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MAPS IN LOCALLY ORIENTABLE SURFACES AND INTEGRALS OVER REAL SYMMETRIC SURFACES

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ABSTRACT. The genus series for maps is the generating series for the number of rooted maps with a given number of vertices and faces of each degree, and a given number of edges. It captures topological information about surfaces, and appears in questions arising in statistical mechanics, topology, group rings, and certain aspects of free probability theory. An expression has been given previously for the genus series for maps in locally orientable surfaces in terms of zonal polynomials. The purpose of this paper is to derive an integral representation for the genus series. We then show how this can be used in conjunction with integration techniques to determine the genus series for monopoles in locally orientable surfaces. This complements the analogous result for monopoles in orientable surfaces previously obtained by Harer and Zagier. A conjecture, subsequently proved by Okounkov, is given for the evaluation of an expectation operator acting on the Jack symmetric function. It specialises to known results for Schur functions and zonal polynomials.

1**. Introduction.** Although the study of embeddings of graphs in surfaces is less well developed for locally orientable surfaces than it is for orientable surfaces, there are compelling algebraic and combinatorial reasons for studying them jointly. From the algebraic point of view it has been shown [5] that the genus series for maps in these two cases corresponds to the instances $b = 1, 0$ of

$$(1+b)t\frac{\partial}{\partial t}\log\Big(\sum_{\theta}\frac{t^{|\theta|}}{\langle J_\theta, J_\theta\rangle_{1+b}}J_\theta(\mathbf{x};1+b)\,J_\theta(\mathbf{y};1+b)\,J_\theta(\mathbf{z};1+b)\Big),$$

at $t = 1$, where θ is summed over all partitions (of integers), $J_\theta(\mathbf{x};\alpha)$ is the Jack symmetric function in the parameter α, and $\langle\ ,\ \rangle_\alpha$ is the usual inner product (see (2)) for Jack functions. From the combinatorial point of view it has been conjectured [6] that, in the above series, b marks a combinatorial statistic positively correlated with a departure from orientability. On the other hand, from the analytic point of view, a representation for the genus series for maps in orientable surfaces by means of an integral over Hermitian complex matrices has been given in [13]. The purpose of this paper is to derive an integral representation of the genus series for maps in locally orientable surfaces. This representation involves real symmetric matrices. These two account for two of the three finite dimensional real division algebras (reals, the complexes and the quaternions).

1.1. *Embeddings.* Throughout this paper we are concerned with 2-cell embeddings of graphs in locally orientable surfaces. Two embeddings of a graph are said to be *equivalent*

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if there is a homeomorphism of the surface that maps vertices to vertices, edges to edges, and preserves the orientation assigned to each edge. The embedded graph is called a *map*. Each edge has two ends, and two sides, so there are four *side-end positions*. A map is *rooted* by distinguishing a side-end position and, throughout, all maps are assumed to be rooted. The partition that lists the degrees of the vertices of the map is called the *vertex partition*. The partition that lists the degrees of the faces of the map is called the *face partition*, where the degree of a face is the number of edges that bound it. For further details the reader is referred to [26] and to the brief account given in [5] that is the starting point for this paper.

Rooted maps occur in a number of contexts. These include the analysis of surfaces [20], the determination of the partition function [1], the determination of the reduced Euler characteristic [8], the generalisation of the work of Farahat and Higman [2] and Macdonald to arbitrary structure constants of the class algebra of the symmetric group ring [4] and, more recently, the combinatorial investigation into free probability theory [27]. Since almost all maps have only the trivial automorphism [23], asymptotic results for maps with a large number of edges can be obtained from a study of rooted maps.

If $l_{\alpha,\beta}^{(n)}$ is the number of maps with n edges, face partition $\alpha \vdash 2n$ and vertex partition $\beta \vdash 2n$, then the *genus series* for maps in locally orientable surfaces is defined to be

$$M(\mathbf{x},\mathbf{y},z) = \sum_{n\geq 1}\sum_{\alpha,\beta\vdash 2n} l_{\alpha,\beta}^{(n)} x_\alpha y_\beta z^n, \tag{1}$$

where $x_1, x_2, \ldots, y_1, y_2, \ldots$ are commuting indeterminates, $\mathbf{x} = (x_1, x_2, \ldots)$, $\mathbf{y} = (y_1, y_2, \ldots)$, $\mathbf{x}_N = (x_1, \ldots, x_N)$, and if $\theta = (\theta_1, \ldots, \theta_m)$ is a partition, then x_θ denotes $x_{\theta_1}\cdots x_{\theta_m}$. We let $p(\mathbf{x}) = \big(p_1(\mathbf{x}), \ldots\big)$, where $p_i(\mathbf{x})$ is the i-th power sum symmetric function of $\mathbf{x}$. The genus of the surface is recoverable from the numbers of vertices, edges and faces, by the Euler-Poincaré theorem.

1.2. *The main result.* The main result of this paper, given as Theorem 1.1 below, is a representation for the genus series by an integral over the vector space W_N of all $N \times N$ real symmetric matrices $\mathbf{M} = [m_{i,j}]_{N\times N} \in W_N$, with measure $e^{-\operatorname{trace}\mathbf{M}^2/4} d\mathbf{M}$ where $d\mathbf{M} = \prod_{1\leq i\leq j\leq N} dm_{i,j}$. The *expectation operator* $\langle\ \rangle_{W_N}$ is defined formally for a polynomial function $f(\mathbf{M})$ of the entries of $\mathbf{M}$ by

$$\langle f(\mathbf{M})\rangle_{W_N} = \frac{\int_{W_N} f(\mathbf{M}) e^{-\frac{1}{4}\operatorname{trace}\mathbf{M}^2} d\mathbf{M}}{\int_{W_N} e^{-\frac{1}{4}\operatorname{trace}\mathbf{M}^2} d\mathbf{M}},$$

and its existence is ensured by the existence of $\int_{\mathbb{R}} e^{-t^2/2} dt$.

THEOREM 1.1. *Let* $\mathbf{X} = \operatorname{diag}(x_1, \ldots, x_N)$. *Then the integral representation for the genus series for maps in locally orientable surfaces is*

$$M\big(p(\mathbf{x}_N), \mathbf{y}, z\big) = 4z\frac{\partial}{\partial z}\log\Big\langle \exp\Big(\sum_{k\geq 1}\frac{\sqrt{z}^k}{2k} y_k \operatorname{trace}\,(\mathbf{X}\mathbf{M})^k\Big)\Big\rangle_{W_N}.$$

In Section 3.2 we give a proof of this result, using an algebraic expression for the genus series in terms of zonal polynomials that was given in [5], and a combinatorial construction for maps using pairings. The necessary background material for this is developed in Sections 2 and 3.1. In Section 4, using the integral representation, we derive an explicit expression for the genus series for monopoles in locally orientable surfaces, given as Theorem 4.2. Both of these results are new. The integral representation complements the result of Jackson [13] in the case of orientable surfaces, using integration over complex Hermitian matrices. The monopole expression complements the one that was obtained by Harer and Zagier [8] in the case of orientable surfaces.

