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Transitive Factorizations in the Symmetric Group, and Combinatorial Aspects of Singularity Theory

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We consider the determination of the number $c_k(\alpha)$ of ordered factorizations of an arbitrary permutation on n symbols, with cycle distribution α, into k-cycles such that the factorizations have minimal length and the group generated by the factors acts transitively on the n symbols. The case $k = 2$ corresponds to the celebrated result of Hurwitz on the number of topologically distinct holomorphic functions on the 2-sphere that preserve a given number of elementary branch point singularities. In this case the monodromy group is the full symmetric group. For $k = 3$, the monodromy group is the alternating group, and this is another case that, in principle, is of considerable interest.

We conjecture an explicit form, for arbitrary k, for the generating series for $c_k(\alpha)$, and prove that it holds for factorizations of permutations with one, two and three cycles (so α is a partition with at most three parts). Our approach is to determine a differential equation for the generating series from a combinatorial analysis of the creation and annihilation of cycles in products under the minimality condition.

1. INTRODUCTION

1.1. Background. This paper has two goals. The first is to provide some general techniques to assist in the solution of the type of enumerative questions about permutation factorization, with transitivity and minimality conditions, that originate in the classical study of holomorphic mappings and branched coverings of Riemann surfaces. Thus, we are concerned with certain *combinatorial* questions that are encountered in aspects of *singularity theory*. The appearance of such questions has long been recognized, and the reader is directed to Arnold [1], for example, for further instances.

Very briefly, the classical construction concerns rational mappings from a Riemann surface to the sphere. Let α be the partition formed by the orders of the poles of this mapping. Each factor in an ordered factorization is associated with a distinguished branch point, and it specifies the sheet transitions imposed in a closed tour of the branch point, starting from an arbitrarily chosen base point on the codomain of the mapping. In the generic case, the sheet transitions are transpositions (2-cycles). The concatenation of the tours for each branch point, from the same base point, in the designated order, gives a sheet transition that is the product of the sheet transitions for each branch point. But this sheet transition is a permutation with α as its cycle type. The transitivity condition ensures that the ramified covering is connected, so the resulting Riemann surface is a ramified covering of a sphere. The minimality condition ensures that the covering surface is also a sphere. The monodromy group is the group freely generated by the sheet transitions.

The particular class of permutation factorization questions that we shall consider in this paper involve as factors only k-cycles, for some fixed, but arbitrary, value of k. The results that we are able to obtain are thus extensions of Hurwitz's [13] result with transpositions as factors, which arose in the singularity theory context described above.

The second goal is to investigate the possibility of determining analogues of Macdonald's 'top' symmetric functions that will be appropriate for accommodating the transitivity condition. A striking common element between the results of this paper on transitive, minimal ordered factorizations, and Macdonald's symmetric functions is the functional equation

$$w = xe^{w^{k-1}}, \tag{1}$$

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that arises in both settings when k-cycles are factors, for apparently different reasons. The nature of this possible connection is explored more fully in Section 1.5.

For the most part we now regard ordered factorizations as discrete structures and we treat them by combinatorial techniques. Throughout, we work in the appropriate ring of formal power series. Thus, for example, the functional Eqn (1) has a unique solution for formal power series in x. Although we have not completely attained the two goals, we have provided a substantial amount of methodology for the first, and concrete evidence for the second. We hope that the results are substantial enough to provoke others to explore further.

1.2. Minimal ordered factorizations. Let $\kappa(\pi)$ denote the number of cycles in $\pi \in \mathfrak{S}_n$. There is an obvious restriction on $\kappa(\pi)$ under permutation multiplication.

PROPOSITION 1.1. *Let $\pi, \pi' \in \mathfrak{S}_n$. Then $(n - \kappa(\pi)) + (n - \kappa(\pi')) \geq (n - \kappa(\pi\pi'))$.*

If $(\sigma_1, \ldots, \sigma_j) \in \mathfrak{S}_n^j$ and $\sigma_1 \cdots \sigma_j = \pi$, then $(\sigma_1, \ldots, \sigma_j)$ is called an *ordered factorization* of π. Immediately from Proposition 1.1, we obtain the inequality

$$\sum_{i=1}^{j} (n - \kappa(\sigma_i)) \geq n - \kappa(\pi). \tag{2}$$

In the case of equality, we call $(\sigma_1, \ldots, \sigma_j) \in \mathfrak{S}_n^j$ a *minimal* ordered factorization of π.

Such factorizations have an elegant theory and many enumerative applications (see, for example, Goulden and Jackson [9]), including permissible commutation of adjacent factors. In particular, [9] contains an explicit construction for a set of symmetric functions (Macdonald's top symmetric functions) that we shall return to in Section 1.5 of the Introduction. Now we turn to the topic of the present paper.

1.3. Minimal, transitive ordered factorizations. We write $\alpha \vdash n$ to indicate that α is a partition of n, and $\mathcal{C}_\alpha$ for the conjugacy class of $\mathfrak{S}_n$ indexed by α. Let $l(\alpha)$ denote the number of parts in α. If $\pi \in \mathcal{C}_\alpha$ then $\kappa(\pi) = l(\alpha)$. An ordered factorization $(\sigma_1, \ldots, \sigma_j)$ is said to be *transitive* if the subgroup of $\mathfrak{S}_n$ generated by the factors acts transitively on $\{1, \ldots, n\}$. We consider the case in which each of the factors is in $\mathcal{C}_{[k,1^{n-k}]}$, and is therefore a pure k-cycle. A transitive ordered factorization of $\pi \in \mathcal{C}_\alpha$ into pure k-cycles with the minimal choice of j consistent with the other conditions is said to be *minimal*. In this case, $j = \mu_k(\alpha)$, where

$$\mu_k(\alpha) = \frac{n + l(\alpha) - 2}{k - 1}, \tag{3}$$

as we shall prove in Proposition 2.1. For example, when $k = 3$, suppressing 1-cycles in the factors,

$$(247)(586)(479)(136)(235) = (1386)(254)(79), \tag{4}$$

and $((247), (568), (479), (136), (235))$ is a minimal, transitive ordered factorization of the permutation $(1386)(254)(79)$ into 3-cycles with five factors (minimality holds in this example since $\mu_3([4, 3, 2]) = 5$).

Such factorizations are encountered in a number of contexts. These include, for example, the topological classification of polynomials of a given degree and a given number of critical values, and the moduli space of covers of the Riemann sphere and properties of the Hurwitz monodromy group, and applications to mathematical physics [2]. The reader is directed to [3, 4, 14] for further background information.

The number of minimal, transitive ordered factorizations of an arbitrary but fixed $\pi \in \mathcal{C}_\alpha$ into pure k-cycles is denoted by $c_k(\alpha)$. Hurwitz [13] conjectured the expression for $c_2(\alpha)$, as a consequence of his study of holomorphic mappings on the sphere, to be

$$c_2(\alpha) = n^{l(\alpha)-3}(n + l(\alpha) - 2)! \prod_{j=1}^{l(\alpha)} \frac{\alpha_j^{\alpha_j}}{(\alpha_j - 1)!}. \tag{5}$$

(See also Strehl [17] for the proof of an identity that completes Hurwitz's treatment.) A shorter and self-contained proof of this result has been given by Goulden and Jackson [10]. The special case $c_2([1^n])$ was derived independently by Crescimanno and Taylor [2]. For related work, in the language of singularity theory, see [16].

The case $k = 3$ is also of considerable interest, since the subgroup generated in this case is the alternating group.

