JOURNAL OF COMBINATORIAL THEORY, Series A 41, 1–10 (1986)

Path Generating Functions and Continued Fractions

I. P. GOULDEN AND D. M. JACKSON

Department of Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

Communicated by the Managing Editors

Received February 7, 1984

This paper extends Flajolet's (*Discrete Math.* **32** (1980), 125–161) combinatorial theory of continued fractions by obtaining the generating function for paths between horizontal lines, with arbitrary starting and ending points and weights on the steps. Consequences of the combinatorial arguments used to determine this result are combinatorial proofs for many classical identities involving continued fractions and their convergents, truncations, numerator and denominator polynomials.

1. INTRODUCTION

A path on the square lattice is an important combinatorial object because it may be used for encoding a variety of configurations such as partitions (Flajolet [1]), permutations (Françon and Viennot [3]), plane partitions (Gessel and Viennot [5]), and random walks (Takács [9]). A path can be thought of as a sequence of steps, each step being an ordered pair of integer increments in the x and y directions. By considering paths on a certain set of steps, Gessel [4] obtained a path-theoretic generalization of the Lagrange Inversion Theorem, and hence a q-analogue of this theorem.

Certain physical processes can also be represented by paths. The most familiar example of these is the ballot sequence. Less familiar examples include the Bolzmann model for heat transfer (Goulden and Jackson [6]) and the analysis of file processing algorithms (Flajolet, Françon, and Vuillemin [2]). In the latter two examples there are physical parameters (resp. the amount of heat, and the number of items in a file) which cannot become negative, so this imposes on the corresponding paths the condition that they cannot cross below the x axis. Moreover, in the former, there is also an upper bound which cannot be exceeded, so the corresponding path cannot cross above a prescribed line parallel to the x axis. It is therefore important to be able to enumerate paths under a variety of conditions.

0097-3165/86 $3.00

In this paper we consider paths with only three steps, called rises, levels, and falls, corresponding, respectively, to the ordered pairs of increments $(1, 1)$, $(1, 0)$, $(1, -1)$. The main purpose of this paper is to derive a general result for enumerating paths, on these steps, which start and end at prescribed points and which are confined to lie between two lines parallel to the x axis.

The generating functions for certain sets of paths have particularly convenient representations as continued fractions. This fact has been used to obtain continued fraction representations of certain special functions by purely combinatorial means (Flajolet [1]).

The other purpose of this paper is to exploit this connexion between continued fractions and paths further to show that many of the classical results about continued fractions are consequences of elementary path decompositions and therefore can be proved combinatorially. For example, a typical such result is that the kth convergent of a continued fraction is equal to P_k/Q_k, where $\{P_k\}$ and $\{Q_k\}$ are sets of polynomials satisfying the same three-term recurrence equation.

In Section 2 we define paths, and show that particular continued fractions associated with the general continued fraction are generating functions for certain fundamental sets of paths. The overall strategy and the general enumerative results are given in Section 3, and Section 4 gives a number of classical identities which may be obtained by equating some of the earlier results.

2. Paths Enumerated by Trunctuations and Convergents

In this section we develop the generating functions for some basic sets of paths which will be used in the next section for more refined results. The following definition is needed.

Definition 2.1. Let $\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \ldots,$ be indeterminates.

(1) $J^{\langle m,n\rangle} = \dfrac{1}{1-\beta_m-}\,\dfrac{\alpha_m}{1-\beta_{m+1}-}\,\cdots\,\dfrac{\alpha_{n-1}}{-1-\beta_n}$ is called a *J-fraction.*

(2) $J^{\langle k\rangle} = J^{\langle 0,k\rangle}$ is called the kth *convergent* of $J = J^{\langle 0,\infty\rangle}$.

(3) $J^{\{k\}} = J^{\langle k,\infty\rangle}$ is called the kth *truncation* of J.

