A CORRESPONDENCE BETWEEN PLANE PLANTED CHROMATIC TREES AND GENERALISED DERANGEMENTS

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§1. *The correspondence*

A tree in which adjacent vertices are of different colours is called a *chromatic tree.* If $\mathbf{L}$ is a matrix with element l_{ij} equal to the number of non-root vertices of colour i and degree j in a chromatic tree t, then $\mathbf{L}$ is called the *chromatic partition* of t.

We demonstrate the existence of a remarkable and tantalising correspondence between two apparently unrelated combinatorial sets, namely plane planted chromatic trees and generalised derangements. The derangement may, of course, be used to construct a graph in which no two vertices of the same colour are joined by an edge, but we cannot guarantee that this graph is in fact a tree. Indeed, we have been unable to give a combinatorial characterisation of this one-to-one correspondence.

The correspondence allows us to enumerate K-chromatic trees with given chromatic partition. This is a new result. The case $K = 2$ has been given previously by Tutte [4].

Throughout this paper we use the following notation: the number of vertices of colour i is $n_i = \sum_{j \geqslant 1} l_{ij}$; the sum of the out-degrees (edges are directed away from the root) of non-root vertices of colour i is $q_i = \sum_{j \geqslant 1} (j-1)l_{ij}$; the number of non-root vertices is $N+1 = n_1 + \ldots + n_K$, where K is the number of colours.

We now define the two combinatorial numbers which appear in the correspondence. Let $\chi_c^{(K)}(\mathbf{L})$ denote the number of K-chromatic plane planted trees with root colour c and chromatic partition $\mathbf{L}$. Let $\rho_c^{(K)}(\mathbf{n}, \mathbf{q})$ denote the number of permutations of q_i indistinguishable objects of type $1 \leqslant i \leqslant K$ in $n_i - 1 + \delta_{ic}$ positions of type $1 \leqslant i \leqslant K$, the remaining positions being of no type, such that no object is in a position of the same type. The following relationship holds between these two numbers (we delay the proof).

THEOREM 1.1.

$$\left(\prod_{i,j} l_{ij}!\right)\chi_c^{(K)}(\mathbf{L}) = \left(\prod_i q_i!\right)\rho_c^{(K)}(\mathbf{n}, \mathbf{q}).$$

Suppose, for example, that we consider 3-chromatic plane planted trees with root colour 1 on 8 non-root vertices. Suppose that there are 4 vertices of colour 1 and degree 1, 1 vertex of colour 2 and degree 1, 2 vertices of colour 2 and degree 3, and 1

Received 14 January, 1980; revised 7 May, 1980.

[BULL. LONDON MATH. SOC., 13 (1981), 28–32]

vertex of colour 3 and degree 4. Thus $n_1 = 4$, $n_2 = 3$, $n_3 = 1$, $q_1 = 0$, $q_2 = 4$, $q_3 = 3$ and $N = 7$. It follows that $\chi_1^{(3)}(\mathbf{L}) = 15$, and these are given in Figure 1.

