Transitive powers of Young–Jucys–Murphy elements are central

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A R T I C L E   I N F O

A B S T R A C T

Although powers of the Young–Jucys–Murphy elements $X_i = (1\,i) + (2\,i) + \cdots + (i-1\,i)$, $i = 1, \ldots, n$, in the symmetric group $\mathfrak{S}_n$ acting on $\{1, \ldots, n\}$ do not lie in the center of the group algebra of $\mathfrak{S}_n$, we show that transitive powers, namely the sum of the contributions from elements that act transitively on $[n]$, are central. We determine the coefficients, which we call star factorization numbers, that occur in the resolution of transitive powers with respect to the class basis of the center of $\mathfrak{S}_n$, and show that they have a polynomiality property. These centrality and polynomiality properties have seemingly unrelated consequences. First, they answer a question raised by Pak [I. Pak, Reduced decompositions of permutations in terms of star transpositions, generalized Catalan numbers and *k*-ary trees, Discrete Math. 204 (1999) 329–335] about reduced decompositions; second, they explain and extend the beautiful symmetry result discovered by Irving and Rattan [J. Irving, A. Rattan, Minimal factorizations of permutations into star transpositions, Discrete Math., in press, math.CO/0610640]; and thirdly, we relate the polynomiality to an existing polynomiality result for a class of double Hurwitz numbers associated with branched covers of the sphere, which therefore suggests that there may be an ELSV-type formula (see [T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein, Hurwitz numbers and intersections on moduli spaces of curves, Invent. Math. 146 (2001) 297–327]) associated with the star factorization numbers.

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1. Introduction and background

We begin with an account of the Main Theorem of this paper and its relationship to the enumeration of a class of ramified covers of the sphere, a question that arises in algebraic geometry.

1.1. Young–Jucys–Murphy elements and the Main Theorem

The *Young–Jucys–Murphy elements* in the group algebra $\mathbb{C}\mathfrak{S}_n$ of the symmetric group $\mathfrak{S}_n$ on $[n] := \{1, \dots, n\}$, are given by

$$X_i = (1\,i) + (2\,i) + \cdots + (i-1\,i), \quad i = 1, \dots, n,$$

where $X_1 = 0$ (see, e.g., [VO] for a detailed description and further references). Let $Z(n)$ denote the center of $\mathbb{C}\mathfrak{S}_n$, $n \geqslant 1$. Then the algebra generated by $Z(1), \dots, Z(n)$ is called the *Gel'fand–Tsetlin algebra*, and one of the key results described in [VO] is the fact that this algebra is also generated by $X_1, \dots, X_n$, despite the fact that X_n is clearly *not* contained in $Z(n)$ for any $n > 2$.

For $A \in \mathbb{C}\mathfrak{S}_n$ and positive integer r, we define the rth *transitive power* $A^{\langle r \rangle}$ of $A = \sum_{\sigma_i \in \mathfrak{S}_n} a_i \sigma_i$ by

$$A^{\langle r \rangle} = \sum a_{i_1} \cdots a_{i_r} \sigma_{i_1} \cdots \sigma_{i_r},$$

where the summation on the right-hand side is restricted to $\sigma_{i_1}, \dots, \sigma_{i_r} \in \mathfrak{S}_n$ such that $\langle \sigma_{i_1}, \dots, \sigma_{i_r} \rangle$, the group generated by $\sigma_{i_1}, \dots, \sigma_{i_r}$, acts transitively on $[n]$. The subject of this paper is the transitive power $X_n^{\langle r \rangle}$ of the Young–Jucys–Murphy element X_n for arbitrary positive integers n, r.

Our main result, Theorem 1.1, is that the transitive powers of X_n, unlike powers, *are* contained in $Z(n)$. Moreover, since a basis for $Z(n)$ is given by the set of all K_α where $\mathsf{K}_\alpha = \sum_{\pi \in \mathcal{K}_\alpha} \pi$, and $\mathcal{K}_\alpha$ is the conjugacy class of $\mathfrak{S}_n$ (naturally) indexed by the partition α of n, then Theorem 1.1 expresses the transitive power $X_n^{\langle r \rangle}$ as an explicit linear combination of the K_α. (Note that an explicit expression for $X_n^{\langle r \rangle}$ can be obtained immediately from the Principle of Inclusion–Exclusion, since the transitivity restriction simply says in this case that the surviving products contain at least one occurrence of $(i\,n)$ as a factor for every $i \in [n-1]$. The resulting expression is

$$X_n^{\langle r \rangle} = \sum_{\gamma \subseteq [n-1]} (-1)^{|\gamma|} X_n(\overline{\gamma})^r,$$

where $X_n(\gamma) = \sum_{j \in \gamma} (j\,n)$. However, we have been unable to use this explicit expression to prove Theorem 1.1.)