An indirect consequence of this work is (Lemma 3.3(2)) that the expectation operator acts remarkably simply on zonal polynomials. An analogous result for Schur functions integrated over complex Hermitian matrices has been given a direct proof in Jackson [14]. These two results are evidence for Conjecture 3.4, involving an analogous expectation operator acting on Jack symmetric functions. This conjecture was subsequently proved by Okounkov [21]. Finally, another application of the integral representation given here appears in [3], where a closed form expression is obtained for the virtual Euler characteristic for the moduli spaces of real algebraic curves. Moreover, Conjecture 3.4 is used there to support in turn a further conjecture concerning the existence of moduli spaces whose virtual Euler characteristics interpolate between the cases of real and complex algebraic curves.

2**. Zonal polynomials and the genus series.** The approach that we adopt makes use of an expression, given as Theorem 2.1 below, that was derived in [5] for the genus series in terms of the zonal polynomials. To state the result the following terminology is needed.

Let $\Lambda_{\mathbb{Q}}$ denote the set of all symmetric functions in $\mathbf{x}$ of bounded degree, with coefficients that are rational. For a partition λ of n (written $\lambda \vdash n$), let $\mathcal{C}_\lambda$ be the conjugacy class of $\mathfrak{S}_n$ with natural index λ, let $l(\lambda)$ be the number of parts of λ, and let $|\lambda|$ be the sum of the parts of λ (so $|\lambda| = n$ in this case). Then, if α is an indeterminate,

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{l(\lambda)} \frac{|\mathcal{C}_\lambda|}{|\lambda|!} \delta_{\lambda,\mu} \tag{2}$$

can be extended bilinearly to an inner product on Λ. Let $\prec$ denote lexicographic ordering on the set of partitions. The zonal polynomials Z_λ are the unique polynomials [12] that are orthogonal with respect to this inner product with $\alpha = 2$, that satisfy the triangularity condition $[m_\mu]Z_\lambda = 0$ for $\lambda \prec \mu$, and that have the normalization $[m_{(1^n)}]Z_\lambda = n!$, where m_μ is a monomial symmetric function.

Let B_n denote the hyperoctahedral group embedded in $\mathfrak{S}_{2n}$ as the stabiliser of a prescribed matching, so $|B_n| = 2^n n!$. The double cosets of $\mathfrak{S}_{2n}$ by B_n are indexed naturally by partitions of n, and the double coset indexed in this way by $\lambda \vdash n$ is denoted by K_λ. Their size is readily determined to be

$$|K_\lambda| = 2^{2n-l(\lambda)} \, |B_{2n}| \;\; |\mathcal{C}_\lambda| \, . \tag{3}$$

The set of all formal sums of the elements of K_λ, for $\lambda \vdash 2n$, span a commutative subalgebra of $\mathbb{C}\mathfrak{S}_{2n}$ called the *double coset algebra*. Let

$$\psi^\lambda(\mu) = \sum_{\sigma \in K_\mu} \chi^{2\lambda}(\sigma),$$

where χ^λ is the character of the ordinary irreducible representation of $\mathfrak{S}_n$ indexed by $\lambda \vdash n$, and 2λ is the partition obtained from λ by multiplying each part by 2. The product of the hook lengths of λ is H_λ. The zonal polynomials can be expanded in the power sum basis by

$$Z_\theta = \frac{1}{|B_{2n}|} \sum_{\mu \vdash 2n} \psi^\theta(\mu) p_\mu. \tag{4}$$

For a concise account of the double coset algebra, and for the properties of this algebra that are used here the reader is referred to [7]. For properties of zonal polynomials that are used here, the reader is referred to the account on Jack symmetric functions given by Stanley [24], since specialisation to zonal polynomials is by setting the Jack parameter equal to 2. For terminology associated with symmetric functions the reader is directed to Macdonald [17].

The following result for the genus series can be obtained by specialising the expression for the genus series of hypermaps given in [5], where a *hypermap* is a map whose faces can be coloured with two colours such that no pair of faces with a common edge have the same colour. The specialisation is by constraining the faces of one colour to have degree two, and then contracting each such face to an edge.

THEOREM 2.1. *The genus series for maps on locally orientable surfaces is*

$$M\Big(p(\mathbf{x}), p(\mathbf{y}), z\Big) = 4z\frac{\partial}{\partial z} \log\Big(1 + \sum_{n\geq 1}\sum_{\theta \vdash 2n} \frac{\psi^\theta(2^n)}{2^{2n}(2n)!\, H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) z^n\Big).$$

Note that this expression for the genus series is with respect to the power sum symmetric function basis in both $\mathbf{x}$ and $\mathbf{y}$, whereas in Theorem 1.1 the genus series is expressed with respect to power sums in $\mathbf{x}_N$, but with respect to $\mathbf{y}$ itself. Although Theorem 2.1 could be expressed directly in terms of $\mathbf{x}$ and $\mathbf{y}$ using the expansion (4), the resulting expression would disguise the simplicity of the presentation in terms of zonal polynomials.

3**. The genus series and the combinatorics of the expectation operator.** In this section we determine the genus series for maps in locally orientable surfaces. The approach that we adopt makes use (in part (1) of Lemma 3.3) of an adaptation of the "fat-graph" construction [1, 10, 22] that involves interconnecting regions homeomorphic to open discs (local orientability) by "ribbons" that are allowed at most one "twist". This operation is represented algebraically by summing over all "pairings", and a result which expresses such a sum in terms of the expectation operator is given in Section 3.1 as Lemma 3.2. Our construction results in the determination of the required map cardinality in terms of a single undetermined scalar. This scalar is then determined, using the results of [5] outlined in Section 2 above, in terms of zonal polynomials. Standard symmetric function results are then used to explicitly determine the integral representation for the genus series.

3.1. *The expectation operator and pairings.* Let P_{2k} be the set of all permutations

$$\big(\rho_1(1), \rho_2(1), \ldots, \rho_1(k), \rho_2(k)\big)$$

of $\{1, \ldots, 2n\}$ such that

$$\rho_1(1) < \cdots < \rho_1(k), \quad \text{and} \quad \rho_1(j) < \rho_2(j), \quad \text{for } j = 1, \ldots, k.$$

An element of P_{2k} is called a *pairing*, and the pairs are $\{\rho_1(j), \rho_2(j)\}$. The number of pairings is

$$|P_{2k}| = \frac{(2k)!}{2^k k!}.$$

PROPOSITION 3.1. *Let* $\mathbf{U}$ *be an* $N \times N$ *symmetric matrix of (algebraically independent) indeterminates. Then*

$$\left\langle e^{\frac{1}{2}\operatorname{trace}\mathbf{UM}} \right\rangle_{W_N} = e^{\frac{1}{4}\operatorname{trace}\mathbf{U}^2}.$$

PROOF. Let $\mathbf{U}$ be an arbitrary real symmetric matrix. Then the change of variables $\mathbf{M} \longmapsto \mathbf{M} - \mathbf{U}$ gives

$$\begin{aligned}\int_{W_N} e^{-\frac{1}{4}\operatorname{trace}\mathbf{M}^2} d\mathbf{M} &= \int_{W_N} e^{-\frac{1}{4}\operatorname{trace}(\mathbf{M}-\mathbf{U})^2} d\mathbf{M} \\ &= e^{-\frac{1}{4}\operatorname{trace}\mathbf{U}^2} \int_{W_N} e^{\frac{1}{2}\operatorname{trace}\mathbf{UM}} e^{-\frac{1}{4}\operatorname{trace}\mathbf{M}^2} d\mathbf{M}\end{aligned}$$

since $\operatorname{trace}\mathbf{MU} = \operatorname{trace}\mathbf{UM}$. The result is established for all real symmetric matrices $\mathbf{U}$, and therefore for the case when the independent elements of $\mathbf{U}$ are algebraically independent indeterminates. ∎

In the next result, we show that a combinatorial sum over pairings naturally arises when applying the expectation operator to a monomial in the entries of $\mathbf{M}$. The result is an adaptation of Wick's Lemma, using an integral representation for the "propagator" given as the righthand side of part (2).