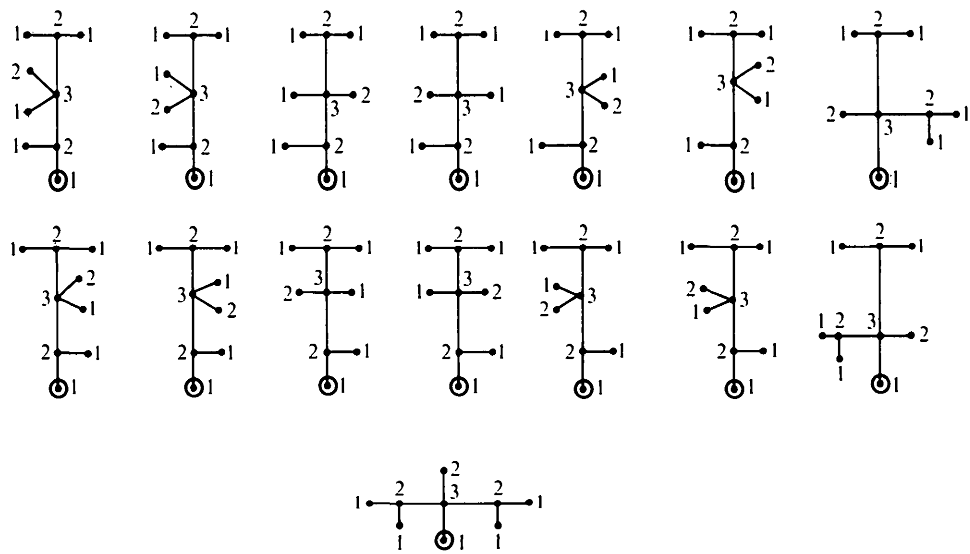

Fig. 1

For the derangements, there are 0, 4 and 3 objects of types 1, 2 and 3 respectively, and 4, 2 and 0 positions of type 1, 2 and 3 respectively. Let * denote a position of no type. Accordingly, the positions $1, 2, \ldots, 7$ have type $1, 1, 1, 1, 2, 2, *$. The derangements are accordingly 3222332, 2322332, 2232332, 2223332 and 2222333, so $\rho_1^{(3)}(\mathbf{n}, \mathbf{q}) = 5$.

Now $\prod_{i,j} l_{ij}! = 4!2!$ and $\prod_i q_i! = 4!3!$ and Theorem 1.1 is confirmed in this case.

In fact the left-hand side and the right-hand side of the equation given in Theorem 1.1 admit a combinatorial interpretation by supposing that vertices of the same colour are distinguishable and that objects of the same type are distinguishable. The two sides of the equation give the number of trees and generalised derangements under this convention. It is reasonable to expect that this one-to-one correspondence may be characterised combinatorially.

As a corollary we obtain the following correspondence between sequences with distinct adjacent elements and generalised derangements, a result previously given by Reilly [3].

COROLLARY 1.2. *The number of sequences over* $\{1, \ldots, K\}$ *beginning with* α, *ending with* β, *with* m_i *occurrences of* i, *for* $1 \leqslant i \leqslant K$, *and with distinct adjacent elements is* $\rho_\alpha^{(K)}(\mathbf{n}, \mathbf{q})$ *where* $n_i = m_i - \delta_{i\alpha}$ *and* $q_i = m_i - \delta_{i\alpha} - \delta_{i\beta}$, $1 \leqslant i \leqslant K$.

Proof. For sequences beginning with α and ending with β we set $c = \alpha$ and $l_{\beta 1} = 1$. Since there are m_i occurrences of i then $l_{i2} = m_i - \delta_{i\alpha} - \delta_{i\beta}$ for $1 \leqslant i \leqslant K$. Thus, from Theorem 1.1, $\chi_\alpha^{(K)}(\mathbf{L})$ counts sequences beginning with α and ending with

β, with distinct adjacent elements, and with m_i occurrences of i for $1 \leqslant i \leqslant K$. But $\prod_{i,j} l_{ij}! = \prod_i q_i!$ in this case, and the result follows.

We have been unable to obtain a combinatorial characterisation even in this strikingly simpler case.

For example, let $K = 3$, $m_1 = 3$, $m_2 = 3$, $n_3 = 2$, and $\alpha = \beta = 1$. Thus $n_1 = 2$, $n_2 = 3$, $n_3 = 2$, $q_1 = 1$, $q_2 = 3$, and $q_3 = 2$. Accordingly there are eight sequences satisfying Corollary 1.2, namely 12123231, 12132321, 12312321, 12321231, 12321321, 12323121, 13212321, 13232121. The eight derangements, with positions of type 11223* in that order are 223321, 223123, 221323, 223312, 323122, 321322, 231322, and 233122.

§2. *The proof of the correspondence*

The following notational conventions are adopted.

If $\mathbf{A}$ is a $K \times K$ matrix with elements a_{ij}, we write $\mathbf{A} = [a_{ij}]_{K \times K}$ and $a_{ij} = [\mathbf{A}]_{ij}$. Let $\mathbf{M} = [m_{ij}]_{K \times K}$ be a non-negative integer matrix. Then

$$\mathbf{A}^{\mathbf{M}} = \prod_{1 \leqslant i,j \leqslant K} a_{ij}^{m_{ij}}, \qquad \mathbf{M}! = \prod_{1 \leqslant i,j \leqslant K} m_{ij}!,$$

and $[{}^{j}_{\mathbf{i}}] = j!(\mathbf{i}!)^{-1}$, the multinomial coefficient, where $j = i_1 + \ldots + i_n$ and $\mathbf{i} = (i_1, \ldots, i_n)$. If $f(\mathbf{x})$ is a formal power series in $\mathbf{x} = (x_1, \ldots, x_n)$ then $[\mathbf{x}^{\mathbf{i}}]f(\mathbf{x})$ denotes the coefficient of $x_1^{i_1} \ldots x_n^{i_n}$ in $f(\mathbf{x})$, $[\mathbf{x}]$ denotes the operator $[x_1 \ldots x_n]$ and $Z_{\mathbf{x}} f(\mathbf{x}) = f(\mathbf{0})$.