1.4. A conjecture and the supporting results. The main conjecture of this paper concerns the form of the generating series for the $c_k(\alpha)$. Let $u, z, p_1, p_2, \ldots$ be indeterminates and let $p_\alpha = p_{\alpha_1} p_{\alpha_2} \ldots$. Let

$$F^{(m)}(u, z; p_1, p_2, \ldots) = \sum_{\substack{n \geq 1 \\ k-1 | n+m-2}} \sum_{\substack{\alpha \vdash n \\ l(\alpha)=m}} c_k(\alpha)\, |\mathcal{C}_\alpha|\, p_\alpha \frac{u^{\mu_k(\alpha)}}{\mu_k(\alpha)!} \frac{z^n}{n!}. \tag{6}$$

Then $F^{(m)}$ is a formal power series in z with coefficients that are polynomial in $u, p_1, p_2, \ldots$, and we will be working in this ring. The choice of this generating series to be exponential in u and z, and ordinary in $p_1, p_2, \ldots$, will become apparent in Section 2.

It is more convenient to work with a symmetrised form of the generating series, defined in terms of the following linear symmetrization operator ψ_m. If $l(\alpha) = m$, let

$$\psi_m(p_\alpha u^i z^j) = \sum_{\sigma \in \mathfrak{S}_m} x_1^{\alpha_{\sigma(1)}} \cdots x_m^{\alpha_{\sigma(m)}} = \left(\prod_{r \geq 1} v_r! \right) m_\alpha(x_1, \ldots, x_m), \tag{7}$$

where m_α is the monomial symmetric function indexed by α, and v_r is the number of parts of α equal to r, for each $r \geq 1$. If $l(\alpha) \neq m$, then the value of ψ_m is 0. Now let

$$P^{(m)}(x_1, \ldots, x_m) = \psi_m(F^{(m)}). \tag{8}$$

In the main conjecture that follows, we let $w_i = w(x_i)$ for $i \geq 1$, where $w(x)$ is the unique power series solution of the functional equation given in (1). Of course, $w_1, \ldots$ are algebraically independent.

CONJECTURE 1.2 (MAIN CONJECTURE). *For* $m \geq 1$,

$$\left(\sum_{i=1}^m x_i \frac{\partial}{\partial x_i} \right)^{3-m} P^{(m)}(x_1, \ldots, x_m) = S^{(m)}(w_1, \ldots, w_m) \prod_{i=1}^m x_i \frac{dw_i}{dx_i},$$

where $S^{(m)}(w_1, \ldots, w_m)$ *is a symmetric polynomial in* $w_1, \ldots, w_m$.

The conjectured form for the series $P^{(m)}$ therefore involves rational expressions in $w_1, \ldots, w_m$. To see this, differentiate (1) with respect to x, to obtain the rational form

$$x \frac{dw}{dx} = \frac{w}{1 - (k-1)w^{k-1}}. \tag{9}$$

Note that the dependence on k not only resides in the coefficients of the symmetric polynomial (which we conjecture to be polynomials in k), but also in the functional Eqn (1). The dependence of the explicit formal power series for w on k through this functional equation is actually straightforward, and is seen immediately by Lagrange's Theorem to be

$$w(x) = \sum_{m\geq 0} \frac{(1+(k-1)m)^{m-1}}{m!} x^{1+(k-1)m}. \tag{10}$$

In this paper, we determine explicitly $P^{(m)}$ for the cases $m = 1, 2, 3$. These are all of a form that supports the above conjecture. Explicit expressions for $S^{(m)}$ in these cases are stated below. Let $V(w_1, \dots, w_j)$ denote the Vandermonde determinant in $w_1, \dots, w_j$, and let $h_i(w_1, \dots, w_j)$ denote the complete symmetric function of degree i in $w_1, \dots, w_j$.

THEOREM 1.3. $S^{(1)}(w_1) = 1$.

THEOREM 1.4. $S^{(2)}(w_1, w_2) = (w_1^{k-1} - w_2^{k-1})^2 / V(w_1, w_2)^2 = h_{k-2}^2(w_1, w_2)$.

THEOREM 1.5. $S^{(3)}(w_1, w_2, w_3) = G^2 / V(w_1, w_2, w_3)^2 = (h_{k-3} + (k-1)h_{2k-4})^2$, *where*

$$G = w_1(1-(k-1)w_1^{k-1})(w_3^{k-1} - w_2^{k-1}) + w_2(1-(k-1)w_2^{k-1})(w_1^{k-1} - w_3^{k-1})$$
$$+ w_3(1-(k-1)w_3^{k-1})(w_2^{k-1} - w_1^{k-1}), \qquad \textit{and} \qquad h_i = h_i(w_1, w_2, w_3).$$

The proofs of these results are given in Section 4. The method is to solve a partial differential equation for $P^{(m)}$ that is obtained in Section 3. This equation is itself deduced by symmetrizing a partial differential equation for $F^{(m)}$ that is obtained in Section 2. The latter is determined by a combinatorial analysis of minimal permutation multiplication. The determination of further cases, at present, seems to be intractable, despite the fact that we have a general solution scheme, as we will discuss in Section 5.

The forms obtained above in the first three cases are remarkably simple, although it has not been possible to conjecture a general form based on this evidence. Although $S^{(1)}$, by default, $S^{(2)}$ and $S^{(3)}$ are perfect squares, we do not believe that this holds in general. Note that $S^{(m)}$ does not restrict to $S^{(m-1)}$ through $w_m = 0$, in the cases $m = 2$ and $m = 3$. Also note that if we substitute $k = 2$ in Theorems 1.4 and 1.5 above, then we immediately obtain $S^{(2)} = S^{(3)} = 1$. In the following result, we demonstrate that this is true when $k = 2$ for an arbitrary choice of m as a direct consequence of Hurwitz's result (5).

LEMMA 1.6. *If* $k = 2$, *then* $S^{(m)}(w_1, \dots, w_m) = 1$ *for* $m \geq 1$.

PROOF. From (5), for $k = 2$, we have

$$\begin{aligned}
\left(\sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i}\right)^{3-m} P^{(m)}(x_1, \dots, x_m) &= \sum_{n\geq 1} \sum_{\substack{\alpha\vdash n \\ l(\alpha)=m}} \frac{|\mathcal{C}_\alpha|}{n!} \left(\prod_{j=1}^{m} \frac{\alpha_j^{\alpha_j+1}}{\alpha_j!}\right) \sum_{\sigma\in\mathfrak{S}_m} x_1^{\alpha_{\sigma(1)}} \cdots x_m^{\alpha_{\sigma(m)}} \\
&= \frac{1}{m!} \sum_{\alpha_1,\dots,\alpha_m\geq 1} \left(\prod_{j=1}^{m} \frac{\alpha_j^{\alpha_j}}{\alpha_j!}\right) \sum_{\sigma\in\mathfrak{S}_m} x_1^{\alpha_{\sigma(1)}} \cdots x_m^{\alpha_{\sigma(m)}} \\
&= \prod_{j=1}^{m} x_j \frac{dw_j}{dx_j}.
\end{aligned}$$

The result now follows. □

We note that, in the case of transpositions, together with Vainshtein [11], we have recently been able to obtain similar results in the case where there are two more than the minimal number of factors. These correspond to holomorphic mappings from the torus.