Let $\alpha = (\alpha_0, \alpha_1, \ldots)$, $\beta = (\beta_0, \beta_1, \ldots)$ and let $\mathbb{Q}$ denote the field of rationals. Then $J^{\langle n,m\rangle} \in \mathbb{Q}[\![\alpha, \beta]\!]$, whence J, $J^{\langle k\rangle}$, $J^{\{k\}} \in \mathbb{Q}[\![\alpha, \beta]\!]$. It is to be understood that all power series we consider are members of this ring, and that two such series, f and g, are equal if and only if $f - g$ is the zero power series in this ring. For a discussion of the ring of formal power series and the combinatorics of the ordinary generating function (in particular the sum and product lemmas) see Goulden and Jackson [7].

If f is a formal power series containing subscripted indeterminates, then $\check{f}$ denotes the formal power series obtained from f by increasing each of the subscripts by one.

DEFINITION 2.2. Let P_k and Q_k, $k \geqslant 1$, satisfy the recurrence equation

$$u_k = (1 - \beta_k)\, u_{k-1} - \alpha_{k-1} u_{k-2}, \qquad k \geqslant 1$$

with $P_{-1} = 0$, $P_0 = 1$ and $Q_{-1} = 1$, $Q_0 = 1 - \beta_0$. Then P_k and Q_k are called the kth *numerator* and *denominator polynomials* for J.

Thus we immediately have $P_k = \check{Q}_{k-1}$.

If $\omega = \omega_1 \cdots \omega_n$ is a sequence over $\{-1, 0, 1\}$, with $n \geqslant 0$, and $k + \omega_1 + \cdots + \omega_i \geqslant 0$, $\forall i$, then we say that $(\omega)_k$ is a *path* with *initial altitude* k and *terminal altitude* $k + \omega_1 + \cdots + \omega_n$. For $i \geqslant 1$, ω_i is a *step* in the path, and for $k + \omega_1 + \cdots + \omega_{i-1} = m$, then ω_i is a *rise, level,* or *fall at altitude* m, if $\omega_i = 1, 0$, or -1, respectively. These are denoted by $(1)_m$, $(0)_m$, $(-1)_m$, respectively. The empty path at altitude k is denoted by ε_k.

The product $(\sigma)_i(\rho)_j$ of the path $(\sigma)_i$ with the path $(\rho)_j$ is defined to be $(\sigma\rho)_i$, if $(\sigma)_i$ has terminal altitude j. If $\mathbf{P}$, $\mathbf{P}_1$,$\mathbf{P}_2$ are sets of paths such that $\mathbf{P} = \{\pi_1\pi_2 \mid (\pi_1, \pi_2) \in \mathbf{P}_1 \times \mathbf{P}_2\}$ then we write $\mathbf{P} = \mathbf{P}_1\mathbf{P}_2$.

Paths may be represented geometrically in the obvious way. Figure 1 gives the path $(1\ 0\ {-1}\ {-1}\ 1\ 0\ 1\ 1)_3$, which has initial altitude 3; terminal altitude 5; rises at altitudes 3, 2, 3, 4; levels at altitudes 4, 3; and falls at altitudes 4, 3.

The path $\check{\pi}$ is formed by adding one to the altitude of each step in the path π. If $\mathbf{S}$ is a set of paths then $\check{\mathbf{S}} = \{\check{\pi} \mid \pi \in \mathbf{S}\}$.

If a path π contains at least one occurrence of a rise at altitude i, then we define

$$R_{i,f}\pi = (\alpha, \beta),$$

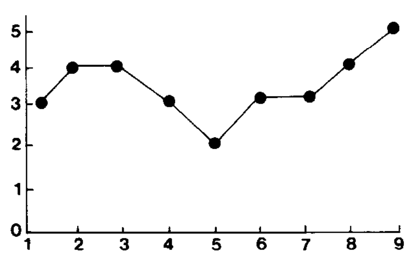

FIG. 1. Geometrical representation of a path.

where α and β are the subpaths of π preceding and following, respectively, the first occurrence of $(1)_i$ in π. We similarly define $R_{i,l}\pi$ in terms of the last occurrence of $(1)_i$. For a path σ which contains at least one occurrence of a fall at altitude i, we define $F_{i,f}\sigma$ and $F_{i,l}\sigma$ similarly, in terms of the first and last occurrences of $(-1)_i$ in σ.