We use the following notation and terminology for partitions. If $\alpha_1, \dots, \alpha_k$ are positive integers with $1 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_k$ and $\alpha_1 + \cdots + \alpha_k = n$, then $\alpha = (\alpha_1, \dots, \alpha_k)$ is a *partition* of n with k *parts*, and we write $\alpha \vdash n$ and $l(\alpha) = k$, for $n, k \geqslant 0$. Let $\alpha \setminus \alpha_j$ denote the partition obtained by removing the single part α_j from α, for any $j = 1, \dots, k$. Let $\alpha \cup m$ denote the partition obtained by inserting a single new part equal to m into α (placed in the appropriate ordered position). Let $2\alpha = (2\alpha_2, \dots, 2\alpha_k)$, and $a_\alpha = a_{\alpha_1} \cdots a_{\alpha_k}$ for any indeterminates $a_1, a_2, \dots$. Let $\mathcal{P}$ denote the set of all partitions, including the empty partition ε, which it a partition of 0 with 0 parts. If α has f_j parts equal to j for each $j \geqslant 1$, then we also use $(1^{f_1} 2^{f_2} \cdots)$ to denote α, and we write $|\operatorname{Aut}\alpha| = \prod_{j \geqslant 1} f_j!$

We now state our main result. The notation $p_i \equiv p_i(\alpha)$ is used to denote the ith power sum of the parts of the partition α, $i \geqslant 1$, and $q_i \equiv q_i(\alpha) := p_i + p_1 - 2$, $i \geqslant 2$, and we define ξ_{2j} and ξ by

$$\sum_{j \geqslant 1} \xi_{2j} x^{2j} := \log(\xi(x)), \quad \text{where } \xi(x) := 2x^{-1}\operatorname{sh}\left(\frac{1}{2}x\right), \tag{1}$$

where *sh* and *ch* denote, respectively, hyperbolic sine and cosine.

Theorem 1.1 *(Main Theorem). For $r \geqslant 0$, $X_n^{\langle r \rangle}$ is contained in the center $Z(n)$ of $\mathbb{C}\mathfrak{S}_n$. Moreover, the resolution of $X_n^{\langle r \rangle}$ with respect to the class basis of $Z(n)$ is*

$$X_n^{\langle r \rangle} = \sum_{\substack{\alpha \vdash n, \\ g \geqslant 0}} a_g(\alpha) \mathsf{K}_\alpha$$

where the range of summation on the right-hand side is restricted by the condition $n + m - 2 + 2g = r$, with $m = l(\alpha)$, and $a_g(\alpha)$ is a polynomial in the parts of α given by

$$a_g(\alpha) = \frac{1}{n!}(n + m - 2 + 2g)! \, \alpha_1 \cdots \alpha_m Q_g(\alpha), \quad \text{where } Q_g := \sum_{\beta \vdash g} \frac{\xi_{2\beta} q_{2\beta}}{|\operatorname{Aut}\beta|}, \; g \geqslant 0.$$

For example, the explicit expressions for small genera $g = 0, \ldots, 5$ are

$$Q_0 = 1, \qquad Q_1 = \frac{1}{24}q_2, \qquad Q_2 = \frac{1}{5760}(-2q_4 + 5q_{2^2}), \qquad Q_3 = \frac{1}{2^3 9!}(16q_6 - 42q_{42} + 35q_{2^3}),$$

$$Q_4 = \frac{1}{3 \cdot 2^7 10!}(-144q_8 + 320q_{62} + 84q_{4^2} - 420q_{42^2} + 175q_{2^4}),$$

$$Q_5 = \frac{1}{3 \cdot 2^8 12!}(768q_{10} - 1584q_{82} - 704q_{64} + 1760q_{62^2} + 924q_{4^2 2} - 1540q_{42^3} + 385q_{2^5}),$$

1.2. Background

1.2.1. Minimal factorizations into star transpositions

We now turn our attention temporarily to another point of view. The transpositions $(1\,a)$, for $a = 2, \ldots, n$, are called *star* transpositions in $\mathfrak{S}_n$, with the distinguished element 1 (it appears in each transposition) referred to as the *pivot* element. An ordered factorization $(\tau_1, \ldots, \tau_r)$ of $\sigma \in \mathfrak{S}_n$ into star transpositions is said to be *transitive* if the group generated by $\tau_1, \ldots, \tau_r$ acts transitively on $[n]$. For a transitive factorization of $\sigma \in \mathcal{K}_\alpha$ into r star transpositions, a result in [GJ0] implies that $r = n + m - 2 + 2g$ for some non-negative integer g, where α has m parts. Thus $r \geqslant n + m - 2$, and we refer to transitive factorizations into $n + m - 2$ star transpositions as *minimal*.

