_{k\geq 1}\Psi_1(\{y(\mathbf{I}-v\mathbf{E})^{-1}\mathbf{A}\}^{k-1}y(\mathbf{I}-v\mathbf{E})^{-1}).\end{aligned}$$

But $\Psi_1(\{y(\mathbf{I}-v\mathbf{E})^{-1}\mathbf{A}\}^{k-1}y(\mathbf{I}-v\mathbf{E})^{-1})$ is the generating function for increasing sequences formed by joining k π_3-paths in increasing order. Each of these paths is enumerated by $y\sum_{j\geq 1} v^{j-1}\gamma_j(\pi_3)$. There are $k!$ ways of ordering k non-overlapping π_3-paths, only one of which produces an increasing sequence, so that $(k!)^{-1}(y\sum_{j\geq 1} v^{j-1}\gamma_j(\pi_3))^k$ gives all increasing paths formed in this way, as well as some other terms. Each of these terms corresponds to a set of π_3-paths with some overlapping, and thus contains a squared x_i for some $i=1, \ldots, n$. However we wish to determine $[x_1 \cdots x_n]\Phi$ so squared x_i's will make no contribution to the solution, and we may replace $\Psi_1((\mathbf{I}-v\mathbf{E}-y\mathbf{A})^{-1})$ by

$$\sum_{k\geq 1} (k!)^{-1}\left(y\sum_{j\geq 1} v^{j-1}\gamma_j(\pi_3)\right)^k.$$

Thus

$$c(n, t, u)=$$

$$[r^u s^t x_1\cdots x_n]\left\{1-(r-1)^{-1}\left(\exp\left\{(r-1)\sum_{k\geq 1}(r(s-1))^{k-1}\gamma_k(\pi_3)\right\}-1\right)\right\}^{-1};$$

and, using Lemma 2.2, we obtain finally

$$c(n, t, u)=[r^u s^t x^n]\sum_{j,k\geq 1}(r-1)^{k+1}r^{-j-1}j^k x^k\{1-r(s-1)x\}^{-k}.$$

6. Extension to other restrictions

The previous results have been concerned with sequences with patterns which place restrictions on pairs of adjacent elements. If we regard a sequence as labelling the vertices of a graph in predetermined order, with edges labelled with π_1, π_2, indicating from which block of $\mathcal{N}_n^2$ the pair of vertex labels must come, then our theory extends immediately. The incidence matrix for a pattern is defined in the same way and has the same additive and multiplicative properties, as long as an ordered pair of patterns is concatenated by identifying a unique pair of vertices. Consider patterns which are formed by concatenating the patterns $\mu_1, \ldots, \mu_6$ in Figure 1. In the pictorial representation, the sequence labels the pattern graph from left to right (the predetermined order).

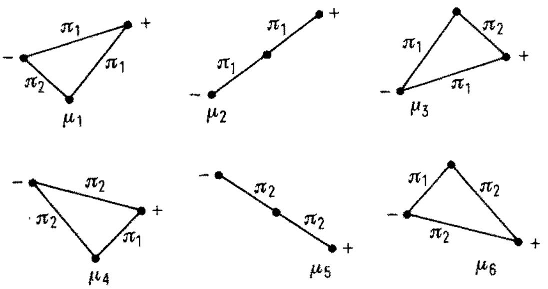

Figure 1

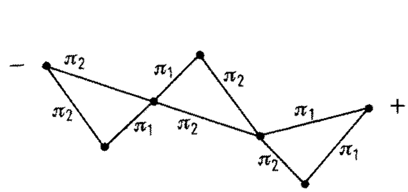

Figure 2

We adopt the convention that $\mu_i\mu_j$ denotes the pattern formed by identifying the vertex of μ_i marked with + with the vertex of μ_j marked with −. Figure 2 gives the pattern $\mu_4\mu_6\mu_1$.

By the conventions described above, the sequences 8167426 and 3235224 have the pattern $\mu_4\mu_6\mu_1$, where π_1 is equal to the set of rises. Note that the pattern of any sequence of length 3 is a unique element of $\{\mu_1, \ldots, \mu_6\}$.