Let $\mathbf{G}(i, j; m, k)$ be the set of all paths from altitude i to altitude j with minimum altitude greater than or equal to m and a maximum altitude less than or equal to k. The set of paths in $\mathbf{G}(i, j; m, k)$ with the further restriction that the maximum altitude is greater than or equal to h is denoted by $\mathbf{G}(i, j; m, k, \geqslant h)$.

Let $G(i, j; m, k)$ be the ordinary generating function for $\mathbf{G}(i, j; m, k)$ with indeterminates α_n marking a rise at altitude n and β_n a level at altitude n, for $n \geqslant 0$. Thus the coefficient of $\alpha_0^{a_0}\alpha_1^{a_1} \cdots \beta_0^{b_0}\beta_1^{b_1} \cdots$, in $G(i, j; m, k)$ is the number of paths in $\mathbf{G}(i, j; m, k)$ with a_n rises at altitude n and b_n levels at altitude n, for $n \geqslant 0$. The monomial associated with the path in Fig. 1 is therefore $\alpha_2\alpha_3^2\alpha_4\beta_3\beta_4$. We do not record the falls since, given the initial and terminal altitude of a path, the number of falls at each altitude can be deduced from the number of rises at each altitude. Similarly we have the following result. Let $c_i = \alpha_0\alpha_1 \cdots \alpha_i$, $i \geqslant 0$, $c_{-1} = 1$.

LEMMA 2.3. $c_{i-1}G(i, j; m, k) = c_{j-1}G(j, i; m, k)$.

Proof. We note that the mapping

$$\mathbf{G}(i, j; m, k) \rightarrow \mathbf{G}(j, i; m, k)\colon (p_1 p_2 \cdots p_l)_i \rightarrow (q_l q_{l-1} \cdots q_1)_j,$$

where $q_s = -p_s$ for $s = 1,..., l$ and $j = i + p_1 + p_2 + \cdots + p_l$, is bijective. The result follows. ∎

Clearly $\check{G}$ is the generating function for $\check{\mathbf{G}}$ for all sets of paths defined above.

The next result gives the generating functions for some important sets of paths which will be used in the next section.

PROPOSITION 2.4. (1) $G(m, m; m, \infty) = J^{\{m\}}$.

(2) $G(0, 0; 0, \infty) = J$.

(3) $G(0, 0; 0, k) = J^{\langle k \rangle}$.

(4) $G(0, 0; 0, k, \geqslant h) = J^{\langle k \rangle} - J^{\langle h-1 \rangle}$.

(5) $G(m, m; m, k) = J^{\langle m,k \rangle}$.

Proof. (1) Let $\mathbf{H}_i = \mathbf{G}(i, i; i, \infty)$ and let $\sigma \in \mathbf{H}_i - \varepsilon_i$.
There are two cases:

Case 1. The first step in σ is a level. Thus $\sigma \in \{(0)_i\}\,\mathbf{H}_i$.

Case 2. The first step in σ is a rise. Then

$$F_{i+1,f}\sigma \in \{(1)_i\}\, \mathbf{H}_{i+1} \times \mathbf{H}_i.$$

These decompositions are reversible, so

$$\mathbf{H}_i = \varepsilon_i \cup ((0)_i \cup (1)_i \mathbf{H}_{i+1}(-1)_{i+1})\, \mathbf{H}_i.$$

If $H_i = G(i, i; i, \infty)$, then

$$H_i = 1 + (\beta_i + \alpha_i H_{i+1})\, H_i$$

so $H_i = 1/(1 - \beta_i - \alpha_i/H_{i+1})$, and the result follows by iterating this for $i \geqslant m$.