To prove the correspondence we use the following specialisation of the multivariate Lagrange theorem.

THEOREM 2.1. (See Jackson and Goulden [2]). *Let $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_\kappa)$ and γ be formal power series in the indeterminates $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\kappa)$ and with no terms with negative exponents. Suppose that $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_\kappa)$ satisfies $\xi_i = \zeta_i \phi_i(\boldsymbol{\xi})$ for $i = 1, \ldots, \kappa$ and that $\phi_i(\boldsymbol{\xi})$ is independent of ξ_j for each $(i,j) \in \mathscr{S} \subseteq \mathscr{N}_\kappa^2$. Then*

$$[\boldsymbol{\zeta}^{\mathbf{v}}]\boldsymbol{\xi}^{\mathbf{r}} = (v_1 \ldots v_\kappa)^{-1} \sum_{\mu} \|\delta_{ij} v_i - \mu_{ij}\| \prod_{i=1}^{\kappa} ([\xi_1^{\mu_{i1}} \ldots \xi_\kappa^{\mu_{i\kappa}}]\phi_i^{v_i})$$

where the summation is over all non-negative integer $\kappa \times \kappa$ matrices such that $\sum_{i=1}^{\kappa} \mu_{ij} = v_j - r_j$, $j = 1, \ldots, \kappa$ and $\mu_{ij} = 0$ for each $(i,j) \in \mathscr{S}$.

Proof of Theorem 1.1. Let $\mathscr{C}_i$ be the set of plane planted K-chromatic ($K > 1$) trees with root colour i. Then we have immediately that

$$\dot{\bigcup_{k=0}^{\infty}} \dot{\bigcup_{\substack{j=1 \\ j \neq i}}^{K}} \mathscr{C}_j^k \cong \mathscr{C}_i. \tag{1}$$

by noting that the vertex adjacent to the root has degree $k+1$, and colour j for $k \geqslant 0$ and $j \neq i$.

Let $[\mathbf{X}]_{ij} = x_{ij}$ and $\mathbf{a} = (a_1, \ldots, a_K)$ and let

$$h_i(\mathbf{a}) = \sum_{\mathbf{L}} \chi_i^{(K)}(\mathbf{L})\mathbf{X}^{\mathbf{L}}\mathbf{a}^{\mathbf{n}}.$$

Thus, from (1) we have

$$h_i(\mathbf{a}) = \sum_{\substack{j=1 \\ j \neq i}}^{K} a_j g_j(h_j(\mathbf{a}))$$

where $g_i(\lambda) = \sum_{k \geqslant 1} x_{ik}\lambda^{k-1}$ for $i = 1, \ldots, K$. Let $\alpha_j = a_j g_j(h_j(\mathbf{a}))$, so $h_i(\mathbf{a}) = A - \alpha_i$ where $A = \alpha_1 + \ldots + \alpha_K$. Thus $\chi_c^{(K)}(\mathbf{L}) = [\mathbf{X}^{\mathbf{L}}\mathbf{a}^{\mathbf{n}}](A - \alpha_c)$, where $\alpha_j = a_j g_j(A - \alpha_j)$. Thus from Theorem 2.1 we have

$$[\mathbf{X}^{\mathbf{L}}\mathbf{a}^{\mathbf{n}}](A-\alpha_c)$$
$$= (n_1 \ldots n_K)^{-1} \sum_{\substack{r=1 \\ r \neq c}}^{K} \sum_{\mathbf{M}} \|\delta_{ij}n_i - m_{ij}\| \left\{ \prod_{i=1}^{K} [x_{i1}^{l_{i1}} x_{i2}^{l_{i2}} \ldots][\alpha_1^{m_{i1}} \ldots \alpha_K^{m_{iK}}] g_i^{n_i}(A - \alpha_i) \right\}$$

where the summation is over all $\mathbf{M}$ such that $\sum_{i=1}^{K} m_{ij} = n_j - \delta_{rj}$ and $m_{jj} = 0$ for $j = 1, \ldots, K$. Thus

$$[\mathbf{X}^{\mathbf{L}}\mathbf{a}^{\mathbf{n}}](A-\alpha_c) = (\mathbf{L}!)^{-1}(\mathbf{n}-\mathbf{1})! \sum_{\substack{r=1 \\ r \neq c}}^{K} \sum_{\mathbf{M}} \|\delta_{ij}n_i - m_{ij}\| \prod_{i=1}^{K} \begin{bmatrix} q_i \\ m_{i1}, \ldots, m_{iK} \end{bmatrix}$$