Pak [P] enumerated minimal factorizations (he called them *reduced decompositions*) into star transpositions for permutations fixing the pivot element 1, with exactly m other cycles, each of length $k \geqslant 2$. More recently, Irving and Rattan [IR] generalized Pak's result by considering minimal factorizations of arbitrary permutations into star transpositions, and proved the following elegant result.

Theorem 1.2. *(See [IR].) For each permutation $\sigma \in \mathcal{K}_\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_1 + \cdots + \alpha_m = n$ and $m, n \geqslant 1$, the number of transitive factorizations of σ into $n + m - 2$ star transpositions is*

$$\frac{(n + m - 2)!}{n!}\alpha_1 \cdots \alpha_m.$$

Because of the apparent asymmetry of these factorizations (i.e., the pivot element 1 appears in *every factor*), the fact that Theorem 1.2 is constant on conjugacy classes is particularly surprising (we shall refer to this fact as the *centrality property* of Theorem 1.2). The proofs given in [P] and [IR] are bijective, involving restricted words and plane trees.

In terms of factorizations into star transpositions, the number $a_g(\alpha)$ given by the Main Theorem clearly can be interpreted as the number of transitive factorizations of each $\sigma \in \mathcal{K}_\alpha$ into $n + m - 2 + 2g$ star transpositions, with pivot element n. We shall call $a_g(\alpha)$ a *star factorization number*. Thus Theorem 1.2 is precisely the case $g = 0$ of Theorem 1.1 (the necessary relabeling of the pivot element is justified by the centrality of these results). The investigation described in this paper answers Pak's [P]

question about an explicit expression for the general case. It was motivated by Irving and Rattan's paper, in our attempt to determine whether the centrality of their remarkable result for star factorizations with a minimum number of factors persisted for star factorizations with an *arbitrary* number of factors.

1.2.2. Connections with algebraic geometry

The connection to algebraic geometry is made through Hurwitz's encoding [H] of an n-sheeted branched cover of the sphere in terms of transpositions that represent the sheet transitions at the elementary branch points. In this context, the transitivity of the factorizations corresponds to the connectedness of the cover. From this perspective, the coefficient $a_g(\alpha)$ in Theorem 1.1 counts genus g branched covers of the sphere in which the branching over the point 0 is specified by α, and there are $n + m - 2 + 2g$ other simple branch points, each of which corresponds to a transition between sheet number n (the *pivot* sheet) and another sheet. For the corresponding transitive factorizations into star transpositions, we therefore also refer to g as the *genus of the factorization* (e.g., Theorem 1.2 counts genus 0 factorizations). For further details about branched covers, see, for example, [GJ0,GJVn, GJV] and [H].

The *double Hurwitz number* $H^g_{(n),\alpha}$ is equal to the number of genus g branched covers of the sphere in which the branching over the points 0 and ∞ is specified by (n) and α, respectively, together with $m - 1 + 2g$ other simple branch points. A scaling of this double Hurwitz number to

$$b_g(\alpha) := \alpha_1 \cdots \alpha_m H^g_{(n),\alpha}$$

gives the number of transitive factorizations of each $\sigma \in \mathsf{K}_\alpha$ into $m - 1 + 2g$ transpositions and a single n-cycle. There is a striking similarity between Theorem 1.1 and the following result, in which the notation $\hat{q}_i := p_i - 1$, $i \geqslant 1$ is used.

Theorem 1.3. *(See [GJV].) For $r \geqslant 0$, the resolution of $\mathsf{K}^r_{(1^{n-2}\,2)}\mathsf{K}_n$ with respect to the class basis of the center $Z(n)$ of $\mathbb{C}\mathfrak{S}_n$ is*

$$\mathsf{K}^r_{(1^{n-2}\,2)}\mathsf{K}_n = \sum_{\substack{\alpha \vdash n, \\ g \geqslant 0}} b_g(\alpha)\mathsf{K}_\alpha, \tag{2}$$

where the range of summation on the right-hand side is restricted by the condition $m - 1 + 2g = r$, with $m = l(\alpha)$, and $b_g(\alpha)$ is a polynomial in the parts of α given by

$$b_g(\alpha) = (m - 1 + 2g)!\, n^{m-2+2g} \alpha_1 \cdots \alpha_m \widehat{Q}_g(\alpha), \quad \text{where } \widehat{Q}_g := \sum_{\beta \vdash g} \frac{\xi_{2\beta}\hat{q}_{2\beta}}{|\operatorname{Aut}\beta|}, \ g \geqslant 0.$$

