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# THE ENUMERATION OF GENERALISED ALTERNATING SUBSETS WITH CONGRUENCES

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We show that a number of problems involving the enumeration of alternating subsets of integers may be solved by a direct method as special cases of the enumeration of configurations which we have termed $(\boldsymbol{l}, \boldsymbol{q})$-sequences.

## 1. Introduction and notational conventions

In this paper we consider a class of enumeration problems which have received attention recently. It contains a number of well-known problems including the Terquem Problem, the Skolem Problem and their various generalisations. Each of these may be obtained by specialising a configuration, defined in Section 2, called an $(\boldsymbol{l}, \boldsymbol{q})$-sequence. Special cases are treated in Section 3.

We denote the sum of the elements of a vector $\boldsymbol{x}$ by $s(\boldsymbol{x})$. The integer part of a real number $y$ is denoted by $[y]$. An empty sum is conventionally zero. A *sequence* over $\mathbb{Z}_N^+=\{1,\ldots,N\}$ is an element of $\bigcup_{i=0}^{\infty}(\mathbb{Z}_N^+)^i$ and $\mathbb{Z}_N=\mathbb{Z}_N^+\cup\{0\}$. The coefficient of $x^n$ in the Laurent series expansion of $f(x)$ at the origin is denoted by $[x^n]f(x)$. Throughout this paper

$$\boldsymbol{l}=(l_1,\ldots,l_n),\qquad \boldsymbol{q}=(q_1,\ldots,q_n),\qquad \boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m),\qquad \boldsymbol{k}=(k_1,\ldots,k_m)$$

are sets of integers where $0\leq l_i<q_i$, $q_i\geq 1$ for $i=1,2,\ldots,n$, and $\alpha_i\geq 1$, $k_i\geq 0$ for $i=1,2,\ldots,m$. Finally, $\mathbf{1}_m$ is the $m$-tuple of all one's.

## 2. The enumeration of $(\boldsymbol{l}, \boldsymbol{q})$-sequences

We begin by defining an $(\boldsymbol{l}, \boldsymbol{q})$-sequence over $\mathbb{Z}_N^+$.

**Definition 2.1**

(1) An $(\boldsymbol{l}, \boldsymbol{q})$-*sequence* is a sequence $(\sigma_1, \dots, \sigma_n)$ over $\mathbf{Z}_N^+$ such that

(i) $\sigma_1 < \sigma_2 < \cdots < \sigma_n$,

(ii) $\sigma_1 \equiv (1 + l_1) \bmod q_1$.

(iii) $\sigma_j - \sigma_{j-1} \equiv (1 + l_j) \bmod q_j$, for $j = 2, 3, \dots, n$.

(2) $\phi_N(\boldsymbol{l}, \boldsymbol{q})$ is the number of such sequences.

The following lemma is the main one, from which all of the subsequent results are derived by various specialisations.

**Lemma 2.2.** *The generating function* $\Phi(x)$ for $\{\phi_N(\boldsymbol{l}, \boldsymbol{q}) : N \geq 0\}$ *is*

$$\Phi(x) = \sum_{N \geq 0} \phi_N(\boldsymbol{l}, \boldsymbol{q}) x^N = x^{s(\boldsymbol{l})+n}(1-x)^{-1} \prod_{i=1}^{n} (1 - x^{q_i})^{-1}.$$

**Proof.** Let $(\sigma_1, \dots, \sigma_n)$ be an $(\boldsymbol{l}, \boldsymbol{q})$-sequence over $\mathbf{Z}_N^+$. Then for $j = 1, 2, \dots, n$ we have $\sigma_j = j + \sum_{i=1}^{j} l_i + \sum_{i=1}^{j} \lambda_i q_i$ where $\lambda_1, \dots, \lambda_n$ are non-negative integers. Thus

$$\phi_N(\boldsymbol{l}, \boldsymbol{q}) = \left| \left\{ (\lambda_1, \dots, \lambda_n) : \sum_{i=1}^{n} \lambda_i q_i \leq N - n - s(\boldsymbol{l}) \right\} \right|$$

so

$$\phi_N(\boldsymbol{l}, \boldsymbol{q}) = \sum_{j=0}^{N-n-s(\boldsymbol{l})} [x^j] \prod_{i=1}^{n} (1 - x^{q_i})^{-1}$$

and the lemma follows.

There are many specialisations of this lemma, each relying on the ease with which $s(\boldsymbol{l})$ may be determined, and on the ease with which the coefficient may be extracted. The following is general enough for the present context.

