JOURNAL OF COMBINATORIAL THEORY, Series A 73, 368–375 (1996)
ARTICLE NO. 0030

Note

Further Determinants with the Averaging Property of Andrews–Burge

I. P. GOULDEN AND D. M. JACKSON

Department of Combinatorics & Optimization, University of Waterloo, Waterloo, Canada N2L 3G1

Communicated by the Managing Editors

Received September 3, 1993

The determinants which arise in the enumeration of symmetry classes of plane partitions are difficult to evaluate. Recently, Andrews and Burge have shown that the determinant of one of these classes has a surprising property, which we call the *averaging* property. We obtain a class of determinants under reasonably general conditions which also possess this property. This class includes the Andrews and Burge example. We also consider conditions under which determinants in this class can be evaluated.

1. INTRODUCTION

Properties of certain determinants and their evaluation play an important role in enumerative combinatorics, and a thorough understanding of them has led to important combinatorial information. One such determinant occurs in connexion with the symmetry classes of plane partitions and was evaluated by Mills, Robbins, and Rumsey [4] in the following theorem. We have trivially recast their statement, reversing the order of the rows and the columns, and exhibiting the polynomiality of the determinant. Their proof relies on a primary result of Andrews [1], evaluating a determinant. Let f^θ denote the degree of the ordinary irreducible representation of the symmetric group which is indexed by the partition θ. Let δ_n denote the partition $(n-1, n-2, ..., 1)$. We use $|a_{i,j}|_n$ to denote the determinant of the $n \times n$ matrix whose (i, j)-entry is $a_{i,j}$ for $i, j = 1, ..., n$.

0097-3165/96 $12.00

THEOREM 1.1. *Let* $m_n(x) = |\binom{x-i-j}{n-2i+j}|_n$. *Then*

$$m_n(x) = \frac{f^{\delta_n}}{\binom{n}{2}!} \prod_{1 \leqslant i \leqslant j \leqslant n-i} (x+2i+j-2n-1)$$

$$\prod_{1 \leqslant j < i \leqslant n-j} \frac{1}{2}(2x+2i-4j-1).$$

While considering a number of related determinants, Andrews and Burge [2] proved the following unexpected result using a ${}_4F_3$ hypergeometric summation theorem.

THEOREM 1.2. *Let*

(1)

$$M_n(x_1, x_2) = \left| \binom{x_1-i-j}{n-2i+j} + \binom{x_2-i-j}{n-2i+j} \right|_n,$$

(2)

$$N_n(x_1, x_2) = \left| \frac{2}{x_1-x_2+1} \left\{ \binom{x_1-i-j+1}{n-2i+j+1} - \binom{x_2-i-j}{n-2i+j+1} \right\} \right|_n$$

Then $M_n(x_1, x_2) = N_n(x_1, x_2) = 2^n m_n(\bar{x})$, *where* $\bar{x} = \frac{1}{2}(x_1 + x_2)$.

In fact Andrews and Burge proved directly that

$$M_n(x_1, x_2) = M_n(\bar{x}, \bar{x}), \qquad N_n(x_1, x_2) = N_n(\bar{x}, \bar{x}), \tag{1}$$

and then used the obvious fact that $M_n(\bar{x}, \bar{x}) = N_n(\bar{x}, \bar{x}) = 2^n m_n(\bar{x})$, to establish the result as a consequence of Theorem 1.1. In view of (1) we say that $M_n(x, y)$ and $N_n(x, y)$ have the *averaging property* of Andrews–Burge, the subject of this paper.

In Section 2 we give a determinant result (Theorem 2.1), in a reasonably general setting, which specialises to Theorem 1.2. Moreover, it yields as special cases further determinants with the averaging property. In Section 3 we show that two of these further determinants have evaluations whose form is similar to Theorem 1.1.

