SIAM J. ALG. DISC. METH.
Vol. 4, No. 2, June 1983

0196-5212/83/0402-0003 $01.25/0

THE HAMMOND SERIES OF A SYMMETRIC FUNCTION AND ITS APPLICATION TO *P*-RECURSIVENESS*

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Abstract. We give a method for determining the exponential generating function for the coefficient of $x_1^p \cdots x_n^p$ in a symmetric function S in the indeterminates $x_1, \cdots, x_n$. This generating function is called the Hammond series of S, and we use it to show that the counting series for certain combinatorial problems satisfy linear recurrence equations with polynomial coefficients. These problems include p-regular labelled graphs and square matrices with row and column sums equal to p.

1. Introduction. Let $[(a_1^p \cdots a_\alpha^p)(b_1^q \cdots b_\beta^q) \cdots]T$ denote the coefficient of $(a_1^p \cdots a_\alpha^p)(b_1^q \cdots b_\beta^q) \cdots$ in the formal power series T which is a symmetric function in each of the sets $\{a_1, \cdots, a_\alpha\}, \{b_1, \cdots, b_\beta\} \cdots$ of commutative indeterminates. We call such coefficients the *regular* coefficients of T. In this paper we present a method for calculating the exponential generating function for regular coefficients, where $p, q, \cdots$ are fixed. We call this power series the Hammond series (or H-series) of T, because of its connection to the Hammond operators.

In the later sections of this paper we use the H-series to determine whether certain sequences of regular coefficients satisfy a linear recurrence equation of fixed order, with polynomial coefficients. Such sequences are called *polynomially-recursive* (or P-recursive). This term is of considerable importance computationally since it means that the nth term of such a sequence may be computed in an amount of time which is linear in n and space which is independent of n (assuming that the time taken to multiply two integers is independent of their size).

We establish P-recursiveness for a sequence by deriving a linear differential equation, with polynomial coefficients, for its H-series. Power series with this property are called *differentially-finite* (or D-finite). The equivalence of D-finiteness and P-recursiveness is discussed in Stanley [6].

Regular coefficients arise in a variety of contexts and the problem of calculating them is a classical one which has been considered by MacMahon [3] in his combinatorial work on symmetric functions. We use the H-series to study two combinatorial configurations, namely

a) p-regular labelled graphs and simple graphs on n vertices for $n = 0, 1, 2 \cdots$ and

b) $n \times n$ matrices with row and column sums p over the nonnegative integers for $n = 0, 1, 2, \cdots$.

This enables us to establish the P-recursiveness of (a) for $p = 4$ and (b) for $p = 3$, an open problem cited by Stanley [6].

The following notation is used. Let $\mathbf{x} = (x_1, x_2, \cdots)$ and $\mathbf{y} = (y_1, y_2, \cdots)$ be sets of indeterminates. If $\mathbf{i} = (i_1, i_2, \cdots)$, then $\mathbf{x}^\mathbf{i}$ denotes $x_1^{i_1} x_2^{i_2} \cdots$ and $[\mathbf{x}^\mathbf{i}]f(\mathbf{x})$ denotes the coefficient of $\mathbf{x}^\mathbf{i}$ in the formal power series $f(\mathbf{x})$. Let $\partial/\partial\mathbf{y}$ denote $(\partial/\partial y_1, \partial/\partial y_2 \cdots)$. We say that $\mathbf{i} \geqq \mathbf{j}$, where $\mathbf{j} = (j_1, j_2, \cdots)$, if $i_1 \geqq j_1, i_2 \geqq j_2, \cdots$.

We begin by considering an arbitrary symmetric formal power series T in the single set $\mathbf{t} = (t_1, t_2, \cdots)$ of commutative indeterminates, since the extension to the multisymmetric case is straightforward. Let $\tau(i_1, i_2, \cdots) = (j_1, j_2, \cdots) = \mathbf{j}$, where j_k is

* Received by the editors March 12, 1982, and in final form June 7, 1982. The work of these authors was supported (in part) by the Natural Sciences and Engineering Research Council of Canada under Grants U0073 and A8235.