LEMMA 3.2. *Let* $1 \le r_1, s_1, \ldots, r_l, s_l \le N$. *Then*

(1) $\langle m_{r_1,s_1} \cdots m_{r_l,s_l} \rangle_{W_N} = 0, \quad l = 2k+1 \ge 1,$

(2) $\langle m_{r_1,s_1} m_{r_2,s_2} \rangle_{W_N} = \delta_{r_1,r_2}\delta_{s_1,s_2} + \delta_{r_1,s_2}\delta_{s_1,r_2},$

(3) $\langle m_{r_1,s_1} \cdots m_{r_l,s_l} \rangle_{W_N} = \sum_{P_{2k}} \prod_{j=1}^{k} \left\langle m_{r_{\omega_1(j)}, s_{\omega_1(j)}} m_{r_{\omega_2(j)}, s_{\omega_2(j)}} \right\rangle_{W_N}, \quad l = 2k \ge 2.$

PROOF. First we consider only the case $r_j \le s_j$ for $j = 1, \dots, l$. Then suppose that $r_j, s_j = a, b$ for $f_{a,b}$ choices of $j = 1 \dots, l$, for each $1 \le a, b \le N$.

Let $\mathbf{U} = [u_{i,j}]_{N\times N}$, and equate coefficients of $u_{r_1,s_1} \cdots u_{r_l,s_l}$ on both sides of Proposition 3.1. This gives

$$\frac{\langle m_{r_1,s_1} \cdots m_{r_l,s_l} \rangle_{W_N}}{2^{f_{1,1}+\cdots+f_{N,N}} \prod_{i,j=1}^{N} f_{i,j}!} = \left[u_{r_1,s_1} \cdots u_{r_l,s_l}\right] \frac{e^{\frac{1}{2}\sum_{1\le\alpha\le\beta\le N} u_{\alpha,\beta}^2}}{\sqrt{2}^{f_{1,1}+\cdots+f_{N,N}}}. \tag{5}$$

If l is odd, the coefficient on the righthand side of (5) is zero. This gives part (1) of the result when $r_j \le s_j$. When $l = 2$, equation (5) gives

$$\langle m_{r_1,s_1} m_{r_2,s_2} \rangle_{W_N} = \begin{cases} 2 & \text{if } r_1 = s_1 = r_2 = s_2, \\ 1 & \text{if } r_1 = r_2, s_1 = s_2, r_1 \ne s_1, \\ 0 & \text{otherwise.} \end{cases}$$

Part (2) of the result follows for $r_j \le s_j$.

When $l = 2k$, we have

$$\left[u_{r_1,s_1} \cdots u_{r_l,s_l}\right] e^{\frac{1}{2}\sum_{1\le\alpha\le\beta\le N} u_{\alpha,\beta}^2} = \frac{1}{\prod_{i,j=1}^{N} f_{i,j}!} \sum_{P_{2k}} \left\{ \prod_{j=1}^{k} \left[u_{r_{\rho_1(j)},s_{\rho_1(j)}} u_{r_{\rho_2(j)},s_{\rho_2(j)}}\right] e^{\frac{1}{2}\sum_{1\le\alpha\le\beta\le N} u_{\alpha,\beta}^2} \right\}.$$

Substituting this on the righthand side of (5) gives part (3) of the result for $r_j \le s_j$.

Now $\mathbf{M}$ is symmetric, so $m_{r_j,s_j} = m_{s_j,r_j}$ in all cases. Moreover, the expressions on the righthand sides of parts (1), (2) and (3) of the result are all symmetric in r_j, s_j, and the result follows for arbitrary r_j, s_j, for $j = 1, \dots, l$. ∎

3.2. *Expectation of symmetric functions and the genus series.* Consider a graph embedded in a locally orientable surface, so an open neighbourhood of each vertex is homeomorphic to an open disc. In each face, and parallel to the bounding edges, draw a line within distance ε of the edge. The two parallel lines on either side of an edge are called the *thick edge* corresponding to the edge of the graph. The segments of a thick edge associated with an edge in an open neighbourhood D_v containing the vertex v are called the *thick half-edge* associated with the edge incident with v, and we say that such a thick half-edge is incident with v. If the degree of v is k, there will be k thick half-edges incident with v, and D_v is called a *disc with k thick half-edges.* A *corner* of a face is a consecutive pair of thick half-edges in cyclic order at v, which is identified with the open region of D_v that is bounded by the two thick half-edges.

The next result involves a combinatorial construction using discs with half-edges attached, independent of the graph from which such discs are obtained in the above description. The result gives an evaluation of the expectation of symmetric functions of real symmetric matrices in terms of zonal polynomials. We use the notation $p_k(\mathbf{M}) = \operatorname{trace} \mathbf{M}^k$, for $k \ge 1$, and as usual $p_\theta(\mathbf{M}) = p_{\theta_1}(\mathbf{M}), \dots, p_{\theta_m}(\mathbf{M})$, for a partition $\theta = (\theta_1, \dots, \theta_m)$. Then $Z_\theta(\mathbf{M})$ is defined using the expansion (4) in terms of the power sums.

LEMMA 3.3. *If* $\nu \vdash 2n$, *then*

$$(1)\ \langle p_\nu(\mathbf{XM})\rangle_{W_N} = \frac{1}{|K_\nu|}\sum_{\theta\vdash 2n}\frac{1}{H_{2\theta}}\psi^\theta(2^n)\psi^\theta(\nu)Z_\theta(\mathbf{x}_N),$$

$$(2)\ \langle Z_\nu(\mathbf{XM})\rangle_{W_N} = \frac{\psi^\nu(2^n)}{2^{2n}(2n)!}Z_\nu(\mathbf{x}_N).$$

PROOF. (1) Suppose the partition $\nu = (\nu_1, \ldots, \nu_m) \vdash 2n$ has m parts, and consider the canonical permutation

$$\sigma = (12\cdots\nu_1)(\nu_1+1\cdots\nu_1+\nu_2)\cdots$$

with m cycles and cycle-type ν (so $\sigma \in C_\nu$). Let D_{ν_i} be a disc with ν_i thick half-edges, $i = 1, \ldots, m$. Associate one of these with each disjoint cycle of σ as follows. The k-cycle $\big(i\ \sigma(i)\ \cdots\ \sigma^{k-1}(i)\big)$ gives a disc with k thick half-edges and the corners labelled $i, \sigma(i), \ldots, \sigma^{k-1}(i)$ in cyclic order in the clockwise circulation of the vertex. Thus $\{j, \sigma(j)\}$ is associated with a unique thick half-edge for each $j = 1, \ldots, 2n$.