2. MAIN THEOREM

In the following determinant result the independence of the argument of the functions G_i will be seen later to be an extension of the averaging property. The result involves the ring $\mathbf{R}[[t]]$ of formal power series with coefficients in an appropriate ring $\mathbf{R}$. The subset of such series with an invertible constant term is denoted by $\mathbf{R}[[t]]_0$. The coefficient operator $[t^k]$ written to the left of $f \in \mathbf{R}[[t]]$ gives the coefficient of t^k in f.

THEOREM 2.1. *Let* $F_m(t)$, $G_m(t) \in \mathbf{R}[[t]]$, $H_m(t) \in \mathbf{R}[[t]]_0$, *for* $m = 1, ..., n$. *Then*

$$\left| [1] \frac{F_j(t_j)}{H_j(t_j)^{n-i}} G_i(H_j(t_j)) \right|_n = \left| [1] \frac{F_j(t_j)}{H_j(t_j)^{n-i}} G_i(0) \right|_n .$$

Proof. Let $a_{i,j}$ denote the (i, j)-element of the matrix on the left-hand side of the enunciation and let $G_i(t) = \sum_{k \geqslant 0} g_{i,k} t^k$. Then

$$a_{i,j} = [1]\, F_j(t_j) \sum_{k \geqslant 0} \frac{g_{i,k}}{H_j(t_j)^{n-k-i}} = [1]\, F_j(t_j) \sum_{k=0}^{n-i} \frac{g_{i,k}}{H_j(t_j)^{n-k-i}}$$

since $F_j(t_j)\{H_j(t_j)\}^{k-n+i} \in \mathbf{R}[[t]]_0$ for $k > n - i$. Thus setting $l = k + i$, we have

$$|a_{i,j}|_n = \left| [1]\, F_j(t_j) \sum_{l=i}^{n} \frac{g_{i,l-i}}{H_j(t_j)^{n-l}} \right|_n = [1] \left| F_j(t_j) \sum_{l=i}^{n} \frac{g_{i,l-i}}{H_j(t_j)^{n-l}} \right|_n$$

since t_j occurs only in column j, for $j = 1, ..., n$, where $[1]\, g(t_1, ..., t_n)$ denotes the constant term in $g(t_1, ..., t_n)$. Then

$$|a_{i,j}|_n = [1]\; |g_{i,j-i}|_n \left| \frac{F_j(t_j)}{H_j(t_j)^{n-i}} \right|_n = [1] \left\{ \prod_{m=1}^{n} g_{m,0} \right\} \left| \frac{F_j(t_j)}{H_j(t_j)^{n-i}} \right|_n$$

$$= \left| [1] \frac{F_j(t_j)}{H_j(t_j)^{n-i}} g_{i,0} \right|_n ,$$

since again t_j occurs only in column j for $j = 1, ..., n$. The result follows since $g_{m,0} = G_m(0)$, for $m = 1, ..., n$. ∎

Note that an equivalent statement of the above result is that, for the scalar λ, the function

$$\left| [1] \frac{F_j(t_j)}{H_j(t_j)^{n-i}} G_i(\lambda H_j(t_j)) \right|_n$$

is independent of λ.

The next result gives a general class of determinants which have the averaging property. The class contains the determinant of Andrews and Burge.

COROLLARY 2.2. (1)

$$\left|\binom{z+k-i+a_j}{n-2i+j}+\binom{z-k-i+a_j}{n-2i+j}\right|_n=\left|2\binom{z-i+a_j}{n-2i+j}\right|_n,$$

(2)

$$\left|\frac{1}{2k+1}\left\{\binom{z+1+k-i+a_j}{n+1-2i+j}-\binom{z-k-i+a_j}{n+1-2i+j}\right\}\right|_n=\left|\binom{z-i+a_j}{n-2i+j}\right|_n.$$