(2) Let $m = 0$ in (1).

(3) $\mathbf{G}(0, 0; 0, k)$ consists of those paths in $\mathbf{G}(0, 0; 0, \infty)$ with no rises at altitude k. The result follows by setting $\alpha_k = 0$ in (2).

(4) Clearly $\mathbf{G}(0, 0; 0, k, \geqslant h) = \mathbf{G}(0, 0; 0, k) - \mathbf{G}(0, 0; 0, h-1)$ and the result follows from (3).

(5) Proof similar to (3), using (1). ∎

3. The Main Result

We now derive, in a series of propositions, the generating functions for progressively more complicated sets of paths, culminating with the main result giving the generating function for paths from a prescribed origin to a prescribed terminus, and lying between two lines parallel to the x axis. The main result specialises to each of the earlier results of this section, under the appropriate substitutions.

Proposition 3.1. (1) $G(k, k; 0, k) = Q_{k-1}/Q_k$.

(2) $G(k, j; 0, k) = Q_{j-1}/Q_k$.

(3) $G(k, 0; 0, k) = 1/Q_k$.

(4) $G(k, 1; 1, k) = 1/P_k$.

(5) $G(0, k; 0, k) = c_{k-1}/Q_k$.

Proof. (1) Let $\mathbf{C}_i = \mathbf{G}(i, i; 0, i)$ and $\sigma \in \mathbf{C}_i - \varepsilon_i$. For $i \geqslant 1$ there are two cases.

Case 1. The first step in σ is a level. Thus

$$\sigma \in \{(0)_i\}\, \mathbf{C}_i.$$

Case 2. The first step in σ is a fall. Then

$$R_{i-1,f}\sigma \in \{(-1)_i\}\, \mathbf{C}_{i-1} \times \mathbf{C}_i.$$

These decompositions are reversible, so

$$\mathbf{C}_i = \varepsilon_i \cup ((0)_i \cup (-1)_i \mathbf{C}_{i-1}(1)_{i-1})\, \mathbf{C}_i.$$

Let $C_i = G(i, i; 0, i)$. Then

$$C_i = 1 + (\beta_i + C_{i-1}\alpha_{i-1})\, C_i$$

so $C_i = (1 - \beta_i - \alpha_{i-1}C_{i-1})^{-1}$, $i \geqslant 1$, and $C_0 = (1 - \beta_0)^{-1}$.

But from Definition 2.2

$$Q_i = (1 - \beta_i)\, Q_{i-1} - \alpha_{i-1}Q_{i-2},$$

so

$$Q_{i-1}/Q_i = (1 - \beta_i - \alpha_{i-1}Q_{i-2}/Q_{i-1})^{-1}, \qquad i \geqslant 1,$$

and $Q_{-1}/Q_0 = (1 - \beta_0)^{-1}$. The result follows by comparing these recurrences.

(2) For $0 \leqslant j < k$ we have

$$F_{j+1,l}\mathbf{G}(k, j; 0, k) = \mathbf{G}(k, j+1; 0, k) \times \mathbf{G}(j, j; 0, j)$$

so $G(k, j; 0, k) = G(k, j+1; 0, k)\, G(j, j; 0, j)$. Applying this successively, we obtain

$$\begin{aligned} G(k, j; 0, k) &= \prod_{i=j}^{k} G(i, i; 0, i) \\ &= \prod_{i=j}^{k} Q_{i-1}/Q_i, \qquad \text{from (1)} \\ &= Q_{j-1}/Q_k, \qquad \text{as required.} \end{aligned}$$

(3) Let $j = 0$ in (2).