where the summation is over all $\mathbf{M}$ satisfying, as well, the further condition $\sum_{j=1}^{K} m_{ij} = q_i$. Let $\boldsymbol{\delta}_r = (\delta_{1r}, \ldots, \delta_{Kr})$. Accordingly

$$[\mathbf{X}^{\mathbf{L}}\mathbf{a}^{\mathbf{n}}](A-\alpha_c)$$
$$= \mathbf{q}!(\mathbf{n}-\mathbf{l})!(\mathbf{L}!)^{-1} \sum_{\substack{r=1 \\ r \neq c}}^{K} [\mathbf{u}^{\mathbf{q}}\mathbf{v}^{\mathbf{n}-\boldsymbol{\delta}_r}] \sum_{\substack{m_{ij} \geqslant 0 \\ \text{for } i \neq j}} (\mathbf{M}!)^{-1} \left\| \delta_{ij}\left(\delta_{rj} + \sum_{l=1}^{K} m_{lj}\right) - m_{ij} \right\| \prod_{1 \leqslant i,j \leqslant K} (u_i v_j)^{m_{ij}}$$

$$= \mathbf{q}!(\mathbf{n}-\mathbf{1})!(\mathbf{L}!)^{-1} \sum_{\substack{r=1 \\ r \neq c}}^{K} [\mathbf{u}^{\mathbf{q}}\mathbf{v}^{\mathbf{n}-\boldsymbol{\delta}_r}] \left\| \delta_{ij}\left(\delta_{rj} + \sum_{l=1}^{K} u_l v_j\right) - u_i v_j \right\| \exp\left\{ \sum_{j=1}^{K} v_j(u - u_j) \right\},$$

(since $\sum_{\mathbf{i} \geqslant \mathbf{0}} g(\mathbf{i})\frac{\mathbf{x}^{\mathbf{i}}}{\mathbf{i}!} = g(\mathbf{x}) \exp \sum_{j=1}^{n} x_j$, when $g(\mathbf{x})$ is multilinear), where $u = u_1 + \ldots + u_K$.

Noting that $\det(\mathbf{I}+\mathbf{A}) = 1+\operatorname{trace}\mathbf{A}$ for $\mathbf{A}$ with rank one, we have

$$[\mathbf{X}^{\mathbf{L}}\mathbf{a}^{\mathbf{n}}](A-\alpha_c) = \mathbf{q}!(\mathbf{n}-\mathbf{1})!(\mathbf{L}!)^{-1}[\mathbf{u}^{\mathbf{q}}\mathbf{v}^{\mathbf{n}-\mathbf{1}}] \sum_{\substack{r=1\\ r\neq c}}^{K} u_r u^{K-2} \exp\left\{\sum_{j=1}^{K} v_j(u-u_j)\right\}$$

so

$$\chi_c^{(K)}(\mathbf{L}) = \mathbf{q}!(\mathbf{L}!)^{-1}[\mathbf{u}^{\mathbf{q}}](u-u_c)^{n_c}u^{K-2}\prod_{\substack{j=1\\ j\neq c}}^{K}(u-u_j)^{n_j-1}.$$

But, by considering the exclusion of objects from each position we have

$$\rho_c^{(K)}(\mathbf{n},\mathbf{q}) = [\mathbf{u}^{\mathbf{q}}](u-u_c)^{n_c}u^{K-2}\prod_{\substack{j=1\\ j\neq c}}^{K}(u-u_j)^{n_j-1}$$

and the result follows.

By expanding the appropriate permanent (see Jackson [1]) for the generalised derangement, it follows from the correspondence that

$$\chi_c^{(K)}(\mathbf{L}) = (\mathbf{L}!)^{-1}\sum_{i=0}^{N}(N-i)![x^i]P(-x)$$

where $\mu(i) = n_i+\delta_{ic}-1$ for $i = 1,\dots,K$ and $P(x) = \prod_{i=1}^{K}\left(\sum_{j\geq 0}\binom{q_i}{j}\binom{\mu(i)}{j}j!x^j\right)$.

The above representation of $\chi_c^{(K)}(\mathbf{L})$ allows us to conclude that this number may be computed in time $O(M(\log M)^2)$ where M is the number of non-root vertices.

Acknowledgement. This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

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