Theorem 1.3 is a restatement of Theorem 3.1 in [GJV], which gives a formula for the double Hurwitz number $H^g_{(n),\alpha}$. (Note that this double Hurwitz number is given by the products on the left-hand side of (2) for which the subgroup generated by the factors acts transitively on $[n]$. However, each term in K_n acts transitively on $[n]$, so the transitivity condition holds for all products on the left-hand side of (2).)

1.2.3. Two relationships between Theorems 1.1 and 1.3

To explore a more direct relationship between Theorems 1.1 and 1.3, we now give two expressions for $a_g(\alpha)$ in terms of the $b_h(\gamma)$'s.

The first is very simple and expresses $a_g(\alpha)$, which enumerates factorizations in $\mathfrak{S}_n$, directly in terms of $b_g(\alpha \cup 1^{n-1})$, which enumerates factorizations in $\mathfrak{S}_{2n-1}$.

Corollary 1.4. *For $g \geqslant 0$ and α a partition of n with m parts, we have*

$$a_g(\alpha) = \frac{1}{n!(2n-1)^{n+m-3+2g}} b_g(\alpha \cup 1^{n-1}).$$

Proof. In the notation of Theorems 1.1 and 1.3, clearly $q_i(\alpha) = \hat{q}_i(\alpha \cup 1^{n-1})$, so $Q_g(\alpha) = \widehat{Q}_g(\alpha \cup 1^{n-1})$. The result follows immediately from Theorems 1.1 and 1.3. $\square$

The second expresses $a_g(\alpha)$ as a linear combination of $b_{g-h}(\alpha)$, $0 \leqslant h \leqslant g$, each of which enumerates factorizations in $\mathfrak{S}_n$.

Corollary 1.5. *For $g \geqslant 0$ and α a partition of n with m parts, we have*

$$a_g(\alpha) = \frac{1}{n!} \sum_{h=0}^{g} \frac{b_{g-h}(\alpha)}{n^{m-2+2g-2h}} \binom{n+m-2+2g}{n-1+2h} \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j \left(\frac{1}{2}(n-1)-j\right)^{n-1+2h}.$$

Proof. In the notation of Theorems 1.1 and 1.3, clearly $q_i(\alpha) = \hat{q}_i(\alpha) + n - 1$. Then from Theorems 1.1 and 1.3, and (1), we have

$$\sum_{g \geqslant 0} Q_g(\alpha)x^{2g} = \exp\left(\sum_{j \geqslant 1} \xi_{2j}q_{2j}(\alpha)x^{2j}\right) = \xi(x)^{n-1} \sum_{g \geqslant 0} \widehat{Q}_g(\alpha)x^{2g}. \tag{3}$$

But, for $h \geqslant 0$, we have (using the notation $[A]B$ to denote the *coefficient* of A in B)

$$[x^{2h}]\xi(x)^{n-1} = [x^{n-1+2h}]\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j \frac{(\frac{1}{2}(n-1)-j)^{n-1+2h}}{(n-1+2h)!}$$

and, together with (3), this gives

$$Q_g(\alpha) = \sum_{h=0}^{g} \frac{\widehat{Q}_{g-h}(\alpha)}{(n-1+2h)!} \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j \left(\frac{1}{2}(n-1)-j\right)^{n-1+2h}.$$

The result follows immediately from Theorems 1.1 and 1.3. $\square$

1.3. Outline

In Section 2, we introduce a generating series for the number of transitive factorizations into star transpositions in arbitrary genus, and prove that it is the unique formal power series solution of a linear partial differential equation that we call the *Join–Cut Equation* for this class of factorizations. The proof is based on a join–cut analysis of these factorizations, since the left-most factor σ either joins two cycles of the product π of the remaining factors to form one cycle or cuts one cycle of π into two, depending on whether the two elements moved by σ are, respectively, in different cycles of π, or in the same cycle. This approach has been applied previously where the factors are arbitrary transpositions, for the genus 0 case in [GJ0], and for higher genera in [GJ1,GJ2,GJVn] and [GJV].