EXAMPLE 6.1 ([14]). We determine the number $c(s, t, n)$ of permutations $\sigma_1 \cdots \sigma_{2n+1}$ on $\mathcal{N}_{2n+1}$ with s rises and t occurrences of $(\sigma_{2j-1}, \sigma_{2j+1})$ with $\sigma_{2j-1} < \sigma_{2j+1}$, $j = 1, \ldots, n$. Let π_1 be the set of rises, and each permutation σ of odd length has as its pattern a unique element of $\{\mu_1, \ldots, \mu_6\}^*$, where $\mu_1, \ldots, \mu_6$ are given in Figure 1. Let the indeterminate r record rises in σ and u record occurrences of $\sigma_{2j-1} < \sigma_{2j+1}$. Then, since the occurrence of μ_1 in the pattern of σ produces a single rise in σ and a single occurrence of $\sigma_{2j-1} < \sigma_{2j+1}$, we record the appearance of μ_1 in the pattern of σ with the monomial ru. Similarly the appearance of μ_2 is recorded by r^2u, μ_3 by ru, μ_4 by r, μ_5 by 1 and μ_6 by r. This is because the two end vertices of a pattern μ_i, $i = 1, \ldots, 6$, always appear in odd positions in a sequence with pattern formed by concatenating these patterns. Thus we have

$$c(s, t, n) = [r^s u^t x_1 \cdots x_{2n+1}]\Psi_1\{(\mathbf{I} - \mathbf{H})^{-1}\}$$

where $\mathbf{H} = ru\mathbf{M}_1 + r^2u\mathbf{M}_2 + ru\mathbf{M}_3 + r\mathbf{M}_4 + \mathbf{M}_5 + r\mathbf{M}_6$ and $\mathbf{M}_i = \mathcal{I}(\mu_i)$ for $i = 1, \ldots, 6$. Now clearly $\langle\mu_1\rangle \cap \langle\mu_4\rangle = \emptyset$ and $\langle\mu_1\rangle \cup \langle\mu_4\rangle = \langle\pi_2\pi_1\rangle$ so that $\mathcal{I}(\mu_1) + \mathcal{I}(\mu_4) = \mathcal{I}(\pi_2\pi_1)$, or $\mathbf{M}_1 + \mathbf{M}_4 = \mathbf{BA}$. Similarly $\mathbf{M}_3 + \mathbf{M}_6 = \mathbf{AB}$, and $\mathbf{M}_2 = \mathbf{A}^2$, $\mathbf{M}_5 = \mathbf{B}^2$ so that, eliminating $\mathbf{M}_4, \mathbf{M}_5, \mathbf{M}_6$, we obtain $\mathbf{H} = r(u-1)(\mathbf{M}_1 + r\mathbf{M}_2 + \mathbf{M}_3) + (r\mathbf{A} + \mathbf{B})^2$. But $\mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = \gamma_1(\pi_1)\mathbf{A}$ since $\langle\mu_1\rangle$, $\langle\mu_2\rangle$, $\langle\mu_3\rangle$ are disjoint sets whose union gives all sequences of length 3 whose terminal elements form a rise and whose interior element is arbitrary, and is thus enumerated by $\gamma_1(\pi_1)$. Eliminating $\mathbf{M}_1 + \mathbf{M}_3$ from $\mathbf{H}$ and using $\mathbf{M}_2 = \mathbf{A}^2$, we obtain

$$\mathbf{H} = \mathbf{Q} - \mathbf{L}_1\mathbf{W}\mathbf{R}_1 - \mathbf{L}_2\mathbf{W}\mathbf{R}_2$$

where $\mathbf{Q} = r(u-1)\gamma_1\mathbf{A} + (r-1)(ru-1)\mathbf{A}^2$, $\mathbf{L}_1 = \mathbf{I}$, $\mathbf{R}_1 = -(r-1)\mathbf{A}$, $\mathbf{L}_2 = r\mathbf{A} + \mathbf{B}$,

$\mathbf{R}_2 = -\mathbf{I}$. We may now apply Theorem 4.1 to the required generating function $\Psi_1\{(\mathbf{I}-\mathbf{H})^{-1}\}$. Specializing to permutations by Lemma 2.2(1) we obtain

$$c(s, t, n) = \left[r^s u^t \frac{x^{2n+1}}{(2n+1)!}\right] g(0)\{(1-g(1))^2 - g(0)(x+g(2))\}^{-1}$$

where

$$g(m) = (r-1)^m \sum_{k\geq 0} \sum_{i=0}^{k} \binom{k}{i} (r(u-1))^i ((r-1)(ru-1))^{k-i} \frac{x^{2k+m+1}}{(2k+m-i+1)!}.$$