1.5. Symmetric functions and minimal ordered factorizations. In [9] (see also [15]) an explicit construction is given for Macdonald's 'top' symmetric functions u_λ, indexed by $\lambda \vdash n$. They have the property that the number of minimal ordered factorizations $(\sigma_1, \ldots, \sigma_j)$ of π, where $\sigma_i \in \mathcal{C}_{\beta_i}$, $i = 1, \ldots, j$, and for each $\pi \in \mathcal{C}_\lambda$, is given by

$$[u_{\lambda-1}]\, u_{\beta_1-1} \cdots u_{\beta_j-1}, \tag{11}$$

where $\beta_i - 1$ is the partition obtained by subtracting one from each part of β_i. Several examples of their use in enumerative questions is given in [9].

The symmetric functions u_λ, where $\lambda \vdash n$, are constructed as follows. Let $H(t; \mathbf{x})$ be the generating series for the complete symmetric functions $h_i(\mathbf{x})$ of degree i in $\mathbf{x} = (x_1, x_2, \ldots)$. Then the functional equation $s = t\,H(t; \mathbf{x})$ has a unique solution $t \equiv t(s, \mathbf{x})$ given by $t = s\,H^\star(s; \mathbf{x})$ where $H^\star(s; \mathbf{x}) = \sum_{j \geq 0} s^j h_j^\star(\mathbf{x})$, and $h_j^\star(\mathbf{x})$ is a symmetric function in $\mathbf{x}$ of total degree j. Let $h_\lambda^\star = h_{\lambda_1}^\star h_{\lambda_2}^\star \cdots$. Then $\{u_\lambda\}$ is defined to be the basis for the symmetric function ring that is dual to the basis $\{h_\lambda^\star\}$ with respect to the inner product for which the monomial and complete symmetric functions are dual (see, e.g., Macdonald [15]).

Thus, for minimal ordered factorizations in which all factors are k-cycles, then in Eqn (11), we have $u_{\beta_i - 1} = u_{k-1}$ for all $i = 1, \ldots, j$. But, as is shown in [9], $u_{k-1} = -p_{k-1}$, so for minimal ordered factorizations in which all factors are k-cycles, we can restrict our attention to a symmetric function algebra in which $p_i = 0$ if $i \neq k - 1$. In this case, we have

$$s = tH(t; \mathbf{x}) = \exp\left(\sum_{m \geq 1} \frac{p_m}{m} t^m\right) = t \exp\left(\frac{-p_{k-1}}{k-1} t^{k-1}\right).$$

Thus, if z is substituted for $\frac{p_{k-1}}{k-1}$, in this equation, we obtain $t = se^{zt^{k-1}}$. But this is precisely the functional Eqn (1), whose solution features so centrally in our results for the transitive case above.

We conclude from this that there must be an important relationship between the transitive case of minimal ordered factorizations for which we have obtained partial results in this paper, and minimal ordered factorizations themselves, that have such an elegant theory based on symmetric functions. Although we have been unable to find a direct link between these two classes, we hope that the results of this paper will provide a good starting point for such a direct link, and a similarly elegant theory for the transitive case.

2. The Partial Differential Equation

In this section we derive a partial differential equation for the generating series

$$\Phi = \sum_{m \geq 1} F^{(m)}, \tag{12}$$

where $F^{(m)}$ is given in (6), by a case analysis of the creation and annihilation of cycles in products of permutations subject to the minimality condition. We begin with a discussion of permutation multiplication. First, we prove the expression (3) for $\mu_k(\pi)$.

Proposition 2.1. *Let $\alpha \vdash n$, and let $\pi \in \mathcal{C}_\alpha$. Then $\mu_k(\pi) = \mu_k(\alpha)$, where*

$$\mu_k(\alpha) = \frac{n + l(\alpha) - 2}{k - 1}.$$

PROOF. Let $(\sigma_1, \ldots, \sigma_j)$ be a minimal, transitive ordered factorization of π into k-cycles. Let π' and π be in the same conjugacy class, so $\pi' = g^{-1}\pi g$ for some $g \in \mathfrak{S}_n$. Then $(g^{-1}\sigma_1 g, \ldots, g^{-1}\sigma_j g)$ is a minimal, transitive ordered factorization of π', so $\mu_k(\pi') = \mu_k(\pi)$, and we denote the common value by $\mu_k(\alpha)$ where $\pi \in \mathcal{C}_\alpha$. Now each k-cycle in $\mathfrak{S}_k$ has a minimal, transitive ordered factorization into $\mu_2([k])$ transpositions, so $\mu_2(\alpha) = \mu_2([k])\,\mu_k(\alpha)$. But [10, Proposition 2.1] $\mu_2(\alpha) = n + l(\alpha) - 2$, and the result follows. □

2.1. A characterization of minimal, transitive ordered factorizations. Next we give a combinatorial characterization of minimal, transitive ordered factorizations. The following lemma characterizes the relationship between σ_1 and $\sigma_2 \cdots \sigma_j$ for a minimal, transitive ordered factorization $(\sigma_1, \ldots, \sigma_j)$ of $\pi \in \mathfrak{S}_n$ into k-cycles. Some notation will be useful, and will be used throughout this section. The multi-graph $\mathcal{D}_{\sigma_1,\ldots,\sigma_j}$ has vertex-set $\{1, \ldots, n\}$, and edges consisting of the edges of the k-cycles in the factorization. Let $\mathcal{V}_1, \ldots, \mathcal{V}_l$ be the vertex-sets of the connected components of $\mathcal{D}_{\sigma_2,\ldots,\sigma_j}$, so $\{\mathcal{V}_1, \ldots, \mathcal{V}_l\}$ is a partition of $\{1, \ldots, n\}$ into non-empty subsets. For $i = 1, \ldots, l$, let α_i consist of all $t \in \{2, \ldots, j\}$ such that all of the k elements on σ_t belong to $\mathcal{V}_i$, so $\{\alpha_1, \ldots, \alpha_l\}$ is a partition of $\{2, \ldots, j\}$. Suppose $\alpha_i = \{\alpha_{i1}, \ldots, \alpha_{is_i}\}$, with $\alpha_{i1} < \cdots < \alpha_{is_i}$, and $\sigma_{\alpha_{i1}} \cdots \sigma_{\alpha_{is_i}} = \pi_i$, for $i = 1, \ldots, l$. Then clearly, by construction,

$$(\sigma_{\alpha_{i1}}, \ldots, \sigma_{\alpha_{is_i}}) \tag{13}$$

is a minimal, transitive ordered factorization of π_i, for $i = 1, \ldots, l$, and we have

$$\pi = \sigma_1 \pi_1 \cdots \pi_l. \tag{14}$$

Moreover, $\sigma_{\alpha_{ab}}$ and $\sigma_{\alpha_{cd}}$ commute for $a \neq c$ and all b, d, since the elements on these k-cycles are disjoint.

For example, in the minimal, transitive factorization given in (4), we have $l = 2$, with $\mathcal{V}_1 = \{1, 2, 3, 5, 6, 8\}$ and $\mathcal{V}_2 = \{4, 7, 9\}$; $\alpha_1 = \{2, 4, 5\}$ and $\alpha_2 = \{3\}$; $\pi_1 = (1386)(25)$ and $\pi_2 = (479)$.

For $\delta \in \mathfrak{S}_n$ and $\mathcal{A} \subseteq \{1, \ldots, n\}$, the *$\mathcal{A}$-restriction* of δ is the permutation on $\mathcal{A}$ obtained by deleting the elements not in $\mathcal{A}$ from the cycles of δ. For example, if $\delta = (1538)(27469)$ and $\mathcal{A} = \{1, 4, 6, 7, 8\}$, then the $\mathcal{A}$-restriction of δ is $(18)(467)$.