(4) $\mathbf{G}(k, 1; 1, k) = \check{\mathbf{G}}(k-1, 0; 0, k-1)$ so

$$\begin{aligned} G(k, 1; 1, k) &= \check{G}(k-1, 0; 0, k-1) \\ &= 1/\check{Q}_{k-1} \text{ from (3)} \\ &= 1/P_k. \end{aligned}$$

(5) From Lemma 2.3

$$\begin{aligned} G(0, k; 0, k) &= c_{k-1}G(k, 0; 0, k) \\ &= c_{k-1}/Q_k \qquad \text{from (3).} \quad \blacksquare \end{aligned}$$

PROPOSITION 3.2. (1) $G(0, 0; 0, k) = P_k/Q_k$.

(2) $G(0, 0; 0, k, \geqslant k) = c_{k-1}/Q_k Q_{k-1}$.

Proof. (1) We have

$$F_{i,f}\mathbf{G}(k, 0; 0, k) = \mathbf{G}(k, 1; 1, k) \times \mathbf{G}(0, 0; 0, k)$$

so $G(k, 0; 0, k) = G(k, 1; 1, k)\, G(0, 0; 0, k)$ and $G(0, 0; 0, k) = G(k, 0; 0, k)/G(k, 1; 1, k) = P_k/Q_k$ from Proposition 3.1(3), (4).

(2) We have

$$F_{k,l}\mathbf{G}(0, 0; 0, k, \geqslant k) = \mathbf{G}(0, k; 0, k) \times \mathbf{G}(k-1, 0; 0, k-1)$$

so $G(0, 0; 0, k, \geqslant k) = G(0, k; 0, k)\, G(k-1, 0; 0, k-1)$ and the result follows from Proposition 3.1(3), (5). ∎

PROPOSITION 3.3. (1) $G(0, j; 0, k) = Q_{j-1}(J^{\langle k \rangle} - J^{\langle j-1 \rangle})$.

(2) $G(i, j; 0, k) = (Q_{i-1} Q_{j-1}/c_{i-1})(J^{\langle k \rangle} - J^{\langle j-1 \rangle})$, $i \leqslant j$.

(3) $G(i, j; 0, k) = (Q_{i-1} Q_{j-1}/c_{i-1})(J^{\langle k \rangle} - J^{\langle i-1 \rangle})$, $i \geqslant j$.

Proof. (1) We have

$$F_{j,l}\mathbf{G}(0, 0; 0, k, \geqslant j) = \mathbf{G}(0, j; 0, k) \times \mathbf{G}(j-1, 0; 0, j-1)$$

so $G(0, 0; 0, k, \geqslant j) = G(0, j; 0, k)\, G(j-1, 0; 0, j-1)$ and

$$G(0, j; 0, k) = \frac{G(0, 0; 0, k, \geqslant j)}{G(j-1, 0; 0, j-1)}.$$

The result follows from Propositions 2.4(4) and 3.1(3).

(2) We have

$$R_{i-1,f}\mathbf{G}(0, j; 0, k) = \mathbf{G}(0, i-1; 0, i-1) \times \mathbf{G}(i, j; 0, k)$$

so $G(0, j; 0, k) = \alpha_{i-1} G(0, i-1; 0, i-1)\, G(i, j; 0, k)$ and

$$G(i, j; 0, k) = \frac{G(0, j; 0, k)}{\alpha_{i-1} G(0, i-1; 0, i-1)}.$$

The result follows from (1) and Proposition 3.1(5).

(3) From Lemma 2.3 we obtain

$$G(i, j; 0, k) = \frac{c_{j-1}}{c_{i-1}}\, G(j, i; 0, k)$$

and the result follows from (2). ∎

PROPOSITION 3.4. (1) $G(m, k; m, k) = c_{k-1}/\alpha_{m-1} Q_k Q_{m-2}(J^{\langle k \rangle} - J^{\langle m-2 \rangle})$.

(2) $G(m, j; m, k) = Q_{j-1}(J^{\langle k \rangle} - J^{\langle j-1 \rangle})/\alpha_{m-1} Q_{m-2}(J^{\langle k \rangle} - J^{\langle m-2 \rangle})$.