In Section 3, we solve the Join–Cut Equation to obtain the generating series for transitive factorizations into star transpositions in arbitrary genus. Then, by determining the coefficients in this generating series, we prove Theorem 1.1 (and hence also give a new proof of Theorem 1.2).

In Section 4, we pose some questions that arise from this investigation, but that we have been unable to resolve.

2. The Join–Cut Equation

Let $\mathfrak{S}_{\mathcal{A}}$ denote the symmetric group on an arbitrary set $\mathcal{A}$. For an arbitrary set $\mathcal{A}$ of size n containing 1 (for convenience, we shall consider star transpositions with pivot element 1), let $\mathcal{K}_{\alpha}^{(i)}$ denote the set of all permutations in $\mathfrak{S}_{\mathcal{A}}$ in which 1 lies on a cycle of length i and the remaining cycle-lengths in the disjoint cycle representation are given by the parts of α, where $\alpha \vdash n - i$, for $n \geqslant i \geqslant 1$. It is straightforward to determine, independently of the choice of $\mathcal{A}$, that

$$\left| \mathcal{K}_{\alpha}^{(i)} \right| = \binom{n-1}{i-1}(i-1)! |\mathcal{K}_{\alpha}| = \frac{(n-1)!}{\alpha_1 \cdots \alpha_k |\operatorname{Aut}\alpha|}, \tag{4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$. Consider a fixed permutation $\sigma \in \mathcal{K}_{\alpha}^{(i)}$ in $\mathfrak{S}_n$, and let $c_g(i, \alpha)$ be the number of transitive factorizations of σ into $n + k - 1 + 2g$ star transpositions (this number is constant for each such σ because of the symmetry of elements $2, \ldots, n$; note that σ lies in the conjugacy class $\mathcal{K}_{\alpha \cup i}$, which has $m = k + 1$ cycles). Let Ψ denote the generating series

$$\Psi(t, u, x; \mathbf{z}, \mathbf{y}) := \sum_{\substack{n \geqslant i \geqslant 1, \\ k, g \geqslant 0}} n t^n \frac{u^{n+k-1+2g}}{(n+k-1+2g)!} x^{2g} z_i \sum_{\substack{\alpha \in \mathcal{P}, \\ \alpha \vdash n-i, \\ l(\alpha)=k}} \left| \mathcal{K}_{\alpha}^{(i)} \right| c_g(i, \alpha) y_\alpha. \tag{5}$$

The following result is the Join–Cut Equation for the set of transitive factorizations into star transpositions. It states that Ψ is annihilated by the partial differential operator

$$\Delta := \frac{\partial}{\partial u} - t \frac{\partial}{\partial t} t \sum_{i \geqslant 1} z_{i+1} \frac{\partial}{\partial z_i} - \sum_{i,j \geqslant 1} z_i y_j \frac{\partial}{\partial z_{i+j}} - x^2 \sum_{i,j \geqslant 1} j z_{i+j} \frac{\partial^2}{\partial z_i \partial y_j}. \tag{6}$$

Theorem 2.1 (Join–Cut Equation). *The generating series $\Psi = \Psi(t, u, x; \mathbf{z}, \mathbf{y})$ is the unique formal power series solution of $\Delta\Psi = 0$, with initial condition $\Psi(t, 0, x; \mathbf{z}, \mathbf{y}) = z_1 t$.*

Proof. Fix a triple (k, g, i) of integers with $k, g \geqslant 0$ and $i \geqslant 1$ to be other than $(0, 0, 1)$. Also fix a partition α with $l(\alpha) = k$ and a permutation $\sigma \in \mathcal{K}_{\alpha}^{(i)}$ in $\mathfrak{S}_n$, where $\alpha \vdash n - i$. Consider a transitive factorization $(\tau_1, \ldots, \tau_r)$ of σ into star transpositions, where $r = n + k - 1 + 2g$. For this factorization, we let $\pi = \tau_2 \cdots \tau_r = \tau_1 \sigma$, and $\tau_1 = (1a)$. There are $c_g(i, \alpha)$ such factorizations of σ, and we obtain a recurrence equation for $c_g(i, \alpha)$ by considering the following case analysis for these factorizations which is based on the left-most factor τ_1.