Proof. Let $H_m(t)=t^2/(1+t)$ and $F_m(t)=t^{n-m}(1+t)^{z-n+a_m}$ for $m=1, ..., n$. Then the two results follow from Theorem 2.1 if there exist choices for $G_i(t)$ such that

$$(1) \qquad G_i(H_m(t))=(1+t)^k+(1+t)^{-k}=2+\sum_{l=1}^{k} c_l^{(k)}s^l,$$

$$(2) \qquad G_i(H_m(t))=\frac{(1+t)^{k+1}-(1+t)^{-k}}{(2k+1)t}=1+\sum_{l=1}^{k} d_l^{(k)}s^l,$$

respectively for i, $m=1, ..., n$, and $k\geqslant 0$, where $s=t^2/(1+t)$. To do this for (1) note that

$$\sum_{k\geqslant 0} u^k\{(1+t)^k+(1+t)^{-k}\}$$

$$=\frac{1}{1-u(1+t)}+\frac{1}{1-u(1+t)^{-1}}$$

$$=\frac{2-(2+s)u}{1-(2+s)u+u^2}=1+\frac{1-u^2}{(1-u)^2-su}=1+\sum_{l\geqslant 0} s^l\,\frac{(1+u)u^l}{(1-u)^{2l+1}}$$

so, by applying $[u^ks^l]$ for $0\leqslant l\leqslant k$, we explicitly determine $c_l^{(k)}=\delta_{k,0}\delta_{l,0}+\binom{k+l}{k-l}+\binom{k+l-1}{k-l-1}$, whence

$$c_l^{(k)}=\begin{cases}2, & 0=l=k,\\ \dfrac{2k}{k+l}\dbinom{k+l}{2l}, & \text{otherwise.}\end{cases}$$

Thus $c_0^{(k)} = 2$ for all $k \geqslant 0$. Similarly, for (2),

$$\sum_{k \geqslant 0} u^k\{(1+t)^{k+1} - (1+t)^{-k}\} = \frac{1+t}{1-u(1+t)} - \frac{1}{1-u(1+t)^{-1}}$$

$$= \frac{t(1+u)}{1-(2+s)u+u^2} = t \sum_{l \geqslant 0} s^l \frac{(1+u)u^l}{(1-u)^{2l+2}}$$

so, by applying $[tu^k s^l]$ for $0 \leqslant l \leqslant k$, $d_l^{(k)} = (1/(2k+1))\{\binom{k+l+1}{k-l} + \binom{k+l}{k-l-1}\}$, whence we explicitly determine $d_l^{(k)} = (1/(2k+1))\binom{k+l}{k-l}$, so $d_0^{(k)} = 1$ for all $k \geqslant 0$. The results now follow. ∎

The left-hand sides of Corollary 2.2 are polynomials in k, so $z+k$ and $z-k$ can be replaced by the indeterminates x_1 and x_2, respectively. As a consequence, on the right-hand side z is replaced by $\bar{x}$. If we set $a_j = -j$ in this result, we obtain Theorem 1.2.

It is readily seen that the $c_l^{(k)}$ appearing in the above proof are the coefficients which appear in the expansion of the Chebyshev polynomials. The existence of G_i in the above instance is not entirely straightforward, as an attempt to find comparable choices quickly reveals.

We now give another class of determinants which has the averaging property.

COROLLARY 2.3.

$$|(x_1 + a_j + i)^{i-1} + (x_2 + a_j + i)^{i-1}|_n = |2(\bar{x} + a_j + i)^{i-1}|_n.$$

Proof. Let $H_m(t) = te^t$, and $F_m(t) = e^{(a_m - 1)t}$, for $m = 1, ..., n$ and let $G_i(H_m(t)) = e^{x_1 t} + e^{x_2 t}$, for i, $m = 1, ..., n$. The result follows from Theorem 2.1, having reversed the order of the rows. Of course, had we wished to determine $G_i(t)$ explicitly, we could have used Lagrange's implicit function theorem. ∎

We observe that the right-hand side of Theorem 2.1 can be written in the alternative form

$$[1] \prod_{k=1}^{n} F_k(t_k)\, G_k(0) \prod_{1 \leqslant i < j \leqslant n} \left(\frac{1}{H_i(t_i)} - \frac{1}{H_j(t_j)}\right),$$

since t_j appears only in column j. The left-hand side can be written in an analogous way if, as in the above examples, the functions G_i are independent of i. This link between determinants and constant terms in Laurent series has been used in many instances. A recent example involving the determinants of Theorems 1.1 and 1.2 occurs in Zeilberger [5].