Now consider the effect of taking all pairings of the $2n$ thick half-edges, and for each pair $\big(i, \sigma(i)\big)$ and $\big(j, \sigma(j)\big)$ of thick half-edges in such a pairing, connecting them in either of the following two ways:

(1) the i side of the first thick half-edge is connected to the j side of the second and the $\sigma(i)$ side of the first is connected to the $\sigma(j)$ side of the second;
(2) the i side of the first thick half-edge is connected to the $\sigma(j)$ side of the second and the $\sigma(i)$ side of the first is joined to the j side of the second.

In all cases the connected thick half-edges give a thick edge joining the vertices corresponding to the discs.

Let A be the set constructed by this procedure. Each member of A corresponds to a collection of rooted maps in locally orientable surfaces, with vertex distribution ν, when taken together over all connected components. The multiplicity with which each collection of maps occurs will be determined indirectly below.

However, first we refine the construction to give a set A^c as follows. Assign colours $1, \ldots, N$, without condition, to each of the $2n$ corners, and suppose that the corner labelled i receives colour c_i, for $i = 1, \ldots, 2n$. The thick half-edges are paired and connected as before, with the additional condition that sides which are connected must have the same colour, in all cases. The resulting elements of A^c are identified simply as elements of A with coloured faces, since the colouring condition forces every corner on a face to have the same colour (which is thus the colour of the face). But parts (2) and (3) of Lemma 3.2 together imply that the number of elements in A^c, for each fixed $c_1, \ldots, c_{2n}$, is

$$\Big\langle \prod_{i=1}^{2n} m_{c_i, c_{\sigma(i)}} \Big\rangle_{W_N}.$$

Now in this expression c_i is the colour of the face containing the corner labelled i, so we conclude that the generating series for A^c, with faces of degree j marked by $p_j(\mathbf{x}_N)$ for

$j \geq 1$ is

$$\sum_{1\leq c_1,\ldots,c_{2n}\leq N} \left\langle \prod_{i=1}^{2n} x_{c_i} m_{c_i,c_{\sigma(i)}} \right\rangle_{W_N} = \langle p_\nu(\mathbf{XM}) \rangle_{W_N}.$$

On the other hand, A^c is, up to a multiplicity to be determined, an unordered collection of rooted maps. From Theorem 2.1, the genus series for maps, up to a multiplicity that depends on the number of edges, is

$$\log\left(1 + \sum_{n\geq 1}\sum_{\theta\vdash 2n} \frac{\psi^\theta(2^n)}{2^{2n}(2n)!\,H_{2\theta}} Z_\theta(\mathbf{x})\, Z_\theta(\mathbf{y}) z^n\right),$$

so the generating series for A^c is, up to a multiplicity that depends on the vertex partition,

$$1 + \sum_{n\geq 1}\sum_{\theta\vdash 2n} \frac{\psi^\theta(2^n)}{2^{2n}(2n)!\,H_{2\theta}} Z_\theta(\mathbf{x}) Z_\theta(\mathbf{y}) z^n.$$

Then

$$\sum_{1\leq c_1,\ldots,c_{2n}\leq N} \left\langle \prod_{i=1}^{2n} x_{c_i} m_{c_i,c_{\sigma(i)}} \right\rangle_{W_N},$$

up to a multiplicity that depends on ν, is equal to

$$\sum_{\theta\vdash 2n} \frac{1}{H_{2\theta}} \psi^\theta(2^n)\psi^\theta(\nu) Z_\theta(\mathbf{x}_N),$$

so we conclude that

$$\langle p_\nu(\mathbf{XM}) \rangle_{W_N} = \alpha_\nu \sum_{\theta\vdash 2n} \frac{1}{H_{2\theta}} \psi^\theta(2^n)\, \psi^\theta(\nu)\, Z_\theta(\mathbf{x}_N), \tag{6}$$

where α_ν is a constant depending only on ν.

To determine α_ν we equate coefficients of x_1^{2n} on each side of (6). For the lefthand side of (6), we obtain

$$\left[x_1^{2n}\right] \langle p_\nu(\mathbf{XM}) \rangle_{W_N} = \frac{\int_{\mathbb{R}} m_{1,1}^{2n} e^{-m_{1,1}^2/4} dm_{1,1}}{\int_{\mathbb{R}} e^{-m_{1,1}^2/4} dm_{1,1}} = \frac{(2n)!}{n!},$$

using integration by parts. For the righthand side, note that (see [24], p. 80)

$$\frac{1}{H_{2\theta}} Z_\theta(1, 0, \ldots) = \begin{cases} 0 & \text{for } \theta \neq (2n), \\ \frac{1}{|B_{2n}|} & \text{for } \theta = (2n) \end{cases}$$

and

$$\psi^{(2n)}(\mu) = \sum_{\sigma\in K_\mu} \chi^{(4n)}(\sigma) = |K_\mu|.$$

Thus for the righthand side of (6) we obtain

$$\begin{aligned}\left[x_1^{2n}\right]\alpha_\nu \sum_{\theta\vdash 2n}\frac{1}{H_{2\theta}}\psi^\theta(2^n)\psi^\theta(\nu)\,Z_\theta(\mathbf{x}_N) &= \alpha_\nu \sum_{\theta\vdash 2n}\frac{1}{H_{2\theta}}\psi^\theta(2^n)\,\psi^\theta(\nu)\;Z_\theta(1,0,\ldots,0)\\ &= \alpha_\nu \frac{1}{|B_{2n}|}\psi^{(2n)}(2^n)\;\psi^{(2n)}(\nu)\\ &= \alpha_\nu \frac{1}{|B_{2n}|}|K_{(2^n)}|\;|K_\nu|.\end{aligned}$$

Equating the coefficients from the two sides gives

$$\alpha_\nu = \frac{|B_{2n}|\,(2n)!}{|K_{(2^n)}|\,|K_\nu|\,n!} = \frac{1}{|K_\nu|}$$

from (3) and the result now follows from (6).

(2) From (4) and part (1) of this result,

$$\begin{aligned}\langle Z_\theta(\mathbf{XM})\rangle_{W_N} &= \frac{1}{|B_{2n}|}\sum_{\mu\vdash 2n}\psi^\theta(\mu)\,\langle p_\mu(\mathbf{XM})\rangle_{W_N}\\ &= \frac{1}{|B_{2n}|}\sum_{\alpha\vdash 2n}\frac{\psi^\alpha(2^n)}{H_{2\alpha}}Z_\alpha(\mathbf{x}_N)\sum_{\mu\vdash 2n}\frac{1}{|K_\mu|}\psi^\theta(\mu)\psi^\alpha(\mu).\end{aligned}$$

But the character sums ψ^θ satisfy the orthogonality relation

$$\frac{1}{H_{2\alpha}}\sum_{\mu\vdash 2n}\frac{1}{|K_\mu|}\psi^\theta(\mu)\psi^\alpha(\mu) = \delta_{\alpha,\theta}$$

and this gives the result directly. ∎

We can now prove the main result, which gives an integral representation for the genus series for maps in locally orientable surfaces.