Case 1. $\tau_1 \neq \tau_j$ for any $j = 2, \ldots, r$. In this case, the element a is a fixed point in π, and $(\tau_2, \ldots, \tau_r)$ is *not* a transitive factorization of π. But, if we let $\pi' \in \mathfrak{S}_{[n] \setminus \{a\}}$, whose disjoint cycle representation is obtained by removing the one-cycle containing a from the disjoint cycles of π, then $(\tau_2, \ldots, \tau_r)$ is a transitive factorization of π'. But σ is obtained from π' by inserting a immediately before 1 in the cycle of π' containing 1. This implies that $\pi' \in \mathcal{K}_{\alpha}^{(i-1)}$ in $\mathfrak{S}_{[n] \setminus \{a\}}$. Note that the transitive factorization $(\tau_2, \ldots, \tau_r)$ of π' has $r - 1 = (n - 1) + k - 1 + 2g$ factors and that this is reversible, so we conclude that the number of such factorizations is $c_g(i - 1, \alpha)$, the contribution from this case.

Case 2. $\tau_1 = \tau_j$ for some $j = 2, \ldots, r$. In this case, $(\tau_2, \ldots, \tau_r)$ is a transitive factorization of π, since for a product of star transpositions in $\mathfrak{S}_n$ to be transitive, it is necessary and sufficient that each of $(12), \ldots, (1n)$ appears at least once as a factor (as observed in [IR]). There are two subcases, based on which disjoint cycles of π contain elements 1 and a.

Subcase 2(a). 1 and a appear on the same cycle of π. In this subcase, that cycle of π is *cut* into two cycles in σ, one containing 1, and the other containing a. Consequently, for each factorization

$(\tau_1, \ldots, \tau_r)$ of σ, we obtain a factorization of π in this subcase by selecting a to be any element on the k cycles of σ not containing 1. We account for the choices of a on these cycles as follows: Suppose the cycles are indexed so they have lengths $\alpha_1, \ldots, \alpha_k$ (the cycles are all non-empty, so they are distinguishable, even if their lengths are equal). If a is on the jth such cycle, of length α_j, then there are α_j choices of a, and the cycle of π containing 1 has length $i + \alpha_j$, so we have $\pi \in \mathcal{K}_{\alpha \setminus \alpha_j}^{(i+\alpha_j)}$ in $\mathfrak{S}_n$. Since the transitive factorization $(\tau_2, \ldots, \tau_r)$ of π has $r - 1 = n + (k - 1) - 1 + 2g$ factors and this is reversible, we conclude that there are $c_g(i + \alpha_j, \alpha \setminus \alpha_j)$ such factorizations, giving a total contribution from this subcase of $\sum_{j=1}^{k} \alpha_j c_g(i + \alpha_j, \alpha \setminus \alpha_j)$.

Subcase 2(b). 1 and a appear on different cycles of π. In this subcase, these cycles of π are *joined* into a single cycle of σ, containing both 1 and a. Consequently, for each factorization $(\tau_1, \ldots, \tau_r)$ of σ, we obtain a factorization of π in this subcase by selecting a to be any other element on the cycle of σ containing 1. We account for these $i - 1$ choices of a as follows: Suppose that the cycle of σ containing 1, in cyclic order, is $(1 j_{i-1} \ldots j_1)$ (i.e., so $\sigma(1) = j_{i-1}$, $\sigma(j_t) = j_{t-1}$, for $t = 2, \ldots, i - 1$, and $\sigma(j_1) = 1$). If $a = j_m$, then π has disjoint cycles $(1 j_{i-1} \ldots j_{m+1})$ (containing 1) and $(j_m \ldots j_1)$, together with all the cycles of σ not containing 1, so we have $\pi \in \mathcal{K}_{\alpha \cup m}^{(i-m)}$ in $\mathfrak{S}_n$, and the transitive factorization $(\tau_2, \ldots, \tau_r)$ of π has $r - 1 = n + (k + 1) - 1 + 2(g - 1)$ factors. Since this is reversible, we conclude that there are $c_{g-1}(i - m, \alpha \cup m)$ such factorizations, giving a total contribution from this subcase of $\sum_{m=1}^{i-1} c_{g-1}(i - m, \alpha \cup m)$.

Adding together the contributions from these disjoint cases, we obtain the linear recurrence equation

$$c_g(i, \alpha) = c_g(i - 1, \alpha) + \sum_{j=1}^{k} \alpha_j c_g(i + \alpha_j, \alpha \setminus \alpha_j) + \sum_{m=1}^{i-1} c_{g-1}(i - m, \alpha \cup m),$$

for $k, g \geqslant 0$, $i \geqslant 1$ (except for the simultaneous choices $k = g = 0$ *and* $i = 1$) and α with $l(\alpha) = k$. The partial differential equation follows by multiplying this recurrence equation by $nt^n \frac{u^{n+k-2+2g}}{(n+k-2+2g)!} x^g z_i |\mathcal{K}_{\alpha}^{(i)}| y_{\alpha}$, and summing over the above range of k, g, i, α.