3. Evaluations

The determinant in Theorems 1.1 and 1.2 is especially interesting because of the averaging property and because it can also be evaluated explicitly. This is the case $a_j = -j$ of the general class of determinants in Corollary 2.2. Although, in general, the determinants in this class cannot be evaluated explicitly, the determinants that arise in the cases $a_j = 0$ and j also can be, by means of the following result.

For integers m, the expression $(x)_m$ denotes the falling factorial $x(x-1)\cdots(x-m+1)$, $m \geqslant 0$, with $(x)_0 = 1$ and $(x)_m = 1/(x-m)_{-m}$ for $m \leqslant -1$.

Lemma 3.1. *Let $L(t) = at + b$, where a, b are arbitrary constants. Then, for integers $t_k \geqslant -k$, $k = 1, ..., n$,*

$$\left|\binom{L(t_i)}{t_i + j}\right|_n = \prod_{1 \leqslant i < j \leqslant n} (t_i - t_j) \cdot \prod_{k=1}^{n} \frac{L(t_k)_{t_k+1}\,(L(-k)+k-1)_{k-1}}{(t_k+n)!}.$$

Proof. Let D denote the determinant on the left-hand side. Then, by rearrangement,

$$D = \frac{1}{\prod_{k=1}^{n} (t_k+n)!}\;|(t_i+n)_{n-j}\,(L(t_i))_{t_i+j}|_n.$$

But $(x)_{t_i+j} = (x - t_i - 1)_{j-1}\,(x)_{t_i+1}$ for all integers t_i, so

$$D = \prod_{k=1}^{n} \frac{(L(t_k))_{t_k+1}}{(t_k+n)!}\;F(t_1, ..., t_n),$$

where $F(t_1, ..., t_n) = |(t_i+n)_{n-j}\,(L(t_i) - t_i - 1)_{j-1}|_n$. Now $(-i+n)_{n-j} = 0$ if $i > j$, and $(-i+n)_{n-i} = (n-i)!$, so $F(-1, ..., -n) = \prod_{k=1}^{n} (n-k)!\,(L(-k)+k-1)_{k-1}$. But $(t_i+n)_{n-j}\;(L(t_i) - t_i - 1)_{j-1} = \sum_{k=1}^{n} t_i^{n-k} a_{k,j}$, $i, j = 1, ..., n$, for some $a_{k,j}$ independent of $t_1, ..., t_n$. Thus $F(t_1, ..., t_n) = |t_i^{n-j}|_n\;|a_{i,j}|_n = V(t_1, ..., t_n)\;|a_{i,j}|_n$, where V is the Vandermonde determinant. Evaluation at $t_k = -k$, $k = 1, ..., n$, gives

$$\begin{aligned} F(-1, ..., -n) = V(-1, ..., -n)\;|a_{i,j}|_n &= \prod_{k=1}^{n} (n-k)!\;|a_{i,j}|_n \\ &= \prod_{k=1}^{n} (n-k)!\;(L(-k)+k-1)_{k-1} \end{aligned}$$

from the above. Thus $|a_{i,j}|_n = \prod_{k=1}^{n} (L(-k)+k-1)_{k-1}$ and the result follows. ∎