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the number of occurrences of k in $(i_1, i_2, \cdots)$. We define the H-series of $T(\mathbf{t})$ to be the formal power series g, in the indeterminates $\mathbf{y} = (y_1, y_2, \cdots)$, such that

$$\left[\frac{\mathbf{y}^{\mathbf{j}}}{\mathbf{j}!}\right] g(\mathbf{y}) = [\mathbf{t}^{\mathbf{i}}]T(\mathbf{t}),$$

where $\mathbf{j}! = j_1! j_2! \cdots$. The H-series, g, of T is denoted by $H(T)$.

For regular coefficients, we observe that

$$[t_1^p \cdots t_n^p]T(\mathbf{t}) = \left[\frac{z^n}{n!}\right] f(z),$$

where $f(z) = (H(T))(0, \cdots, 0, z, 0, \cdots)$, where z occurs as the pth argument. The H-series therefore enables us to obtain a univariate exponential generating function for the regular coefficients of the multivariate generating function T.

2. The *H*-series. Let $s_k = t_1^k + t_2^k + \cdots$ and let $\mathbf{s} = (s_1, s_2, \cdots)$, where s_k is called a *power sum symmetric function*. Now the symmetric power series $T(\mathbf{t})$ can be expressed uniquely in terms of the power sum symmetric functions, to give $T(\mathbf{t}) = G(\mathbf{s}(\mathbf{t}))$. We adopt the notational convention that the H-series, $H(T)(\mathbf{y})$, of $T(\mathbf{t})$ may also be denoted by $H(G)(\mathbf{y})$ without ambiguity, since G and T are used only in this context.

The next theorem enables us to express the H-series for $\partial G/\partial s_n$ and $s_n G$ in terms of the H-series for G. We shall use this theorem later to deduce a system of differential equations for $H(G)(\mathbf{y})$ from a system for $G(\mathbf{s})$. It happens that the latter system is often easy to derive.

THEOREM (H-series). *Let $g(\mathbf{y})$ be the H-series for a symmetric function $T(\mathbf{t})$ and let $T(\mathbf{t}) = G(\mathbf{s}(\mathbf{t}))$. Then*

1) $$\left(H\left(\frac{\partial G}{\partial s_n}\right)\right)(\mathbf{y}) = \sum_{\mathbf{i} \geq \mathbf{0}} (-1)^{m-1} \frac{1}{\mathbf{i}!}(m-1)! \frac{\partial^{\mathbf{i}}}{\partial \mathbf{y}^{\mathbf{i}}} g(\mathbf{y}),$$

where $m = i_1 + i_2 + \cdots$ and the summation is over $\mathbf{i} = (i_1, i_2, \cdots)$ such that $i_1 + 2i_2 + \cdots = n$.

2) $$(H(s_n G))(\mathbf{y}) = \left\{y_n + \sum_{i \geq 1} y_{n+i} \frac{\partial}{\partial y_i}\right\} g(\mathbf{y}).$$

Proof. 1) Let $A_{\mathbf{i}}(\mathbf{t})$ denote the *monomial symmetric function* defined by

$$A_{\mathbf{i}}(\mathbf{t}) = \sum_{\substack{\mathbf{j} \geq 0 \\ \tau(\mathbf{j}) = \mathbf{i}}} \mathbf{t}^{\mathbf{j}} = [\mathbf{x}^{\mathbf{i}}] \prod_{k \geq 1} (1 + x_1 t_k + x_2 t_k^2 + \cdots).$$

Since $T(\mathbf{t})$ is a symmetric function in $t_1, t_2, \cdots$, there exist $c(\mathbf{i})$, independent of $\mathbf{t}$, such that

$$T(\mathbf{t}) = \sum_{\mathbf{i} \geq \mathbf{0}} c(\mathbf{i}) A_{\mathbf{i}}(\mathbf{t})$$

whence

$$H(T)(\mathbf{y}) = \sum_{\mathbf{i} \geq \mathbf{0}} c(\mathbf{i}) \frac{\mathbf{y}^{\mathbf{i}}}{\mathbf{i}!}.$$