LEMMA 2.2. *Let $(\sigma_1, \ldots, \sigma_j)$ be a minimal, transitive ordered factorization of $\pi \in \mathfrak{S}_n$ into k-cycles, and let $\pi_1, \ldots, \pi_l$ be constructed as above. Then:*

(1) σ_1 has at least one element in common with each of $\pi_1, \ldots, \pi_l$.
(2) The elements of σ_1 in common with π_i lie on a single cycle of π_i, for $i = 1, \ldots, l$.
(3) Let $\mathcal{U}$ denote the k-subset of $\{1, \ldots, n\}$ consisting of the elements on the k-cycle σ_1. Let γ denote the $\mathcal{U}$-restriction of σ_1, let ρ denote the $\mathcal{U}$-restriction of $\pi_1 \cdots \pi_l$, and let τ denote the $\mathcal{U}$-restriction of π (so $\gamma\rho = \tau$). Then $(k - \kappa(\tau)) + (k - \kappa(\rho)) = k - \kappa(\gamma)$, so (τ, ρ^{-1}) is a minimal, ordered factorization of γ.

PROOF. Since $(\sigma_1, \ldots, \sigma_j)$ is a transitive factorization of π, then $\mathcal{D}_{\sigma_1,\ldots,\sigma_l}$ is connected. Thus the single k-cycle in $\mathcal{D}_{\sigma_1}$ has at least one vertex in each of the connected components of $\mathcal{D}_{\sigma_2,\ldots,\sigma_l}$, and this establishes part (1).

For part (2), from part (1) it suffices to prove that $\kappa(\rho) = l$. Now, from (14) and the fact that $(\sigma_{\alpha_{i1}}, \ldots, \sigma_{\alpha_{is_i}})$ is a minimal, transitive ordered factorization of π_i, for $i = 1, \ldots, l$, we have

$$\mu_k(\pi) = 1 + \mu_k(\pi_1) + \cdots + \mu_k(\pi_l). \tag{15}$$

But, from Proposition 2.1

$$\mu_k(\pi) = \frac{n + \kappa(\pi) - 2}{k-1} \qquad \text{and} \qquad \mu_k(\pi_i) = \frac{|\mathcal{V}_i| + \kappa(\pi_i) - 2}{k-1},$$

for $i = 1, \ldots, l$. Thus $n + \kappa(\pi) - 2 = k - 1 + \sum_{i=1}^{l}(|\mathcal{V}_i| + \kappa(\pi_i) - 2)$, from (15). But $n = \sum_{i=1}^{l} |\mathcal{V}_i|$, so

$$\kappa(\pi) - \sum_{i=1}^{l} \kappa(\pi_i) = k + 1 - 2l. \tag{16}$$

Now let ρ_i be the $\mathcal{U}$-restriction of π_i, for $i = 1, \ldots, l$, so $\pi = \sigma_1\pi_1 \cdots \pi_l$ restricts to $\tau = \gamma\rho$, where $\rho = \rho_1 \cdots \rho_l$. Thus, $\kappa(\pi) - \kappa(\tau) = \sum_{i=1}^{l}(\kappa(\pi_i) - \kappa(\rho_i)) = \sum_{i=1}^{l} \kappa(\pi_i) - \kappa(\rho)$, and together with (16) this gives

$$\kappa(\tau) - \kappa(\rho) = \kappa(\pi) - \sum_{i=1}^{l} \kappa(\pi_i) = k + 1 - 2l. \tag{17}$$

On the other hand, since γ, ρ and τ act on a k-set and $\tau\rho^{-1} = \gamma$ we have from Proposition 1.1 that $(k - \kappa(\tau)) + (k - \kappa(\rho^{-1})) \geq (k - \kappa(\gamma))$. But $\kappa(\gamma) = 1$ and $\kappa(\rho^{-1}) = \kappa(\rho)$, so $\kappa(\tau) + \kappa(\rho) \leq k + 1$, and in addition, from part (1) we have $\kappa(\rho) \geq l$. It follows that $\kappa(\tau) - \kappa(\rho) \leq k + 1 - 2\kappa(\rho) \leq k + 1 - 2l$. Combining this with (17) gives $\kappa(\rho) = l$. Together with part (1), this establishes part (2).

Part (3) follows immediately from $\kappa(\rho) = l$, $\kappa(\gamma) = 1$ and (17). □

2.2. The tree bijection. We now use this characterization as a construction for deriving a partial differential equation for Φ with arbitrary k, given in Theorem 2.3 below. The terms in the equation are indexed by the set $\mathcal{T}_k$ of plane, vertex two-coloured (black, white), edge-rooted trees with k edges, $k \geq 1$, together with canonical labellings of the vertices and edges, described as follows.

Let T be such a tree. Now T is the boundary of an unbounded region of the plane. Describe the boundary by moving along the edges, keeping the region on the left, beginning along the root edge from its incident black vertex to its incident white vertex. Each edge is encountered twice, once from black vertex to white vertex, and once from white vertex to black vertex. Assign the labels $1, \ldots, k$ to the edges, in the order that they are encountered from black to white vertex. Let $B(T)$ and $W(T)$ denote, respectively, the number of black vertices and white vertices in T. The black vertices and white vertices are labelled $b_1, \ldots, b_{B(T)}$ and $w_1, \ldots, w_{W(T)}$, respectively, where for each colour, the subscripts are in increasing order of the smallest label on the edges incident with the vertices they index. For $j = 1, \ldots, B(T)$, let β_j be the set of labels on the edges incident with vertex b_j. Similarly, let ω_j be the set of labels on the edges incident with vertex w_j. Then $\{\beta_1, \ldots, \beta_{B(T)}\}$ and $\{\omega_1, \ldots, \omega_{W(T)}\}$ are set partitions of $\{1, \ldots, k\}$, and the blocks are indexed in increasing order of their smallest elements. Let $i_1, \ldots, i_k$ be positive integer-valued indexed variables, and define

$$\theta(b_r) = \sum_{s \in \beta_r} i_s, \qquad \theta(w_r) = \sum_{s \in \omega_r} i_s.$$

The set $\mathcal{T}_k$ is in one-to-one correspondence with minimal ordered factorizations of a k-cycle, as was shown in [8, Theorem 2.1]. The correspondence, which we shall refer to as the *tree bijection*, is described as follows: λ_B and λ_W are permutations in $\mathfrak{S}_k$. There is one cycle in the disjoint cycle representation of λ_B for each black vertex in T; the elements on this cycle are the labels of the edges incident with the black vertex, where the order of these labels on the cycle is the clockwise order in which the corresponding edges are encountered around the vertex. Similarly, there is one cycle of λ_W for each white vertex in T, giving the labels on incident edges in clockwise order around the vertex. Then (λ_B, λ_W) is a minimal ordered factorization of $(1 \dots k)$. In terms of the vertex labelling of T given above, note that the cycles of λ_B and λ_W are the elements of the β_j's and ω_j's, respectively, arranged in a particular cyclic order.

2.3. The partial differential equation for Φ. The next result gives a non-linear, inhomogeneous partial differential equation for Φ, defined in (12), where Φ_r denotes $r\,\frac{\partial \Phi}{\partial p_r}$.