COROLLARY 3.2. (1)

$$\left|\binom{x-i}{n-2i+j}\right|_n = \frac{f^{\delta_n}}{\binom{n}{2}!} \prod_{1 \leqslant i \leqslant j \leqslant n-i} (x-j) \prod_{1 \leqslant j < i \leqslant n-j} \frac{1}{2}(2x-2i+1),$$

(2)

$$\left|\binom{x-i+j}{n-2i+j}\right|_n = \frac{f^{\delta_n}}{\binom{n}{2}!} \prod_{1 \leqslant i \leqslant j \leqslant n-i} (x-j+1) \prod_{1 \leqslant j < i \leqslant n-j} \frac{1}{2}(2x-2i+3).$$

Proof. (1) Set $L(t) = \frac{1}{2}(2x-n+t)$ and $t_i = n-2i$, for $i = 1, ..., n$ in Corollary 3.1. The result follows from the degree formula (see, e.g., [3, p. 64]) since $t_i = \theta_i - i$ for $\theta = \delta_n$.

(2) Since $\binom{L(t_i)+j}{t_i+j} = \binom{-(L(t_i)-t_i+1)}{t_i+j}(-1)^{t_i+j}$ the result follows from (1). ∎

We conclude with a number of points which are prompted by the form of the evaluations. From the above proof, Corollary 3.2(1) is the special case $\theta = \delta_n$ of the more general result

$$\left|\binom{\frac{1}{2}(z+\theta_i-i)}{\theta_i-i+j}\right|_n = \frac{f^\theta}{|\theta|!} \prod_{k=1}^{n} \left(\frac{1}{2}(z+\theta_k-k)\right)_{\theta_k-k+1} \left(\frac{1}{2}(z+k-2)\right)_{k-1}$$

This is obtained by setting $L(t) = \frac{1}{2}(t+z)$ in Lemma 3.1, where θ is a partition of $|\theta|$ with at most n parts (since δ_n is a partition of $\binom{n}{2}$). Similarly, with $L(t) = z$ in Lemma 3.1, we obtain the determinant evaluation

$$\left|\binom{z}{\theta_i-i+j}\right|_n = \frac{f^\theta}{|\theta|!} \prod_{i=1}^{n} \prod_{1 \leqslant j \leqslant \theta_i} (z+i-j).$$

This is well known (see, e.g., [3, pp. 28–29]) in symmetric functions, since it can also be interpreted as the principal specialization of the Schur function indexed by θ, and the right-hand side is usually written in the form

$$\frac{f^\theta}{|\theta|!} \prod_{(i,j) \in \theta} (z - \mathrm{ct}(i,j)),$$

where $\mathrm{ct}(i,j) = j-i$, the *content* of the (i,j)-cell in the Ferrers graph of θ. In this context, the right-hand sides of Corollary 3.2 and Theorem 1.1 also contain content-like products over the Ferrers graph of the *staircase* partition δ_n, for a suitably revised content function. It is therefore natural to ask:

(i) Is there a determinant result indexed by an arbitrary partition θ, for which Theorem 1.1 is the case $\theta = \delta_n$?

(ii) Can this result, and the others in this paper, be determined through the use of symmetric functions or the tableaux they count?

Successful answers to these questions, as a by-product, may lead to a simpler and more direct derivation of Theorem 1.1, although we have not been able to carry this out.

Acknowledgments

This work was supported by grants from the Natural Sciences and Engineering Research Council of Canada.

References

1. G. E. Andrews, Plane partitions (III): The weak Macdonald conjecture, *Invent. Math.* **53** (1979), 193–225.
2. G. E. Andrews and W. H. Burge, Determinant identities, *Pacific J. Math.* **158** (1993), 1–14.
3. I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Clarendon Press, Oxford, 1979.
4. W. H. Mills, D. P. Robbins, and H. Rumsey, Enumeration of a symmetry class of plane partitions, *Discrete Math.* **67** (1987), 43–55.
5. D. Zeilberger, Proof of the alternating sign matrix conjecture, *Electronic J. Combinatorics*, to appear.