**Corollary 2.3.** *Let p be a positive integer. Then*

$$\phi_N(\boldsymbol{l}, p\mathbf{1}_n) = \binom{[(N + n(p-1) - s(\boldsymbol{l}))/p]}{n}.$$

**Proof.** From Lemma 2.2. we have

$$\phi_N(\boldsymbol{l}, p\mathbf{1}_n) = [x^{N-n-s(\boldsymbol{l})}](1-x^p)^{-(n+1)} \sum_{i=0}^{p-1} x^i = [x^{N-n-s(\boldsymbol{l})}] \sum_{i=0}^{p-1} \sum_{j=0}^{\infty} \binom{n+j}{n} x^{jp+i}$$

and the result follows.

The quantity $s(\boldsymbol{l})$ may be evaluated in several circumstances. The following definition furnishes a context which is general enough for present purposes.

**Definition 2.4.** Let $(\sigma_1, \ldots, \sigma_n)$ be an arbitrary $(\boldsymbol{l}, p\mathbf{1}_n)$-sequence over $Z_N^+$ and let

$$R_j = \left\{ \sigma_i : 1 + \sum_{k=1}^{j-1} \alpha_k \leq i \leq \sum_{k=1}^{j} \alpha_k, \qquad j = 1, \ldots, m \right\}$$

where

$$s(\boldsymbol{\alpha}) = n, \quad \text{and} \quad 0 \leq k_i < p \quad \text{for} \quad i = 1, 2, \ldots, m.$$

(1) An $(\boldsymbol{\alpha}, \boldsymbol{k}, p)$-*sequence* over $Z_N^+$ is an $(\boldsymbol{l}, p\mathbf{1}_n)$-*sequence* such that

(i) If $a \in R_1$, then $a \equiv (1 + k_1) \bmod p$,

(ii) If $a, b \in R_i$, then $b - a \equiv 0 \bmod p$ for each $i$, $1 \leq i \leq m$,

(iii) if $a \in R_i$ and $b \in R_{i+1}$ then $b - a \equiv (1 + k_{i+1}) \bmod p$, for each $i$, $1 \leq i \leq m-1$.

(2) $\theta_N(\boldsymbol{\alpha}, \boldsymbol{k}, p)$ is the number of such sequences

The enumeration of these sequences is given in the following corollary.

**Corollary 2.5.**

$$\theta_N(\boldsymbol{\alpha}, \boldsymbol{k}, p) = \binom{[\{N + (p-1)m - s(\boldsymbol{k})\}/p]}{s(\boldsymbol{\alpha})}$$

**Proof.** Let $l_{t(j)} = k_j$ where $t(j) = 1 + \sum_{i=1}^{j-1} \alpha_i$ and $l_i = p - 1$ where $1 + \sum_{k=1}^{j-1} \alpha_k < i \leq \sum_{k=1}^{j} \alpha_k$ and $j = 1, 2, \ldots, m$. This identifies $(\boldsymbol{l}, p\mathbf{1}_n)$-sequences over $\mathbb{Z}_N^+$ with $(\boldsymbol{\alpha}, \boldsymbol{k}, p)$-sequences over $\mathbb{Z}_N^+$.

Thus

$$s(\boldsymbol{l}) = \sum_{j=1}^{m} k_j + (p-1) \sum_{i=1}^{m} (\alpha_i - 1) = (p-1)(s(\boldsymbol{\alpha}) - m) + s(\boldsymbol{k})$$

and the result follows from Corollary 2.3.

## 3. Applications of $(\boldsymbol{\alpha}, \boldsymbol{k}, p)$-sequences

We now obtain several specialisations of $(\boldsymbol{\alpha}, \boldsymbol{k}, p)$-sequences.

**Corollary 3.1.** (The Moser and Abramson [1] generalisation of the Terquem Problem). *The number of sequences* $(a_1, \ldots, a_m)$ *such that* $1 \leq a_1 < a_2 < \cdots < a_m \leq N$ *and*

$$a_1 \equiv (1 + k_1) \bmod p,$$
$$a_j \equiv (a_{j-1} + 1 + k_j) \bmod p, \qquad j = 2, 3, \ldots, m,$$

is

$$\binom{[\{N + (p-1)m - s(\boldsymbol{k})\}/p]}{m}$$

**Proof.** Let $\alpha = 1_m$, so $s(\alpha) = m$. The result follows from Definition 2.4 and Corollary 2.5.

The Skolem Problem [1, 2] is obtained with $k = 0$, while the Terquem Problem [1, 2, 6] is obtained from this with $p = 2$.

**Corollary 3.2.** *The number of sequences $(a_1, \ldots, a_n)$, $n = s(\alpha)$, such that $1 \leq a_1 < \cdots < a_n \leq N$ in which the first $\alpha_1$ elements have the same parity, the next $\alpha_2$ have opposite parity and so on, alternating parities between successive blocks containing $\alpha_j$ elements for $j = 1, \ldots, m$ is*

$$\binom{[\frac{1}{2}(N+m)]}{n} + \binom{[\frac{1}{2}(N+m-1)]}{n}.$$

**Proof.** There are two cases, namely $a_1 \equiv 1 \pmod 2$ and $a_1 \equiv 0 \pmod 2$. For the first, let $k = 0$ so $s(k) = 0$. For the second put $k_1 = 1$, $k_2 = \cdots = k_m = 0$ so $s(k) = 1$. Moreover $p = 2$. The result follows immediately by applying Corollary 3.1 in each case and adding the results.