The initial condition follows from the fact that there is a single, empty factorization with no factors, of the single permutation (with 1 as a fixed point) in $\mathfrak{S}_1$. Thus we have $c_0(1, \varepsilon) = 1$. $\quad\square$

3. A proof of Theorem 1.1

3.1. An explicit solution to the Join–Cut Equation

The next result gives the explicit solution of the Join–Cut Equation in terms of the series ξ defined in (1) and $W \equiv W(t, u, x : \mathbf{z})$ where

$$W := \sum_{\ell \geqslant 1} z_{\ell} \xi(\ell u x) \xi(u x)^{\ell-2} u^{\ell-1} t^{\ell}.$$

Theorem 3.1. *Let* $Z := t \frac{\partial}{\partial t} W(t, u, x : \mathbf{z})$ *and* $Y := \xi(u x)^2 u^2 W(t, u, x : \mathbf{y})$. *Then* $\Psi = Z e^Y$.

Proof. It is a straightforward matter to show that the Join–Cut Equation with the given boundary condition has a unique solution. The remainder of the proof is a verification that Δ annihilates Ψ and that the boundary condition is satisfied.

The operator Δ is a linear combination of four differential operators. It is straightforward to obtain the four expressions for the application of each of these operators to Ψ. Let $\hat{x} := u x$ for brevity. Then the expressions are:

$$e^{-Y}\frac{\partial \Phi}{\partial u} = \sum_{\ell \geqslant 1} \ell z_\ell \xi(\hat{x})^{\ell-3} u^{\ell-2} t^\ell \big((\ell-1)\xi(\ell\hat{x})\xi(\hat{x}) + \ell\hat{x}\xi'(\ell\hat{x})\xi(\hat{x}) + (\ell-2)\hat{x}\xi(\ell\hat{x})\xi'(\hat{x})\big)$$

$$+ Z \sum_{m \geqslant 1} y_m \xi(\hat{x})^{m-1} u^m t^m \big((m+1)\xi(m\hat{x})\xi(\hat{x}) + m\hat{x}\xi'(m\hat{x})\xi(\hat{x}) + m\hat{x}\xi(m\hat{x})\xi'(\hat{x})\big),$$

$$e^{-Y}t\frac{\partial}{\partial t}t\sum_{i \geqslant 1} z_{i+1}\frac{\partial \Phi}{\partial z_i} = \sum_{i \geqslant 1} i z_{i+1}\xi(i\hat{x})\xi(\hat{x})^{i-2}u^{i-1}t^{i+1}\bigg(i+1+\sum_{m \geqslant 1} m y_m \xi(m\hat{x})\xi(\hat{x})^m u^{m+1}t^m\bigg),$$

$$e^{-Y}\sum_{i,j \geqslant 1} z_i y_j \frac{\partial \Phi}{\partial z_{i+j}} = \sum_{i,j \geqslant 1} (i+j) z_i y_j \xi\big((i+j)\hat{x}\big)\xi(\hat{x})^{i+j-2}u^{i+j-1}t^{i+j},$$

$$e^{-Y}x^2\sum_{i,j \geqslant 1} j z_{i+j}\frac{\partial^2 \Phi}{\partial z_i y_j} = x^2 \sum_{i,j \geqslant 1} ij z_{i+j}\xi(i\hat{x})\xi(j\hat{x})\xi(\hat{x})^{i+j-2}u^{i+j}t^{i+j}$$

$$= \sum_{\ell \geqslant 1} z_\ell \xi(\hat{x})^{\ell-2}u^{\ell-2}t^\ell S_\ell, \tag{7}$$

where, with $r := \exp(\frac{1}{2}\hat{x})$, we have

$$S_\ell = \sum_{\substack{i,j \geqslant 1, \\ i+j=\ell}} \big(r^i - r^{-i}\big)\big(r^j - r^{-j}\big) = (\ell-1)\big(r^\ell + r^{-\ell}\big) - 2\frac{r^{\ell-1} - r^{-\ell+1}}{r - r^{-1}}.$$

Now let $\theta := \frac{1}{2}\hat{x}$, and substituting this expression for S_ℓ in (7), we obtain the revised fourth expression

$$e^{-Y}x^2\sum_{i,j \geqslant 1} j z_{i+j}\frac{\partial^2 \Phi}{\partial z_i y_j} = 2\sum_{\ell \geqslant 1} z_\ell \xi(\hat{x})^{\ell-2}u^{\ell-2}t^\ell\bigg((\ell-1)\mathrm{ch}(\ell\theta) - \frac{\mathrm{sh}((\ell-1)\theta)}{\mathrm{sh}(\theta)}\bigg).$$