7. Circular sequences and a logarithmic connection

Let s be a directed cycle on p vertices, embedded in the plane, such that an element of $\mathcal{N}_n$ is assigned to each vertex. Such a configuration is called a *circular sequence* of length p on $\mathcal{N}_n$. We represent a circular sequence by a linear sequence which begins with the label of an arbitrary vertex and then lists successive labels along the directed cycle. Accordingly 53214 and 21453 represent the same circular permutation on $\mathcal{N}_5$. Since edges are directed, we may define π_1-paths as before. For certain bipartitions there may be circular sequences each of whose pairs of consecutive elements belongs to π_1. These circular sequences are called *π_1-cycles.* If no π_1-cycles exist then π_1 is said to be *cycle-free*. For example the set of rises is cycle-free.

We enumerate circular permutations by combinatorially identifying the required number as the coefficient of $x_1 \cdots x_n$ in trace $\log(\mathbf{I}-\mathbf{H})^{-1}$. The solution follows immediately by applying Theorem 4.1(2), with an appropriate expansion

$$\mathbf{H} = \mathbf{Q} - \sum_{i=1}^{s} \mathbf{L}_i \mathbf{W} \mathbf{R}_i.$$

We may also enumerate circular sequences by using the cycle index polynomial for the cyclic group in conjunction with trace $\log(\mathbf{I}-\mathbf{H})^{-1}$. The circular version of the maximal decomposition theorem is the first example of this method.

THEOREM 7.1 ([9]). *Let $g_1, g_2, \ldots$ and $f_1, f_2, \ldots$ be commutative indeterminates, with $\mathbf{f} = (f_1, f_2, \ldots)$, $F(x) = 1 + \sum_{i\geq 1} f_i x^i$ and $\mathbf{m} = (m_1, m_2, \ldots)$. Then*

(1) *The number of circular permutations on $\mathcal{N}_n$ with m_i maximal π_1-paths of length $i \geq 1$, and t (either 0 or 1) π_1-cycles of length l is*

$$[x_1 \cdots x_n \mathbf{f}^{\mathbf{m}} g_l^t]\{\log(F^{-1} \circ \boldsymbol{\gamma})^{-1} + \psi\}$$

where $\psi = \text{trace } \log (F(\mathbf{A}))^{-1} + \sum_{i\geq 1} i^{-1} g_i \text{ trace } \mathbf{A}^i$, $\mathbf{A} = \mathcal{I}(\pi_1)$ *and* $\boldsymbol{\gamma} = (\gamma_1(\pi_1), \gamma_2(\pi_1), \ldots)$.

(2) *If* π_1 *is cycle-free then* $\psi = 0$.

Proof. (1) There are $[x_1 \cdots x_n]n^{-1}$ trace $\mathbf{A}^n$ circular permutations which are π_1-cycles, where we divide by n since each permutation is counted once in trace $\mathbf{A}^n$ for each of the n possible cyclic rotations. Each other circular permutation may be decomposed into k maximal π_1-paths, for some $k \geq 1$. The contribution of these permutations, recording a maximal π_1-path of length i by f_i, is $[x_1 \cdots x_n]k^{-1}$ trace $\{(f_1\mathbf{I} + f_2\mathbf{A} + \cdots)\mathbf{B}\}^k$, by considering one of the k possible starting points of some maximal π_1-path. We sum over $k \geq 1$ to obtain the solution as $[x_1 \cdots x_n \mathbf{f}^{\mathbf{m}} g_{\mathbf{i}}^{\mathbf{j}}]\Phi$ where

$$\Phi = \text{trace} \log (\mathbf{I} - f(\mathbf{A})\mathbf{B})^{-1} + \sum_{i\geq 1} i^{-1} g_i \text{ trace } \mathbf{A}^i$$

and $f(x) = f_1 + f_2 x + f_3 x^2 + \cdots$. But $f(\mathbf{A})\mathbf{B} = f(\mathbf{A})\mathbf{W} - f(\mathbf{A})\mathbf{A}$ and the result follows from Theorem 4.2(2) with $\mathbf{H} = f(\mathbf{A})\mathbf{B}$, $\mathbf{Q} = -f(\mathbf{A})\mathbf{A}$, $\mathbf{L}_1 = -f(\mathbf{A})$ and $\mathbf{R}_1 = \mathbf{I}$, since $\mathbf{I} - \mathbf{Q} = F(\mathbf{A})$.