**Corollary 3.3.** ($(\alpha, \beta)$-alternating subsets). *The number of sequences $(a_1, \ldots, a_n)$, of length $n = (\alpha + \beta)l + j$, $0 < j \leq \alpha + \beta$ such that $1 \leq a_1 < \cdots < a_n \leq N$ in which the first $\alpha$ elements have the same parity, the next $\beta$ have the opposite parity and so on, alternating parities between successive blocks of $\alpha$, $\beta$, $\alpha$, $\beta, \ldots$ elements, where the final block is of length i since it may be a fragment of a block of length $\alpha$ or a block of length $\beta$ is*

$$\binom{[\frac{1}{2}(N+1)] + (n-i)/(\alpha+\beta)}{n} + \binom{[\frac{1}{2}N] + (n-i)/(\alpha+\beta)}{n} \quad \text{if} \quad 0 < i \leq \alpha$$

*or*

$$\binom{[\frac{1}{2}N] + 1 + (n-i)/(\alpha+\beta)}{n} + \binom{[\frac{1}{2}(N+1)] + (n-i)/(\alpha+\beta)}{n} \quad \text{if} \quad \alpha < i \leq \alpha + \beta.$$

**Proof.** Let

$$\alpha_i = \begin{cases} \alpha & \text{if} \quad i \equiv 1 \bmod 2, \\ \beta & \text{if} \quad i \equiv 0 \bmod 2. \end{cases}$$

If there are $m = 2l + 1$ blocks then $n = l(\alpha + \beta) + i$ so $m = 2(n-i)/(\alpha+\beta) + 1$ where $0 < i \leq \alpha$. If there are $m = 2l + 2$ blocks, then $n = l(\alpha+\beta) + \alpha + j$ so $m = 2(n - \alpha - j)/(\alpha + \beta) + 2$ where $0 < j \leq \beta$. The results follows by substituting these values into Corollary 3.2.

The sequences enumerated in Corollary 3.3. have been called $(\alpha, \beta)$-*alternating* *subsets.* The solution for $(\alpha, 1)$-subsets is given by Tanny [4], and the solution for $(\alpha, \beta)$-subsets by Tanny [5] and Reilly [3].

**Corollary 3.4.** (The Moser and Abramson [1] "circular" generalisation of the Terquem Problem). *The number of sequences* $(a_1, \ldots, a_m)$ *such that* $1 \leq a_1 < \cdots < a_m \leq N$ *and*

$$a_j \equiv (a_{j-1} + 1 + k_j) \bmod p, \qquad j = 2, 3, \ldots, m,$$

*is*

$$\frac{pu + mv + m}{u + m} \cdot \binom{u + m}{m}$$

*where* $u = [(N - m - K)/p]$ *and* $v = N - m - K - pu$ *and* $K = k_2 + \cdots + k_m$.

**Proof.** There are $p$ cases, namely $a_1 \equiv (1 + t) \bmod p$, $t = 0, 1, \ldots, p - 1$. Let $k_1 = t$ so $s(\boldsymbol{k}) = t + K$ so from Corollary 3.1 the number of sequences is

$$\sum_{t=0}^{p-1} \binom{[(N - m - K)/p - t/p] + m}{m}.$$

But

$$\left[\frac{N - m - K}{p} - \frac{t}{p}\right] = \begin{cases} u & \text{if } t \leq v, \\ u - 1 & \text{if } t > v. \end{cases}$$

Accordingly the number of sequences is

$$\sum_{t=0}^{v} \binom{u + m}{m} + \sum_{t=v+1}^{p-1} \binom{u - 1 + m}{m}$$

and the results follows.

Many other specialisations of Lemma 2.1 remain, but no effort is made here to list them.

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## References

[1] W.O.J. Moser, and M. Abramson, Generalizations of Terquem's problem, J. Combinatorial Theory 7 (1969) 171–180.

[2] E. Netto, Lehrbuch der Combinatorik, 2nd ed. (Leipzig, 1927; reprinted Chelsea, New York, 1958).

[3] J.W. Reilly, Generalized alternating subsets by generating function methods, Ars Combinatoria (to appear).
[4] S.M. Tanny, Generating functions and generalized alternating subsets, Discrete Math. 13 (1975) 55–65.
[5] S.M. Tanny, An application of a generalization of Terquem's problem, Can. Math. Bull. 18 (1975) 769–770.
[6] O. Terquem, Sur un symbole combinatoire d'Euler et son utilité dans l'analyse, J. Math. Pures Appl. 4 (1839) 177–184.