Combining these four expressions, and recalling the definition (6) of the partial differential operator Δ, we have

$$e^{-Y}\Delta\Phi = \sum_{\ell \geqslant 1} z_\ell \xi(\hat{x})^{\ell-3}u^{\ell-2}t^\ell T_\ell + \sum_{\ell,m \geqslant 1} z_\ell y_m \xi(\hat{x})^{\ell+m-3}u^{\ell+m-1}t^{\ell+m}U_{\ell,m}, \tag{8}$$

where T_ℓ, for $\ell \geqslant 1$ and $U_{\ell,m}$, for $\ell, m \geqslant 1$, are explicit polynomials in hyperbolic cosines and hyperbolic sines of multiples of θ, and in θ, using (1). It is readily shown, using the addition formulae for hyperbolic sines and cosines that $T_\ell = 0$ for $\ell \geqslant 1$ and, similarly, that $U_{\ell,m} = 0$ for $\ell, m \geqslant 1$. Thus, from (8), we have $\Delta\Phi = 0$. But $\xi(0) = 1$, so $\Phi|_{u=0} = z_1 t$ and we conclude from Theorem 2.1 and the uniqueness of the solution of the Join–Cut Equation that $\Psi = \Phi$, giving the result. $\quad\square$

3.2. An expression for the coefficients of Ψ

It is now straightforward to determine the coefficients in the generating series Ψ, and thus obtain a proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that α is a partition of $n - i$ with k parts. Then for all $n \geqslant i \geqslant 1$, $k, g \geqslant 0$, Theorem 3.1 and (1) gives ($[A]B$ denotes the coefficient of A in B)

$$\left[t^n u^{n+k-1+2g} x^{2g} z_i y_\alpha\right]\Psi = \frac{i}{|\operatorname{Aut}\alpha|}\left[u^{2g}x^{2g}\right]\xi(iux)\xi(ux)^{i-2}\prod_{j=1}^{l(\alpha)}\xi(\alpha_j ux)\xi(ux)^{\alpha_j}$$

$$= \frac{i}{|\operatorname{Aut}\alpha|}\left[x^{2g}\right]\xi(x)^{n-2}\xi(ix)\prod_{j=1}^{l(\alpha)}\xi(\alpha_j x)$$

$$= \frac{i}{|\operatorname{Aut}\alpha|}\left[x^{2g}\right]\exp\left(\sum_{j\geqslant 1}\xi_{2j}q_{2j}(\alpha\cup i)x^{2j}\right),$$

so, together with (4) and (5), this gives

$$c_g(i,\alpha) = \frac{(n+k-1+2g)!}{n!}\alpha_1\cdots\alpha_k i\sum_{\beta\vdash g}\frac{\xi_{2\beta}q_{2\beta}(\alpha\cup i)}{|\operatorname{Aut}\beta|}.$$

But this is symmetric in $\alpha_1,\dots,\alpha_k, i$, and the result follows immediately by renaming $\alpha\cup i$ as α, which has $m=k+1$ parts. $\square$

4. Further questions

The following questions arise in the light of the results of this paper:

(1) Is it possible to find a simple proof of the centrality in Theorem 1.1, without evaluating the class coefficients $a_g(\alpha)$? This might follow from a decomposition for Young–Jucys–Murphy elements, or from a more elementary argument in the symmetric group.
(2) Is it possible to give a direct proof of Corollary 1.4 or 1.5 – i.e., to establish these relationships between $a_g(\alpha)$ and $b_g(\alpha)$ without appealing, as we have, to the explicit formulae? This would be particularly interesting, since $b_g(\alpha)$, as defined, is clearly central. Such a proof might involve Young–Jucys–Murphy elements, or a more elementary argument in the symmetric group, or the geometry of branched covers. Presumably such a proof would contain a solution to Question 1 above.
(3) In [GJV], the polynomiality of $b_g(\alpha)$ (in the parts of α) in Theorem 1.3, was the basis for a conjectured ELSV-type formula (see [ELSV]) for $H_{(n),\alpha}^g$, involving a Hodge integral over some, unspecified, moduli space. Does the polynomiality of $a_g(\alpha)$ in Theorem 1.1 also lead to a similar ELSV-type formula when $a_g(\alpha)$ is rescaled as a covering number?

Acknowledgments

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