Proof of Theorem 1.1. Let $M\big(p(\mathbf{x}_N),\mathbf{y},z\big)$ be temporarily denoted by M_N. Then from Theorem 2.1 and (4), with $\mathbf{x}$ replaced by $\mathbf{x}_N$ and $p(\mathbf{y})$ by $\mathbf{y}$, we get

$$\begin{aligned}M_N &= 4z\frac{\partial}{\partial z}\log\Big(1+\sum_{n\geq 1}\frac{z^n}{|B_{2n}|^2}\sum_{\mu\vdash 2n}\mathbf{y}_\mu\sum_{\theta\vdash 2n}\frac{\psi^\theta(2^n)\psi^\theta(\mu)}{H_{2\theta}}Z_\theta(\mathbf{x}_N)\Big)\\ &= 4z\frac{\partial}{\partial z}\log\Big(1+\sum_{n\geq 1}\frac{z^n}{|B_{2n}|^2}\sum_{\mu\vdash 2n}|K_\mu|\,\mathbf{y}_\mu\;\langle p_\mu(\mathbf{XM})\rangle_{W_N}\Big),\end{aligned}$$

from part (1) of Lemma 3.3. Let $\mu=(1^{a_1}2^{a_2}\cdots)$ so $a_1+2a_2+\cdots=2n$. But from (3), the size of the double coset indexed by μ is $|K_\mu| = |B_{2n}|\,|C_\mu|\,2^{2n-l(\mu)}$ so

$$M_N = 4z\frac{\partial}{\partial z}\log\left(1+\sum_{n\geq 1}z^n\sum_{\substack{a_1,a_2,\ldots\geq 0\\ a_1+2a_2+\cdots=2n}}\left\langle\prod_{j\geq 1}\frac{1}{a_j!}\left(\frac{1}{2j}y_j\operatorname{trace}(\mathbf{XM})^j\right)^{a_j}\right\rangle_{W_N}\right).$$

From Lemma 3.2(1), the terms of odd degree in $\mathbf{M}$ contribute zero to the sum, and the result follows. ∎

The expectation of the zonal polynomial given in part (2) of Lemma 3.3 can be made particularly striking, by reexpressing the righthand side to obtain

$$\langle Z_\theta(\mathbf{XM})\rangle_{W_N} = Z_\theta(\mathbf{x}_N)\Big([p_{(2^n)}]Z_\theta\Big). \tag{7}$$

The simplicity of this result suggests that it might be possible to prove it directly using analytic properties of the zonal polynomials, which, together with Theorem 2.1, would give a proof of the integral representation of the genus series (the main theorem) that avoids the combinatorial and topological constructions used in the above proof. Indeed, there is an analogous result [14] associated with orientable surfaces, namely

$$\langle s_\theta(\mathbf{XM})\rangle_{V_N} = s_\theta(\mathbf{x}_N)\Big([p_{(2^n)}]s_\theta\Big), \tag{8}$$

where s_θ is the Schur symmetric function (orthonormal with respect to the inner product $\langle\ ,\ \rangle_1$), and the expectation operator $\langle\ \ \rangle_{V_N}$ is defined by

$$\langle p_\theta(\mathbf{M})\rangle_{V_N} = \frac{\int_{V_N} p_\theta(\mathbf{M})e^{-\frac{1}{2}\operatorname{trace}\mathbf{M}^2}d\mathbf{M}}{\int_{V_N} e^{-\frac{1}{2}\operatorname{trace}\mathbf{M}^2}d\mathbf{M}}$$

with $d\mathbf{M} = (\prod_{j<k}\Im dm_{j,k})(\prod_{j\le k}\Re dm_{j,k})$, and V_N is the vector space of all $N\times N$ Hermitian complex matrices. Macdonald [18] has obtained such a proof using the orthogonality result of James [15] for integrating zonal polynomials over positive definite matrices.

Note that (8) is given in [14] only for the case $\mathbf{X} = \mathbf{I}$, but the proof given there can be adapted easily to arbitrary $\mathbf{X}$. The introduction of the matrix $\mathbf{X}$ in this way into the results of this section combinatorially allows us to mark the degrees of faces. It also makes an interesting departure from, for example, the model of Kontsevich [16], who considers integrals over complex Hermitian matrices with an arbitrary matrix Λ introduced to modify the weight in the expectation operator from $e^{-\operatorname{trace}\mathbf{M}^2/2}$ to $e^{-\operatorname{trace}\Lambda\mathbf{M}^2/2}$.

Results (7) and (8) suggest that there may be a comparable result for Jack functions. For this purpose we introduce the expectation operator

$$\langle f(\lambda)\rangle_{\mathbb{R}^N} = \frac{\int_{\mathbb{R}^N}|V(\lambda)|^{2\gamma}e^{-\frac{\gamma}{2}p_2}f(\lambda)\,d\lambda}{\int_{\mathbb{R}^N}|V(\lambda)|^{2\gamma}e^{-\frac{\gamma}{2}p_2}\,d\lambda}.$$

Then we make the following conjecture.

CONJECTURE 3.4.

$$\langle J_\theta(\lambda;1/\gamma)\rangle_{\mathbb{R}^N} = J_\theta(1_N;1/\gamma)\,([p_2^m]J_\theta),$$

where 1_N is the vector with N 1's, and $\theta\vdash 2m$.

This correctly specialises to (7) and (8) through the Weyl integration theorems to diagonalise the families of matrices W_N and V_N, respectively. In addition, we have confirmed this conjecture computationally for $N = 4, \theta\vdash 6$, with $\gamma = 2$ and 3. The integration was reduced to the known moments of the normal distribution by expanding the Vandermonde determinant and the Jack function into monomials.

This conjecture was subsequently proved by Okounkov [21].

4**. The genus series for monopoles.** A *monopole* is a rooted map with a single vertex. The *genus series* for *monopoles* in locally orientable surfaces with n edges, for $n \geq 0$, is

$$F_n(u) = \sum_{k\geq 1} f_{n,k} u^k$$

where $f_{n,k}$ is the number of monopoles in locally orientable surfaces with n edges and k faces. Since the set of all maps with n edges is finite, $F_n(u)$ is a polynomial in u. In this section we determine the genus series for monopoles by applying Theorem 1.1 and then explicitly carrying out the integration over real symmetric matrices, taking advantage of the polynomiality and using the following transformation. This result is obtained by means of the orthogonal group, the diagonalising group for W_N. For a proof, see, for example, [9].