THEOREM 2.3. *Let* $\mathbf{i} = (i_1, \dots, i_k)$ *where* $i_1, \dots, i_k \geq 1$. *Then*

$$\frac{1}{k} \sum_{T \in \mathcal{T}_k} \sum_{\mathbf{i}} \left(\prod_{j=1}^{B(T)} p_{\theta(b_j)} \right) \left(\prod_{j=1}^{W(T)} \Phi_{\theta(w_j)} \right) = \frac{\partial \Phi}{\partial u}. \tag{18}$$

PROOF. For each fixed k, Φ is the generating series for minimal, transitive ordered factorizations $(\sigma_1, \dots, \sigma_j)$ of π into k-cycles for all permutations $\pi \in \mathfrak{S}_n$, $n \geq 1$. The series is exponential in z, recording n, and u, recording $\mu_k(\pi)$, which is the number of factors in the factorization. The series is ordinary in p_j, recording the number of cycles of length j in π, $j \geq 1$. (Note that $c_k(\alpha)$ gives the number of factorizations for each π in $\mathcal{C}_\alpha$, so the coefficient $c_k(\alpha)|\mathcal{C}_\alpha|$ in Φ accounts for factorizations of all such π.)

Consider modifying Φ to obtain the generating series $\hat{\Phi}$ for the same set, but not recording the left-most factor σ_1 by u. The result is $\frac{\partial \Phi}{\partial u}$, since $\frac{\partial}{\partial u} \frac{u^h}{h!} = \frac{u^{h-1}}{(h-1)!}$, $h \geq 1$. This gives the right-hand side of the equation.

We now determine another expression for $\hat{\Phi}$, to obtain the left-hand side of the equation. This is carried out by reconstructing the cycle lengths of π from the left-most factor σ_1 and the cycle lengths of $\pi_1, \dots, \pi_l$, where we are using the notation of Lemma 2.2. Let $u_1, \dots, u_k$ be distinct elements of $\{1, \dots, n\}$, in any permuted order. We now consider the contributions to $\hat{\Phi}$ of all factorizations with $\sigma_1 = (u_1 \dots u_k)$, and since the k-cycles σ_1 are created exactly k times in these ordered lists, we will divide the resulting generating function by k to obtain an expression for $\hat{\Phi}$. In the notation of Lemma 2.2, we have $\mathcal{U} = \{u_1, \dots, u_k\}$. Then from Lemma 2.2(3) and the tree bijection described above, $(\tau, \rho^{-1}) = (\lambda_B, \lambda_W)$ for some unique tree T in $\mathcal{T}_k$ (replace i in the construction for factorizations of the canonical cycle $(1 \cdots k)$ by u_i, for $i = 1, \dots, k$). Moreover, the black vertex-degrees are given by the cycle lengths of τ, and the white vertex-degrees are given by the cycle lengths of ρ^{-1}.

We now observe that, in the product $\gamma\rho$ (recall that γ is the $\mathcal{U}$-restriction of σ_1), cycles with lengths equal to the degrees of the white vertices are annihilated, and combined to create cycles of lengths equal to the degrees of the black vertices. This observation permits us to reconstruct the cycle lengths of π from σ_1 and the cycle lengths of $\pi_1, \dots, \pi_l$.

Now τ is the $\mathcal{U}$-restriction of π, and from Lemma 2.2(2), each cycle of ρ is the $\mathcal{U}$-restriction of a single cycle in some unique π_j. We call this cycle the *active* cycle of π_j in the construction below. Suppose that on the active cycle containing element u_j, there are $i_j - 1$ elements of $\{1, \dots, n\}$ between u_j and the next element of $\mathcal{U}$ (which may be u_j itself) around the cycle. Then $i_1, \dots, i_k \geq 1$.

In terms of the tree T, there is one active cycle corresponding to each white vertex of T, so $l = W(T)$, and for convenience, we suppose that the subscripts on $\pi_1, \dots, \pi_l$ are chosen so that the active cycle of π_j corresponds to vertex w_j, for $j = 1, \dots, W(T)$. Then the cycle in π_j that is annihilated has length $\theta(w_j)$. Moreover, once we identify this cycle, then the elements of $\mathcal{U}$ on the cycle are all uniquely determined by specifying any of the $\theta(w_j)$ elements on the cycle to be a canonically chosen element of $\mathcal{U}$ (say, the element of $\mathcal{U}$ with smallest index on the cycle), since the other elements are then determined by the values of $i_1, \dots, i_k$. But, from (13), $(\sigma_{\alpha_{j1}}, \dots, \sigma_{\alpha_{js_j}})$ is a minimal, transitive ordered factorization of π_j, so the contribution to $\hat{\Phi}$ from the annihilated cycles is $\prod_{j=1}^{W(T)} \Phi_{\theta(w_j)}$. Also, a cycle of length $\theta(b_j)$ is created for each black vertex b_j in T, so the contribution from cycles that are created is $\prod_{j=1}^{B(T)} p_{\theta(b_j)}$.

Multiplying these contributions together, dividing by k, and summing over all $T \in \mathcal{T}_k$ and $i_1, \dots, i_k \geq 1$ gives the left-hand side of the equation. This is an expression for $\hat{\Phi}$; the contributions from the cycle lengths as recorded by $p_1, \dots$ is explained by the analysis above. The effect of labelling the elements on which $\pi_1, \dots, \pi_l$ act with disjoint subsets of $\{1, \dots, n\}$ (where $\pi \in \mathfrak{S}_n$) is accounted for because the series is exponential in z. The effect of shuffling the factors of $\pi_1, \dots, \pi_l$ into disjoint subsets of positions chosen from $\{2, \dots, \mu_k(\pi)\}$ is accounted for because the series is exponential in u (here we use the fact that factors from different π_j's commute since the π_j's act on disjoint sets of elements). □

Note that, if p_j is the power sum symmetric function of degree j in an infinite set of ground variables, then $j\partial/\partial p_j = p_j^\star$, where $p_j^\star$ is the adjoint of premultiplication by p_j (see, e.g., [15] for details). The partial differential equation therefore can be rewritten in a form that exhibits the symmetry between black and white vertices, by writing Φ_j as $p_j^\star \Phi$.

2.4. Examples. As examples, we now give the cases $k = 2$ and $k = 3$ of Eqn (18). For $k = 2$, there are two two-coloured trees on k edges. Both are paths of edge-length two; in one the vertex of degree 2 is black, in the other it is white. Since both trees have a single edge-rooting, Eqn (18) for $k = 2$ is

$$\tfrac{1}{2} \sum_{i_1, i_2 \geq 1} (\Phi_{i_1} \Phi_{i_2} p_{i_1+i_2} + \Phi_{i_1+i_2} p_{i_1} p_{i_2}) = \frac{\partial \Phi}{\partial u}. \tag{19}$$

This is the equation given in [10], where we demonstrated that a series conjectured from numerical computations satisfied this equation uniquely.

When $k = 3$ there are three two-coloured trees with k edges. Two of these have a single vertex of degree 3, adjacent to three vertices of degree 1; in one the vertex of degree 3 is black, in the other it is white. Both of these have a single edge-rooting. The third tree is a path of edge-length three, and this tree has three edge-rootings, so Eqn (18) for $k = 3$ is (after permuting the summation indices arising from the three edge-rootings of the third tree)

$$\sum_{i_1, i_2, i_3 \geq 1} \left(\tfrac{1}{3} \Phi_{i_1} \Phi_{i_2} \Phi_{i_3} p_{i_1+i_2+i_3} + \tfrac{1}{3} \Phi_{i_1+i_2+i_3} p_{i_1} p_{i_2} p_{i_3} + \Phi_{i_1} \Phi_{i_2+i_3} p_{i_1+i_2} p_{i_3} \right) = \frac{\partial \Phi}{\partial u}. \tag{20}$$

We do not know of any method for solving this equation for Φ when $k = 3$ explicitly, and have not been able to conjecture the solution from numerical computations, as we could for $k = 2$. However, as we show in the next section, we are able to determine the low degree terms of Φ in the p's, for arbitrary k.