Proof. We have

$$R_{m-1,l}\mathbf{G}(0, j; 0, k) = \mathbf{G}(0, m-1; 0, k) \times \mathbf{G}(m, j; m, k)$$

so $G(0, j; 0, k) = \alpha_{m-1} G(0, m-1; 0, k)\, G(m, j; m, k)$ and

$$G(m, j; m, k) = \frac{G(0, j; 0, k)}{\alpha_{m-1} G(0, m-1; 0, k)}.$$

(1) follows from Proposition 3.1(5) with $j = k$, and 3.3(1).

(2) follows from Proposition 3.3(1) ∎

THEOREM 3.5. (1) $G(i, j; m, k) = Q_{i-1} Q_{j-1}(J^{\langle k \rangle} - J^{\langle j-1 \rangle})(J^{\langle i-1 \rangle} - J^{\langle m-2 \rangle})/c_{i-1}(J^{\langle k \rangle} - J^{\langle m-2 \rangle})$, $i \leqslant j$.

(2) $G(i, j; m, k) = Q_{i-1} Q_{j-1}(J^{\langle k \rangle} - J^{\langle i-1 \rangle})(J^{\langle j-1 \rangle} - J^{\langle m-2 \rangle})/c_{i-1}(J^{\langle k \rangle} - J^{\langle m-2 \rangle})$, $i \geqslant j$.

Proof. (1) We have

$$R_{i-1,f}\mathbf{G}(m, j; m, k) = \mathbf{G}(m, i-1; m, i-1) \times \mathbf{G}(i, j; m, k)$$

so $G(m, j; m, k) = \alpha_{i-1} G(m, i-1; m, i-1)\, G(i, j; m, k)$ and

$$G(i, j; m, k) = \frac{G(m, j; m, k)}{\alpha_{i-1} G(m, i-1; m, i-1)}.$$

The result follows from Proposition 3.4(1) and (2).

(2) The result follows from (1) and Lemma 2.3. ∎

Note that all previous results are corollaries of the above theorem, with appropriate initial values. We conclude with two further results which will be used in the next section for deriving classical identities.

PROPOSITION 3.6. (1) $G(m, m; m, k) = Q_{m-1}(J^{\langle k \rangle} - J^{\langle m-1 \rangle})/\alpha_{m-1} Q_{m-2}(J^{\langle k \rangle} - J^{\langle m-2 \rangle})$.

(2) $G(m, m; m, \infty) = Q_{m-1}(J - J^{\langle m-1 \rangle})/\alpha_{m-1} Q_{m-2}(J - J^{\langle m-2 \rangle})$.

Proof. (1) Let $j = m$ in Proposition 3.4(2).

(2) Let $k \to \infty$ in (1). The result follows since $\lim_{k \to \infty} J^{\langle k \rangle} = J$. ∎

PROPOSITION 3.7. (1) $G(0, j; 0, \infty) = c_{j-1} JJ^{\{1\}} \cdots J^{\{j\}}$.

(2) $G(0, 0; 0, \infty, \geqslant j) = (c_{j-1}/Q_{j-1}) JJ^{\{1\}} \cdots J^{\{j\}}$.

Proof. (1) We have trivially

$$R_{i,l}\mathbf{G}(i, j; 0, \infty) = \mathbf{G}(i, i; i, \infty) \times \mathbf{G}(i+1, j; 0, \infty),$$

so $G(i, j; 0, \infty) = \alpha_i J^{\{i\}} G(i+1, j; 0, \infty)$ from Proposition 2.4(1). The result follows by iterating this equation for $i = 0, 1, \ldots, j-1$.

(2) Clearly

$$F_{j,l}\mathbf{G}(0, 0; 0, \infty, \geqslant j) = \mathbf{G}(0, j; 0, \infty) \times \mathbf{G}(j-1, 0; 0, j-1),$$

so $G(0, 0; 0, \infty, \geqslant j) = G(0, j; 0, \infty)\, G(j-1, 0; 0, j-1)$ and the result follows from (1) and Proposition 3.1(3). ∎

4. IDENTITIES

We conclude by using the above combinatorial results to obtain a number of well known relationships between J, $J^{\{k\}}$, $J^{\langle k \rangle}$, P_k, and Q_k. For a classical approach to these and other results, see Perron [8].