PROPOSITION 4.1. *Let* $\lambda = (\lambda_1, \ldots, \lambda_N)$*, and* $V(\lambda) = \prod_{1\leq i<j\leq N}(\lambda_j - \lambda_i)$ *be the Vandermonde determinant. Then for a polynomial* g*,*

$$\left\langle g\big(p(\mathbf{M})\big)\right\rangle_{W_N} = \frac{\int_{\mathbb{R}^N} |V(\lambda)| e^{-p_2/4} g(p)\, d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)| e^{-p_2/4}\, d\lambda},$$

where p_k *denotes* $p_k(\lambda)$*.*

The absolute value of the Vandermonde determinant in this result presents problems, but these are surmounted in the monopole case below by a number of appeals to symmetry. In the proof we make extensive use of techniques that have been employed in mathematical physics; a good source for such techniques is Mehta [19]. In particular, the series $\phi_j(x)$ that appears in the proof is a "wave function", but is used here simply as a convenience. Hermite polynomials also arise, since they are closely related to wave functions; for general properties of the Hermite polynomials see Szegö [25]. The result that is obtained is of interest in its own right, since it is the counterpart of the monopole series in the orientable case obtained by Harer and Zagier [8] (see also [11, 13, 16, 22]), in a study of singularities on orientable surfaces.

Some notation is needed. We write $\mathbf{A} = [a_{i,j}]_{m,n}$ to mean that $\mathbf{A}$ is a block matrix whose (i, j)-block is the matrix $a_{i,j}$, with indexing $i = 1, \ldots, m$, and $j = 1, \ldots, n$. If $a_{i,j}$ is a 1×1 matrix then we write it as a scalar.

THEOREM 4.2. *The genus series* $F_n(u)$ *for monopoles in locally orientable surfaces with* n *edges, for* $n \geq 0$*, is*

$$n! \sum_{k=0}^{n} 2^{2n-k} \sum_{r=0}^{n} \binom{n-\frac{1}{2}}{n-r}\binom{k+r-1}{k}\binom{\frac{1}{2}(u-1)}{r} + \frac{(2n)!}{2^n n!}\sum_{k=0}^{n} 2^k \binom{n}{k}\binom{u-1}{k+1}.$$

PROOF. From Theorem 1.1 we obtain immediately

$$\sum_{\alpha\vdash 2n} l^{(n)}_{\alpha,(2n)}\, p_\alpha(\mathbf{x}_N) = [y_{2n} z^n] M\big(p(\mathbf{x}_N), \mathbf{y}, z\big) = \left\langle \text{trace } (\mathbf{XM})^{2n} \right\rangle_{W_N}.$$

Thus, replacing the x_i's by 1, we obtain $F_n(N) = \left\langle \operatorname{trace} \mathbf{M}^{2n} \right\rangle_{W_N}$, and applying Proposition 4.1 gives

$$F_n(N) = \frac{\int_{\mathbb{R}^N} |V(\lambda)| e^{-p_2/4} p_{2n}\, d\lambda}{\int_{\mathbb{R}^N} |V(\lambda)| e^{-p_2/4}\, d\lambda}.$$

It is convenient to change variables in these integrals by $\lambda_i \longmapsto \sqrt{2}\lambda_i$ for $i = 1, \ldots, N$, so

$$F_n(N) = 2^n N \frac{I_n(N)}{I_0(N)} \tag{9}$$

where

$$I_n(N) = \int_{\mathbb{R}^N} |V(\lambda)| e^{-p_2/2} p_{2n}\, d\lambda, \tag{10}$$

and, in this context, $p_0 = N$. But $F_n(N)$ is a polynomial in N, so it is sufficient to consider only the case where $N = 2m$, to obtain the series as a polynomial in m, and then to replace m formally in this by $N/2$. Because of polynomiality, the resulting expression holds for all N, and thus N can be replaced by the indeterminate u.

The first part of our strategy in determining $I_n(2m)$ is to introduce a set of polynomials that are orthogonal with respect to the measure. Thus consider the *Hermite polynomials* $\left(H_j(x)\right)_{j\geq 0}$, defined by the recurrence equation $H_{j+1} = 2xH_j - 2jH_{j-1}$ for $j \geq 0$, with initial conditions $H_0 = 1, H_{-1} = 0$. These are orthogonal with respect to the measure $e^{-x^2}dx$ on $\mathbb{R}$. Then

$$\phi_i(x) = \frac{H_i(x)e^{-x^2/2}}{\sqrt{a_i}}, \quad i \geq 0,$$

where $a_i = 2^i i! \sqrt{\pi}$, for $i \geq 0$, satisfy the orthonormality relation

$$\int_{\mathbb{R}} \phi_i(x)\phi_j(x)\, dx = \delta_{i,j}, \quad i, j \geq 0.$$

Let

$$\Phi_i(y) = \int_{-\infty}^{y} \phi_i(x)\, dx,$$

and note that $\phi_i(\pm\infty) = \Phi_i(-\infty) = 0$. We now re-express the Vandermonde determinant in terms of the submatrices

$$\mathbf{v}_i(x) = \begin{bmatrix} \Phi_i(x) \\ \phi_i(x) \end{bmatrix} \quad \text{and} \quad \mathbf{v}_i'(x) = \begin{bmatrix} \phi_i(x) \\ \phi_i'(x) \end{bmatrix},$$

and subsequent operations on matrices fully respect the partitioning of matrices into these submatrices. First, note that $2^{-j}H_j(x)$ is a monic polynomial of degree j in x, so

$$\begin{aligned} V(\lambda)e^{-p_2/2} &= \det\left[\lambda_j^{i-1}\right]_{2m,2m} e^{-p_2/2} \\ &= \det\left[2^{-(i-1)}H_{i-1}(\lambda_j)\right]_{2m,2m} e^{-p_2/2} \\ &= \left(\prod_{i=1}^{2m} \frac{\sqrt{a_{i-1}}}{2^{i-1}}\right) \det\left[\phi_{i-1}(\lambda_j)\right]_{2m,2m}. \end{aligned}$$

But

$$\phi_{j+1}(x) = -\sqrt{\frac{2}{j+1}}\,\phi_j'(x) + \sqrt{\frac{j}{j+1}}\,\phi_{j-1}(x) \tag{11}$$

for $j \geq 0$, and $\phi_{-1}(x) = 0$. Then

$$\begin{aligned}\det\left[\phi_{i-1}(\lambda_j)\right]_{2m,2m} &= \det\begin{bmatrix}\phi_{2i-2}(\lambda_j)\\ \phi_{2i-1}(\lambda_j)\end{bmatrix}_{m,2m}\\ &= (-1)^m\sqrt{\prod_{i=1}^{m}\frac{2}{2i-1}}\,\det\left[\mathbf{v}_{2i-2}'(\lambda_j)\right]_{m,2m}\end{aligned}$$

where we have substituted (11) into the even numbered rows, and used row operations. Thus

$$V(\lambda)e^{-p_2/2} = d_{2m}\det\left[\mathbf{v}_{2i-2}'(\lambda_j)\right]_{m,2m}, \tag{12}$$

where

$$d_{2m} = (-1)^m 4^{m(1-m)}\sqrt{\frac{m!}{(2m)!}\prod_{i=1}^{2m}a_{i-1}}.$$

Now we consider the integration. First we take advantage of symmetry to restrict the region of integration to the canonical cone

$$R_{2m} = \left\{(\lambda_1,\ldots,\lambda_{2m}) : \lambda_1 < \cdots < \lambda_{2m}\right\}.$$