3. Restriction of the Differential Equation by Grading

In this section we determine a partial differential equation for $P^{(m)}$, defined in (8), that can be used recursively to construct $P^{(m)}$ for all $m \geq 1$. It is obtained by applying the symmetrization operator ψ_m given in (7) to the partial differential Eqn (18) given in Theorem 2.3. Some notation is needed for this purpose. For $i \geq 1$, let

$$h_i^+(x_1, \dots, x_r) = \sum_{\substack{j_1, \dots, j_r \geq 1 \\ j_1 + \cdots + j_r = i}} x_1^{j_1} \cdots x_r^{j_r},$$

and let $h_0^+(x_1, \dots, x_r) = 1$, so $h_i^+(x_1, \dots, x_r)$ is the sum of the terms of the complete symmetric function of degree i in $x_1, \dots, x_r$ with positive exponents on all variables. Let

$$H^+(t; x_1, \dots, x_r) = \sum_{i \geq 0} h_i^+(x_1, \dots, x_r) t^i.$$

If $a(t) = \sum_{i \geq 0} a_i t^i$ is a formal power series, let

$$a(t) \circ H^+(t; x_1, \dots, x_r) = \sum_{i \geq 0} a_i h_i^+(x_1, \dots, x_r).$$

This is essentially the Hadamard product of $a(t)$ and $H^+(t; x_1, \dots, x_r)$ with respect to t. Throughout, the Hadamard product will be taken exclusively with respect to the indeterminate t.

For $l = 1, \dots, m$, let $\Xi_{m,l}$ be the symmetrization operator defined for formal power series in $x_1, \dots, x_m$ by

$$\Xi_{m,l} f(x_1, \dots, x_m) = \sum_{\sigma} f(x_{\sigma(1)}, \dots, x_{\sigma(m)}),$$

where the summation is over $\sigma \in \mathfrak{S}_m$, with the restriction that $\sigma(l+1) < \cdots < \sigma(m)$. Thus, there are $\frac{m!}{(m-l)!}$ terms in this summation. Let $\delta = \{\delta_1, \dots, \delta_m\}$, where $1 \leq \delta_1 < \cdots < \delta_m$. Then, similarly to ψ_m, we define ψ_δ by

$$\psi_\delta(p_\alpha z^i u^j) = \sum_{\sigma \in \mathfrak{S}_m} x_{\delta_1}^{\alpha_{\sigma(1)}} \cdots x_{\delta_m}^{\alpha_{\sigma(m)}},$$

if $l(\alpha) = m$, and is 0 if $l(\alpha) \neq m$. We denote $(x_{\delta_1}, \dots, x_{\delta_m})$ by x_δ, and $(t, x_{\delta_1}, \dots, x_{\delta_m})$, with minor abuse of notation, by (t, x_δ).

THEOREM 3.1. *Let $\chi(w_r) = \{x_j : b_j$ is adjacent to $w_r\}$. Then, for $m \geq 1$,*

$$\frac{1}{k} \sum_{T \in \mathcal{T}_k} \Xi_{m,B(T)} \sum_{\zeta} \prod_{r=1}^{W(T)} \left(t \frac{\partial}{\partial t} P^{(|\zeta_r|+1)}(t, x_{\zeta_r}) \right) \circ H^+(t; \chi(w_r))$$
$$= \frac{1}{k-1} \left(x_1 \frac{\partial}{\partial x_1} + \cdots + x_m \frac{\partial}{\partial x_m} + m - 2 \right) P^{(m)}(x_1, \dots, x_m),$$

where the sum is over all ordered (set) partitions $\zeta = (\zeta_1, \dots, \zeta_{W(T)})$ of $\{B(T)+1, \dots, m\}$, and blocks in the ordered set partition may be empty. (If T has more than m black vertices, then the term in the summation corresponding to T is 0.)

PROOF. The left-hand side of the partial differential Eqn (18) is $\frac{1}{k} \sum_{T \in \mathcal{T}_k} L_T$, where

$$L_T = \sum_{\mathbf{i}} \left(\prod_{r=1}^{B(T)} p_{\theta(b_r)} \right) \left(\prod_{s=1}^{W(T)} \Phi_{\theta(w_s)} \right).$$

Then, for $m \geq k$,

$$\psi_m L_T = \sum_{\mathbf{i}} \Xi_{m,B(T)} \left(\left(\prod_{r=1}^{B(T)} x_r^{\theta(b_r)} \right) \sum_{\zeta} \left(\prod_{s=1}^{W(T)} \psi_{\zeta_s}(\Phi_{\theta(w_s)}) \right) \right).$$

If s is an edge label of T, let $\Omega(s)$ be the index of the black vertex incident with the edge labelled s, and let $\mathcal{I}(v)$ be the set of all labels of edges incident with the vertex v. Then

$$\prod_{r=1}^{B(T)} x_r^{\theta(b_r)} = \prod_{r=1}^{W(T)} \prod_{s\in\mathcal{I}(w_r)} x_{\Omega(s)}^{i_s},$$

so from (12),

$$\psi_m L_T = \sum_{\mathbf{i}} \Xi_{m,B(T)} \sum_{\zeta} \prod_{r=1}^{W(T)} \left(\left(\prod_{s\in\mathcal{I}(w_r)} x_{\Omega(s)}^{i_s} \right) \psi_{\zeta_r}\ \theta(w_r) \frac{\partial}{\partial p_{\theta(w_r)}} F^{(|\zeta_r|+1)} \right).$$

But, using (8),

$$\psi_{\zeta_r}\ j \frac{\partial}{\partial p_j} F^{(|\zeta_r|+1)} = [t^j] t \frac{\partial}{\partial t} P^{(|\zeta_r|+1)}(t, x_{\zeta_r}),$$

and we thus obtain

$$\psi_m L_T = \Xi_{m,B(T)} \sum_{\zeta} \prod_{r=1}^{W(T)} \left(t \frac{\partial}{\partial t} P^{(|\zeta_r|+1)}(t, x_{\zeta_r}) \right) \circ H^{+}(t; \chi(w_r)).$$

Clearly, for the right-hand side of the partial differential Eqn (18), we have

$$\psi_m \frac{\partial \Phi}{\partial u} = \frac{1}{k-1} \left(x_1 \frac{\partial}{\partial x_1} + \cdots + x_m \frac{\partial}{\partial x_m} + m - 2 \right) P^{(m)}(x_1, \ldots, x_m),$$

and the result now follows. □

In order to use Theorem 3.1 in practice, to determine $P^{(m)}$ in terms of k as a parameter, it is convenient to refine the statement of the result to involve a smaller and more manageable set of trees. Thus, for $m \geq 2$, let $\mathcal{S}_m$ be the set of plane, vertex two-coloured (black, white) trees with at least one edge, at most m black vertices, and no monovalent white vertices. These trees are not edge-rooted, and we let $|\mathrm{aut}\,(S)|$ denote the number of automorphisms of such a tree S. Let $B(S)$ and $W(S)$ denote the numbers of black and white vertices in S, respectively, and label these vertices $b_1, \ldots, b_{B(S)}$ and $w_1, \ldots, w_{W(S)}$, respectively, arbitrarily. Let $G(t, z) = \sum_{i\geq 1} g_i(t) z^{i-1}$, and define the linear operator $\Gamma_{m,l}$ to act on any finite product of g's by

$$\Gamma_{m,l} \prod_{r=1}^{s} g_{i_r}(a_r) = \sum_{\zeta} \prod_{r=1}^{s} a_r \frac{\partial}{\partial a_r} P^{(i_r)}(a_r, x_{\zeta_r}),$$

where the summation is over all ordered set partitions $\zeta = (\zeta_1, \ldots, \zeta_s)$ of $\{l+1, \ldots, m\}$, with $|\zeta_r| = i_r - 1$, for $r = 1, \ldots, s$, and $\sum_{r=1}^{s} i_r - 1 = m - l$, for any $s \geq 1$. If $\sum_{r=1}^{s} i_r - 1 \neq m - l$, then the value of $\Gamma_{m,l}$ is 0.