COROLLARY 4.1. (1) $J^{\langle k \rangle} = P_k/Q_k$.

(2) $J^{\langle k \rangle} - J^{\langle k-1 \rangle} = c_{k-1}/Q_k Q_{k-1}$.

(3) $P_k Q_{k-1} - P_{k-1} Q_k = c_{k-1}$.

(4) $J^{\{m\}} = Q_{m-1}(J - J^{\langle m-1 \rangle})/\alpha_{m-1} Q_{m-2}(J - J^{\langle m-2 \rangle})$.

(5) $J - J^{\langle k-1 \rangle} = (c_{k-1}/Q_{k-1}) JJ^{\{1\}} \cdots J^{\{k\}}$.

(6) $P_k Q_{k-2} - P_{k-2} Q_k = c_{k-2}(1 - \beta_k)$.

Proof. (1) Equate the expressions for $G(0, 0; 0, k)$ in Proposition 2.4(3) and Proposition 3.2(1).

(2) Equate the expressions for $G(0, 0; 0, k, \geqslant k)$ in Proposition 2.4(4) and Proposition 3.2(2).

(3) Direct from (1) and (2).

(4) Equate the expressions for $G(m, m; m, \infty)$ in Proposition 2.4(1) and Proposition 3.6(2).

(5) Equate the expressions for $G(0, 0; 0, \infty, \geqslant j)$ in Proposition 2.4(4) and Proposition 3.7(2).

(6) From (1)

$$\frac{P_k}{Q_k} - \frac{P_{k-2}}{Q_{k-2}} = J^{\langle k\rangle} - J^{\langle k-2\rangle}$$
$$= (c_{k-1}Q_{k-2} + c_{k-2}Q_k)/Q_kQ_{k-1}Q_{k-2}$$

from (2). The result follows from the Definition 2.2. ∎

Note that the left-hand side of Corollary 4.1(4) is independent of $\alpha_0, \ldots, \alpha_{m-1}$ and $\beta_0, \ldots, \beta_m$ while this is not obviously the case for the right-hand side. Other identities of this sort can be obtained from the main theorem by other specializations.

ACKNOWLEDGMENT

This work was supported by the Natural Sciences and Engineering Research Council of Canada (Grants U0073 and A8235).

REFERENCES

1. P. FLAJOLET, Combinatorial aspects of continued fractions, *Discrete Math.* **32** (1980), 125–161.
2. P. FLAJOLET, J. FRANÇON, AND J. VUILLEMIN, Sequence of operations analysis in dynamic data structures, *J. Algorithms* **1** (1980), 111–141.
3. J. FRANÇON AND G. VIENNOT, Permutations selon leur pics, creux, doubles montées et double descentes, nombres d'Euler et nombres de Genocchi, *Discrete Math.* **28** (1979), 21–35.
4. I. M. GESSEL, A noncommutative generalization and q-analogue of the Lagrange inversion formula, *Trans. Amer. Math. Soc.* **257** (1980), 455–481.
5. I. M. GESSEL AND G. VIENNOT, Determinants and plane partitions, preprint.
6. I. P. GOULDEN AND D. M. JACKSON, Distributions, continued fractions and the Ehrenfest urn model, *J. Combin. Theory Ser. A*, to appear.
7. I. P. GOULDEN AND D. M. JACKSON, "Combinatorial Enumeration," Wiley, New York, 1983.
8. O. PERRON, "Die Lehre von den Kettenbruchen," Leipzig and Berlin, 1929.
9. L. TAKÁCS, "Combinatorial Methods in the Theory of Stochastic Processes," Wiley, New York, 1967.