Since the integrand of (10) is a symmetric function of λ, we thus obtain

$$I_n(2m) = (2m)!\int_{R_{2m}}|V(\lambda)|e^{-p_2/2}p_{2n}\,d\lambda.$$

Let $p_k^{(e)} = \lambda_2^k + \lambda_4^k + \cdots + \lambda_{2m}^k$ and $p_k^{(o)} = \lambda_1^k + \lambda_3^k + \cdots + \lambda_{2m-1}^k$, so $p_k = p_k^{(o)} + p_k^{(e)}$. Then

$$\int_{R_{2m}}|V(\lambda)|e^{-p_2/2}p_{2n}^{(o)}\,d\lambda = \int_{R_{2m}}|V(\lambda)|e^{-p_2/2}p_{2n}^{(e)}\,d\lambda$$

by the change of variables $\lambda_i \longmapsto -\lambda_{2m+1-i}$ for $i = 1,\ldots,2m$. From this additional symmetry and the fact that $|V(\lambda)| = V(\lambda)$ for $\lambda \in R_{2m}$, we obtain

$$\begin{aligned}I_{2n}(2m) &= 2(2m)!\int_{R_{2m}}V(\lambda)e^{-p_2/2}p_{2n}^{(e)}\,d\lambda,\\ &= 2(2m)!\,d_{2m}E_{m,n},\end{aligned} \tag{13}$$

where

$$E_{m,n} = \int_{R_{2m}}\det\left[\mathbf{v}_{2i-2}'(\lambda_j)\right]_{m,2m}p_{2n}^{(e)}\,d\lambda,$$

from (12). Now we integrate over "alternate variables": by integrating over λ_{2j-1} from λ_{2j-2} to λ_{2j}, with the convention that $\lambda_0 = -\infty$, and then setting $\mu_i = \lambda_{2i}$, for $i = 1,\ldots,m$ and $\mu = (\mu_1,\ldots,\mu_m)$, and denoting $p_k(\mu)$ by p_k, we obtain

$$\begin{aligned}E_{m,n} &= \int_{R_m}\det\left[\mathbf{v}_{2i-2}(\mu_j) - \mathbf{v}_{2i-2}(\mu_{j-1}), \mathbf{v}_{2i-2}'(\mu_j)\right]_{m,m}p_{2n}\,d\mu\\ &= \int_{R_m}\det\left[\mathbf{v}_{2i-2}(\mu_j), \mathbf{v}_{2i-2}'(\mu_j)\right]_{m,m}p_{2n}\,d\mu,\end{aligned}$$

where the second equality is by column operations within the matrix. But this integrand is a symmetric function of μ, so resymmetrising, but now with m variables, gives

$$\begin{aligned} E_{m,n} &= \frac{1}{m!}\int_{\mathbb{R}^m} \det\left[\mathbf{v}_{2i-2}(\mu_j), \mathbf{v}'_{2i-2}(\mu_j)\right]_{m,m} p_{2n}\, d\mu \\ &= \frac{1}{(m-1)!}\int_{\mathbb{R}^m} \det\left[\mathbf{v}_{2i-2}(\mu_j), \mathbf{v}'_{2i-2}(\mu_j)\right]_{m,m} \mu_m^{2n}\, d\mu. \end{aligned} \tag{14}$$

The matrix $[\mathbf{v}_{2i-2}(\mu_j), \mathbf{v}'_{2i-2}(\mu_j)]_{m,m}$ is such that columns $2j-1$ and $2j$ involve the same variable. We therefore carry out m simultaneous Laplace expansions of the determinant, one for each of the paired columns. There are three types of 2×2 submatrices that arise in consequence, namely $[\mathbf{v}_i(x), \mathbf{v}_k(x)]^t$, $[\mathbf{v}'_i(x), \mathbf{v}'_k(x)]^t$, and $[\mathbf{v}_i(x), \mathbf{v}'_k(x)]^t$. Now, integrating an odd function over $\mathbb{R}$ gives 0, so

$$\int_{\mathbb{R}} \det\left[\mathbf{v}_i(x), \mathbf{v}_k(x)\right]^t dx = \int_{\mathbb{R}} \det\left[\mathbf{v}'_i(x), \mathbf{v}'_k(x)\right]^t dx = 0,$$

while by the orthonormality relation for the $\phi_i(x)$ and integration by parts,

$$\int_{\mathbb{R}} \det\left[\mathbf{v}_i(x), \mathbf{v}'_k(x)\right]^t dx = -2\delta_{i,k}.$$

Thus the Laplace expansions give

$$E_{m,n} = (-2)^{m-1}\sum_{i=1}^{m}\int_{\mathbb{R}}\left(\Phi_{2i-2}(x)\phi'_{2i-2}(x) - \phi^2_{2i-2}(x)\right)x^{2n}\,dx. \tag{15}$$

The integral can be simplified in the following way. Integrating (11) yields

$$\sqrt{j+1}\,\Phi_{j+1} = -\sqrt{2}\;\phi_j + \sqrt{j}\;\Phi_{j-1}, \quad j \geq 0, \tag{16}$$

where $\Phi_{-1} = 0$. But substituting for ϕ'_{2i-2} by means of (11) and then using (16), we get

$$\begin{aligned} \Phi_{2i-2}\phi'_{2i-2} &= \Phi_{2i-2}\left(-\sqrt{\tfrac{1}{2}(2i-1)}\;\phi_{2i-1} + \sqrt{\tfrac{1}{2}(2i-2)}\;\phi_{2i-3}\right) \\ &= \begin{cases} -\phi^2_{2i-3} - \sqrt{\tfrac{1}{2}(2i-1)}\;\Phi_{2i-2}\phi_{2i-1} + \sqrt{\tfrac{1}{2}(2i-3)}\;\Phi_{2i-4}\phi_{2i-3}, & i \geq 2, \\ -\frac{1}{\sqrt{2}}\Phi_0\Phi_1, & i = 1, \end{cases} \end{aligned}$$

to give a telescoping sum from which it follows that

$$\sum_{i=1}^{m}\left(\phi^2_{2i-2} - \Phi_{2i-2}\phi'_{2i-2}\right) = \sqrt{\tfrac{1}{2}(2m-1)}\Phi_{2m-2}\phi_{2m-1} + \sum_{j=0}^{2m-2}\phi_j^2.$$

We now return to the determination of $F_n(N)$. It follows from (9) and (13) that $F_n(2m) = 2^n(2m)E_{m,n}/E_{m,0}$. But $E_{m,0} = (-2)^m m$, from (15), by orthonormality and integration by parts. Thus, from (15) and the telescoping sum above,

$$F_n(2m) = 2^n(I_{m,n} + K_{m,n}), \tag{17}$$

where

$$I_{m,n} = \sqrt{\tfrac{1}{2}(2m-1)}\int_{\mathbb{R}} x^{2n}\Phi_{2m-2}(x)\phi_{2m-1}(x)\,dx, \quad K_{m,n} = \sum_{j=0}^{2m-2}\int_{\mathbb{R}}\phi_j^2(x)x^{2n}\,dx.$$