COROLLARY 3.2. *Let $d(v)$ be the degree of vertex v. Then, for $m \geq 1$,*

$$\sum_{S\in\mathcal{S}_m} \frac{\Xi\Gamma_{m,B(S)}}{|\mathrm{aut}\,(S)|} [y^k z^m] \left(\prod_{j=1}^{B(S)} z \left(\frac{y}{1 - yG(x_j, z)} \right)^{d(b_j)} \right) \left(\prod_{r=1}^{W(S)} G(t, z) \circ H^{+}(t; \chi(w_r)) \right)$$
$$+ \frac{1}{k}\ \Xi\Gamma_{m,1}\ [z^{m-1}] G(x_1, z)^k = \frac{1}{k-1} \left(x_1 \frac{\partial}{\partial x_1} + \cdots + x_m \frac{\partial}{\partial x_m} + m - 2 \right) P^{(m)}(x_1, \ldots, x_m),$$

where $\Xi\Gamma_{m,l}$ denotes $\Xi_{m,l}\Gamma_{m,l}$.

PROOF. We begin by re-expressing the left-hand side, L, of the partial differential equation given in Theorem 3.1, using the operator $\Gamma_{m,l}$ defined above. Thus we obtain

$$L = \frac{1}{k} \sum_{T \in \mathcal{T}_k} \Xi \Gamma_{m,B(T)} \, [z^{m-B(T)}] \prod_{r=1}^{W(T)} G(t,z) \circ H^+(t;\chi(w_r)).$$

But symmetry implies that the vertices of T can be labelled $b_1, \ldots, b_{B(T)}$ and $w_1, \ldots, w_{W(T)}$ arbitrarily. In particular, all edge-rootings of the same plane tree give the same contribution to L; thus let $\hat{\mathcal{T}}_k$ consist of plane, vertex two-coloured (black, white) trees with k edges, with vertices labelled arbitrarily, and let $\epsilon(T)$ be the number of edge-rootings of $T \in \hat{\mathcal{T}}_k$. Then

$$L = \sum_{T \in \hat{\mathcal{T}}_k} \frac{\epsilon(T)}{k} \, \Xi \Gamma_{m,B(T)} \, [z^{m-B(T)}] \prod_{r=1}^{W(T)} G(t,z) \circ H^+(t;\chi(w_r)).$$

Now note that if w_r is monovalent (of degree 1), then $\chi(w_r) = \{x_j\}$, where b_j is the unique black vertex adjacent to w_r, so we obtain $G(t,z) \circ H^+(t,\chi(w_r)) = G(x_j,z)$. Thus the contribution to L for T with a single black vertex, adjacent to k monovalent white vertices (a black-centred *star*), is $\Xi\Gamma_{m,1} \; [z^{m-1}]G(x_1,z)^k/k$. Moreover, each other tree T in $\hat{\mathcal{T}}_k$ is constructed from a unique tree S in $\mathcal{S}_m$, by embedding a black-centred star in some subset of the corners of S at each black vertex, and identifying the black centre vertex with that black vertex, so that there is a total of k edges in the resulting plane tree T. (The trees with more than m black vertices contribute 0 to L. A *corner* is an open region of the face bounded by two edges that are encountered consecutively when traversing the outer face.) The generating series for the possible embeddings at each corner at vertex b_j is $(1 - yG(x_j,z))^{-1}$, where y records the number of additional edges in this construction, and there are $d(b_j)$ such corners at vertex b_j. The number of edges in S is given by $\sum_{j=1}^{B(S)} d(b_j)$, where $B(S) = B(T)$. Thus

$$\begin{aligned} L = &\sum_{S \in \mathcal{S}_m} \alpha \; \Xi\Gamma_{m,B(S)} \, [y^k z^m] \left(\prod_{j=1}^{B(S)} z \left(\frac{y}{1 - yG(x_j,z)} \right)^{d(b_j)} \right) \left(\prod_{r=1}^{W(S)} G(t,z) \circ H^+(t;\chi(w_r)) \right) \\ &+ \frac{1}{k} \, \Xi\Gamma_{m,1} \, [z^{m-1}] G(x_1,z)^k, \end{aligned}$$

where $\alpha = \epsilon(T)\beta/k$, and β gives the number of times that T is constructed by embedding black-centred stars in all possible ways. But $\epsilon(T)\beta = k/|\mathrm{aut}\,(S)|$, and the result follows. □

The following result gives an explicit expression for the Hadamard product of a formal power series with H^+. This will be useful in the next section to evaluate the Hadamard products that arise when we apply Corollary 3.2 for small values of m.

PROPOSITION 3.3. *Let $f(t)$ be a formal power series in t. Then*

$$f(t) \circ H^+(t; x_1, \ldots, x_k) = \sum_{i=1}^{k} f(x_i) \prod_{\substack{1 \le p \le k \\ p \ne i}} \frac{x_p}{x_i - x_p},$$

where empty products are equal to 1.

4. PROOFS OF THE SUPPORTING THEOREMS

It is convenient to identify the trees in $\mathcal{S}_3$ in order to identify how the terms in the partial differential equations for various choices of $m \le 3$ arise. Let S_1 be the plane tree with two black monovalent vertices adjacent to a single white vertex of degree 2. Let S_2 be the plane tree with three black monovalent vertices, adjacent to a single white vertex of degree 3. Let S_3 be the path on four edges, with vertices alternately black and white, where the monovalent vertices are both black.

4.1. Proof of Theorem 1.3. Consider the case $m = 1$ in Corollary 3.2. Then since contributions on the left-hand side come only from the last term, we obtain the differential equation

$$\frac{1}{k}\left(x_1 \frac{dP^{(1)}}{dx_1}\right)^k = \frac{1}{k-1}\left(x_1 \frac{d}{dx_1} - 1\right)P^{(1)}$$

for $P^{(1)}$. To solve this equation, differentiate the equation with respect to x_1 and multiply by x_1. Then, with $f = x_1 dP^{(1)}/dx_1$, we obtain

$$x_1 \frac{df}{dx_1} = \frac{f}{1-(k-1)f^{k-1}}.$$

It is now straightforward to determine, for formal power series in x, that $f = w_1$, by comparing this differential equation with (9), and using the initial condition $f(0) = 0$. The result follows immediately. □