(2) If π_1 is cycle-free then trace $\mathbf{A}^k = 0$ for all $k \geq 1$.

Accordingly, the generating functions for the linear and circular versions of the maximal decomposition theorem are very closely related. We shall refer to this striking relationship between the linear and circular versions of sequence enumeration problems as a *logarithmic connection.* Another example of this connection is given in the next example, where we consider a circular version of Example 4.4.

EXAMPLE 7.2 ([8]). The number of circular permutations on $\mathcal{N}_{4k}$ whose pattern consists of alternating pairs of π_1's and π_2's is $[x_1 \cdots x_{4k}]k^{-1} \text{trace} (\mathbf{A}^2\mathbf{B}^2)^k = [x_1 \cdots x_{4k}] \text{trace} \log (\mathbf{I} - \mathbf{A}^2\mathbf{B}^2)^{-1}$. If π_1 is the set of rises, then, by Theorem 4.2(2) and Example 4.2, the required number is $[x_1 \cdots x_{4k}] \log (G_0^2 - G_1 G_3)^{-1}$ where $G_j = \sum_{l\geq 0} \gamma_{4l+j}(\pi_1)$. Accordingly, by Lemma 2.2 the required number of permutations on $\mathcal{N}_n$ is

$$\left[\frac{x^n}{n!}\right] \log (\tfrac{1}{2} + \tfrac{1}{2} \cos x \cosh x)^{-1}.$$

Finally we calculate the Ménage numbers m_n, (Lucas [16]) the number of ways of seating n couples around a circular table so that no members of the same sex are adjacent and no man is adjacent to his wife.

EXAMPLE 7.3. If $\mathbf{D}=\mathbf{J}-\mathbf{I}$, $\mathbf{X}=\operatorname{diag}(x_1,\ldots,x_n)$, $\mathbf{Y}=\operatorname{diag}(y_1,\ldots,y_n)$ where $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ are commutative indeterminates then we have

$$m_n=[x_1\cdots x_n y_1\cdots y_n]\{\operatorname{trace}(\mathbf{XDYD})^n+\operatorname{trace}(\mathbf{YDXD})^n\}.$$

Here x_i and y_i record the occurrence of the man and woman in couple i. $\mathbf{X}$ and $\mathbf{Y}$ are separated by the matrix $\mathbf{D}$ to ensure that a man is not adjacent to his wife. They alternate to ensure that no members of the same sex are adjacent. We extract the coefficient of $x_1\cdots x_n y_1\cdots y_n$ so that each man and woman occurs once, and consider two cases since the arrangement may be rooted on a man or a woman. Thus $m_n=2n[x_1\cdots x_n y_1\cdots y_n]\operatorname{trace}\log(\mathbf{I}-\mathbf{XDYD})^{-1}$ where $\mathbf{XDYD}=\mathbf{XY}-\mathbf{XJY}+\mathbf{XDYJ}$. Accordingly, from Theorem 4.1(2) we have

$$[x_1\cdots x_n y_1\cdots y_n]\operatorname{trace}\log\{\mathbf{I}-\mathbf{XDYD}\}^{-1}$$
$$=[x_1\cdots x_n y_1\cdots y_n]\{\operatorname{trace}\log(\mathbf{I}-\mathbf{XY})^{-1}+\log|\mathbf{M}_1|^{-1}\}$$

where $|\mathbf{M}_1|=\begin{vmatrix}1+w & -vw\\ u & 1+w-uv\end{vmatrix}$, where $w=\operatorname{trace}\mathbf{XY}=x_1y_1+\cdots+x_ny_n$, $u=\operatorname{trace}\mathbf{X}=x_1+\cdots+x_n$, $v=\operatorname{trace}\mathbf{Y}=y_1+\cdots+y_n$ and $\mathbf{M}_1$ is obtained from $\mathbf{M}$ of Theorem 4.1(2) by deleting squared terms in x_i and y_i, $i=1,\ldots,n$, since we are only interested in extracting the coefficient of $x_1\cdots x_n y_1\cdots y_n$. Thus $m_n=2n[x_1\cdots x_n y_1\cdots y_n]\{w+\log((1+w)^2-uv)^{-1}\}$ so

$$m_n=2(n!)\sum_{k=0}^{n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!\text{ routinely.}$$

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