The proof is concluded by making use of classical expansions associated with Hermite polynomials, which are listed as they are applied, for completeness. The two integrals are considered in turn. First,

$$\begin{aligned} I_{m,n} &= \frac{1}{2a_{2m-2}}\int_{\mathbb{R}}\int_{-\infty}^{y} y^{2n}H_{2m-1}(y)H_{2m-2}(x)e^{-x^2/2}e^{-y^2/2}\,dx\,dy \\ &= \frac{1}{2a_{2m-2}}\int_{\mathbb{R}} H_{2m-2}(x)e^{-x^2/2}\int_x^{\infty} y^{2n}H_{2m-1}(y)e^{-y^2/2}\,dy\,dx \end{aligned} \tag{18}$$

by reversing the order of integration. But it is readily checked that, for $k \geq 1$ and odd,

$$H_k(x) = \sum_{j=0}^{(k-1)/2} b_{j,k}x^{2j+1}, \text{ where } b_{j,k} = \frac{(-1)^{(k-1-2j)/2}2^{2j+1}k!}{\left(\frac{1}{2}(k-1-2j)\right)!\,(2j+1)!},$$

while, for $k \geq 0$,

$$x^k = \sum_{l=0}^{\lfloor k/2\rfloor} c_{l,k}H_{k-2l}(x), \text{ where } c_{l,k} = \frac{k!}{2^k l!\,(k-2l)!},$$

and, for $i \geq 1$ and odd,

$$\int_x^{\infty} z^i e^{-z^2/2}\,dz = \sum_{s=0}^{(i-1)/2} d_{s,i}\,x^{i-2s-1}e^{-x^2/2}, \text{ where } d_{s,i} = (i-1)(i-3)\cdots(i+1-2s).$$

Now apply these expansions to evaluate expression (18) as follows: first expand $H_{2m-1}(y)$ in powers of y using the b's, then carry out the inner integration, giving d's, and finally express the resulting powers of x in terms of the H's using the c's. Orthonormality of the ϕ's then gives

$$I_{m,n} = \tfrac{1}{2}\sum_{j=0}^{m-1} b_{j,2m-1}\sum_{s=0}^{n+j} d_{s,2n+2j+1}c_{l,2n+2j-2s}$$

where $n+j-s-l = m-1$, and $0 \leq l \leq n+j-s$. Now transform the summation variables with $a = m-j-1$ and $b = j-s-m+n+1$, with the assumption that $m > n \geq 0$, to obtain

$$\begin{aligned} I_{m,n} &= \tfrac{1}{2}\sum_{\substack{a,b\geq 0\\ a+b\leq n}} b_{m-a-1,2m-1}\,c_{b,2b+2m-2}\,d_{n-a-b,2m+2n-2a-1} \\ &= n!\sum_{\substack{a,b\geq 0\\ a+b\leq n}}(-2)^{n-a-b}\binom{\frac{1}{2}-m}{b}\binom{m-\frac{1}{2}}{a}\binom{a-m}{n}, \end{aligned}$$

on rearrangement of the series. Thus we have

$$I_{m,n} = n!\sum_{k=0}^{n}(-2)^{n-k}\alpha_{n,k}(m-\tfrac{1}{2}) \tag{19}$$

where

$$\alpha_{n,k}(p) = \sum_{\substack{a,b\geq 0\\ a+b=k}} \binom{p}{a}\binom{-p}{b}\binom{a-\frac{1}{2}-p}{n},$$

for $n, k \geq 0$. Then

$$\begin{aligned}\sum_{n,k\geq 0} \alpha_{n,k}(p)x^n y^k &= (1+y)^{-p}(1+x)^{-\frac{1}{2}-p}\Big(1+y(1+x)\Big)^p \\ &= (1+x)^{-\frac{1}{2}}\left(1-\frac{x}{(1+x)(1+y)}\right)^p \\ &= \sum_{r\geq 0}\binom{p}{r}(-x)^r(1+x)^{-r-\frac{1}{2}}(1+y)^{-r},\end{aligned}$$

so, evaluating the coefficient of $x^n y^k$, we obtain

$$\alpha_{n,k}(p) = (-1)^{n+k}\sum_{r=0}^{n}\binom{n-\frac{1}{2}}{n-r}\binom{k+r-1}{k}\binom{p}{r},$$

and substituting this into (19) gives, finally

$$I_{m,n} = n!\sum_{k=0}^{n}2^{n-k}\sum_{r=0}^{n}\binom{n-\frac{1}{2}}{n-r}\binom{k+r-1}{r}\binom{m-\frac{1}{2}}{r}.$$

For the second integral $K_{m,n}$ we use the expansion, for $j \geq 0$,

$$H_j^2(x) = \sum_{k=0}^{j} e_{k,j}H_{2j}(x), \quad \text{where} \quad e_{k,j} = \frac{2^{j-k}j!}{k!}\binom{j}{k},$$

so

$$K_{m,n} = \sum_{j=0}^{2m-2}\sum_{k=0}^{j}\frac{a_{2k}}{a_j}e_{k,j}c_{l,2n},$$

where $n - l = k$, $0 \leq l \leq n$, whence

$$K_{m,n} = \frac{(2n)!}{2^{2n}n!}\sum_{k=0}^{n}2^k\binom{n}{k}\sum_{j=k}^{2m-2}\binom{j}{k} = \frac{(2n)!}{2^{2n}n!}\sum_{k=0}^{n}2^k\binom{n}{k}\binom{2m-1}{k+1}.$$

The result now follows from (17), by combining the two evaluated integrals $I_{m,n}$ and $K_{m,n}$ and by the polynomiality of the result in m, so m can be replaced by $u/2$ where u is an indeterminate. ■

For example the result for $n = 0, 1, 2$ produces the expressions $F_0(u) = u, F_1(u) = u + u^2, F_2(u) = 5u + 5u^2 + 2u^3$, so there are, for example, 5 maps in locally orientable surfaces with one vertex, 2 edges, and 1 face. In fact, we can be more specific than this since the genus series for monopoles in orientable surfaces with n edges obtained in [8] is

$$G_n(u) = \frac{(2n)!}{2^n n!}\sum_{k=0}^{n}2^k\binom{n}{k}\binom{u}{k+1}.$$

But locally orientable surfaces include both orientable and nonorientable surfaces. It follows that the genus series for monopoles in nonorientable surfaces with n edges is $F_n(u) - G_n(u)$. For example, we have $G_2(u) = u + 2u^3$, so, for orientable surfaces, there are 2 monopoles with 2 edges and 3 faces (they are in the *sphere*), and 1 with 2 edges and 1 face (it is in the *torus*). Moreover, $F_2(u) - G_2(u) = 4u + 5u^2$. Thus, for nonorientable surfaces, there are 4 monopoles with 2 edges and 1 face (these are therefore in the *Klein bottle*) and 5 monopoles with 2 edges and 2 faces (these are therefore in the *projective plane*).

The approach that has been used in the above result can be applied to determine the genus series for other classes of maps. For example, in the case of dipoles, which are maps with two vertices of equal degree, the integration proceeds as in the above result for monopoles until we reach (14). At this stage, there is a slightly modified constant outside, and the monomial μ_m^{2n} is replaced by $\mu_{m-1}^n\mu_m^n$. Thus in the ensuing Laplace expansion, the last two paired columns are treated specially, which results in an iterated sum of integrals which are quartic in the ϕ's, in place of (15). We have been unable to simplify the resulting multiple summations in an attractive manner.

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