4.2. Proof of Theorem 1.4. Consider the case $m = 2$ in Corollary 3.2. Now contributions on the left-hand side come from the last term, and from the tree S_1. Thus, substituting the expression for $P^{(1)}$ from Theorem 1.3, and applying Proposition 3.3 to evaluate the Hadamard products, we obtain

$$\begin{aligned} w_1^{k-1}x_1\frac{\partial P^{(2)}}{\partial x_1} + w_2^{k-1}x_2\frac{\partial P^{(2)}}{\partial x_2} + \frac{x_2w_1 - x_1w_2}{x_1 - x_2}\,\frac{w_1^{k-1} - w_2^{k-1}}{w_1 - w_2} \\ = \frac{1}{k-1}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right)P^{(2)}. \end{aligned}$$

so, rearranging, we have

$$\begin{aligned} \frac{1}{k-1}\left((1-(k-1)w_1^{k-1})x_1\frac{\partial}{\partial x_1} + (1-(k-1)w_2^{k-1})x_2\frac{\partial}{\partial x_2}\right)P^{(2)} \\ = \frac{x_2w_1 - x_1w_2}{x_1 - x_2}\,\frac{w_1^{k-1} - w_2^{k-1}}{w_1 - w_2}. \end{aligned}$$

It is now straightforward to verify that

$$P^{(2)}(x_1, x_2) = \log\left(\frac{w_1 - w_2}{x_1 - x_2}\right) - \frac{w_1^k - w_2^k}{w_1 - w_2},$$

by confirming that it satisfies the above differential equation, and the initial condition $P^{(2)}(0, 0) = 0$. (Note that the constant term in the expansion of $(w_1 - w_2)/(x_1 - x_2)$ as a formal power series in x_1, x_2 is 1, so the logarithm exists.) Finally, apply the operator $x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ to $P^{(2)}$, and the result follows. □

4.3. Proof of Theorem 1.5. Consider the case $m = 3$ in Corollary 3.2. Then contributions on the left-hand side come from the last term, together with the trees S_1, S_2, S_3. Substituting the expression for $P^{(1)}$ from Theorem 1.3, it follows that

$$\begin{aligned}
&\Xi_{3,1} w_1^{k-1} x_1 \frac{\partial}{\partial x_1} P^{(3)} \\
&\quad + \Xi_{3,1}(k-1) w_1^{k-2} \left(x_1 \frac{\partial}{\partial x_1} P^{(2)}(x_1, x_2) \right) \left(x_1 \frac{\partial}{\partial x_1} P^{(2)}(x_1, x_3) \right) \\
&\quad + \Xi_{3,2} \left(\frac{\partial}{\partial w_1} \frac{w_1^{k-1} - w_2^{k-1}}{w_1 - w_2} \right) \left(x_1 \frac{\partial}{\partial x_1} P^{(2)}(x_1, x_3) \right) (w(t) \circ H^+(t; x_1, x_2)) \\
&\quad + \frac{1}{2}\, \Xi_{3,2} \left(\frac{w_1^{k-1} - w_2^{k-1}}{w_1 - w_2} \right) \left(t \frac{\partial}{\partial t} P^{(2)}(t, x_3) \circ H^+(t; x_1, x_2) \right) \\
&\quad + 2h_{k-3}(w_1, w_2, w_3)(w(t) \circ H^+(t; x_1, x_2, x_3)) \\
&\quad + \frac{1}{2}\, \Xi_{3,3} \left(\frac{\partial}{\partial w_2} h_{k-3}(w_1, w_2, w_3) \right) (w(t) \circ H^+(t; x_1, x_2))\, (w(t) \circ H^+(t; x_2, x_3)) \\
&= \frac{1}{k-1} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + 1 \right) P^{(3)}.
\end{aligned}$$

The third and fourth expressions on the left-hand side arise from S_1, and the fifth and sixth expressions arise from S_2 and S_3, respectively. Note that, in the fifth expression, we have used the fact that all six terms that arise in the symmetrization by $\Xi_{3,3}$ are equal. (In general, there will be at least $|\mathrm{aut}\,(S)|$ symmetries among the first l variables that arise when applying $\Xi_{m,l}$, where $l = B(S)$.) Now apply Proposition 3.3 to evaluate the Hadamard products, and use the fact that

$$(1 - (k-1) w^{k-1}) x \frac{\partial}{\partial x} = w \frac{\partial}{\partial w} \tag{21}$$

(this latter follows from (9)). Simplifying (with the help of Maple), we obtain

$$\begin{aligned}
&\frac{1}{k-1} \left(\sum_{i=1}^{3} w_i \frac{\partial}{\partial w_i} + 1 \right) P^{(3)} = (k-1)(w_1^{k-2} A_{12} A_{13} + w_2^{k-2} A_{21} A_{23} + w_3^{k-2} A_{31} A_{32}) \\
&\quad + \frac{w_1^{k-1} - w_2^{k-1}}{(w_1 - w_2)^2} (w_2 A_{13} - w_1 A_{23}) + \frac{w_1^{k-1} - w_3^{k-1}}{(w_1 - w_3)^2} (w_3 A_{12} - w_1 A_{32}) \\
&\quad + \frac{w_2^{k-1} - w_3^{k-1}}{(w_2 - w_3)^2} (w_3 A_{21} - w_2 A_{31}),
\end{aligned}$$

where

$$A_{ij} = \frac{w_i w_j}{1 - (k-1) w_i^{k-1}} \frac{w_i^{k-1} - w_j^{k-1}}{(w_i - w_j)^2}.$$

The solution to this equation is given in Theorem 1.5, and has been verified with the aid of Maple, giving the desired result. □

5. Computational Comments and Conjectures

We have shown in Section 4 that $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ can each be obtained as the solutions to first-order linear partial differential equations. We believe that $P^{(m)}$, for $m \geq 4$, can be obtained in a similar way as the solution of such an equation. Moreover, we conjecture that the equation for any $m \geq 3$, (obtained from Corollary 3.2, and applying (21) as described for $m = 1, 2, 3$ in Section 4) after multiplying through by $k - 1$, is of the form

$$\left(\sum_{i=1}^{m} w_i \frac{\partial}{\partial w_i} + (m-2)\right) P^{(m)} = R_m(w_1, \ldots, w_m),$$

where R_m is a rational function in $w_1, \ldots, w_m$, obtained from $P^{(1)}, \ldots, P^{(m-1)}$. That is, there is no dependency of R_m on $x_1, \ldots, x_m$ except through (10). Now let $Q^{(m)}(t)$ be obtained by substituting tw_i for w_i in $P^{(m)}$ for $1 = 1, \ldots, m$. Then the above partial differential equation is transformed into the first-order linear ordinary differential equation

$$\frac{d}{dt}(t^{m-2} Q^{(m)}(t)) = t^{m-3} R_m(tw_1, \ldots, tw_m), \tag{22}$$

which can be solved routinely, in theory. In practice, this is precisely how we obtained $P^{(3)}$, with the aid of Maple, in Section 4. However, even in this case, the simplification of the equation was difficult; we provided human help by proving that the rational expression on the right-hand side of the equation is independent of the x's, and then replaced each x_i by w_i to evaluate it. This explains how the A_{ij} arise, as $x_i \frac{\partial}{\partial x_i} P^{(2)}(x_i, x_j)$ evaluated at $x_i = w_i$ and $x_j = w_j$.

For $m = 4$, the expressions became too big to be tractable, and we have not found a convenient way of circumventing this. We conjecture that, for each $m \geq 3$, $P^{(m)}$ is a rational function of $w_1, \ldots, w_m$, whose denominator is consistent with Conjecture 1.2, using (9). (Note that for $m = 2$, the right-hand side of the equation, as obtained in the Proof of Theorem 1.4, is not a rational function of w_1, w_2 alone, but rather involves x_1, x_2 as well.)

Note added in press. An independent proof of (5), from the point of view of singularity theory, has been given in [5].

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