Let $E_n(\mathbf{x})=\sum_{\mathbf{i}\geq\mathbf{0}}(-1)^{m-1}(m-1)!\mathbf{x}^{\mathbf{i}}/\mathbf{i}!$, where m and the range of summation are defined in 1). Then $\sum_{n\geq1}E_n(\mathbf{x})z^n=\log(1+zx_1+z^2x_2+\cdots)$, so that

$$\sum_{\mathbf{i}\geq\mathbf{0}}A_{\mathbf{i}}(\mathbf{t})\mathbf{x}^{\mathbf{i}}=\prod_{j\geq1}(1+x_1t_j+x_2t_j^2+\cdots)=\exp\sum_{j\geq1}\log(1+x_1t_j+x_2t_j^2+\cdots)$$
$$=\exp\sum_{j\geq1}\sum_{n\geq1}E_n(\mathbf{x})t_j^n=\exp\sum_{n\geq1}E_n(\mathbf{x})s_n(\mathbf{t}).$$

From this we obtain

$$\sum_{\mathbf{i}\geq\mathbf{0}}\mathbf{x}^{\mathbf{i}}\frac{\partial}{\partial s_n}A_{\mathbf{i}}(\mathbf{t})=\frac{\partial}{\partial s_n}\sum_{\mathbf{i}\geq\mathbf{0}}\mathbf{x}^{\mathbf{i}}A_{\mathbf{i}}(\mathbf{t})=\frac{\partial}{\partial s_n}\exp\sum_{n\geq1}E_n(\mathbf{x})s_n$$
$$=E_n(\mathbf{x})\exp\sum_{n\geq1}E_n(\mathbf{x})s_n=E_n(\mathbf{x})\sum_{\mathbf{j}\geq\mathbf{0}}A_{\mathbf{j}}(\mathbf{t})\mathbf{x}^{\mathbf{j}}.$$

The application of $[\mathbf{x}^{\mathbf{i}}]$ to this equation yields

$$\frac{\partial}{\partial s_n}A_{\mathbf{i}}(\mathbf{t})=\sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{i}}A_{\mathbf{j}}(\mathbf{t})[\mathbf{x}^{\mathbf{i}-\mathbf{j}}]E_n(\mathbf{x}).\tag{2.1}$$

Thus

$$H\left(\frac{\partial G}{\partial s_n}\right)=\sum_{\mathbf{i}\geq\mathbf{0}}c(\mathbf{i})H\left(\frac{\partial}{\partial s_n}A_{\mathbf{i}}(\mathbf{t})\right),$$

so from (2.1) we have

$$H\left(\frac{\partial G}{\partial s_n}\right)=\sum_{\mathbf{i}\geq\mathbf{0}}c(\mathbf{i})\sum_{\mathbf{j}\leq\mathbf{i}}\{[\mathbf{x}^{\mathbf{i}-\mathbf{j}}]E_n(\mathbf{x})\}H(A_{\mathbf{j}}(\mathbf{t}))$$
$$=\sum_{\mathbf{i}\geq\mathbf{0}}c(\mathbf{i})\sum_{\mathbf{j}\leq\mathbf{i}}[\mathbf{x}^{\mathbf{i}-\mathbf{j}}]E_n(\mathbf{x})\mathbf{y}^{\mathbf{j}}/\mathbf{j}!$$
$$=\sum_{\mathbf{i}\geq\mathbf{0}}c(\mathbf{i})\sum_{\mathbf{j}\leq\mathbf{i}}[\mathbf{x}^{\mathbf{i}-\mathbf{j}}]E_n(\mathbf{x})\frac{\partial^{\mathbf{i}-\mathbf{j}}}{\partial\mathbf{y}^{\mathbf{i}-\mathbf{j}}}(\mathbf{y}^{\mathbf{i}}/\mathbf{i}!).$$

Now let $\mathbf{i}-\mathbf{j}=\mathbf{k}\geq\mathbf{0}$ since $\mathbf{i}\geq\mathbf{j}$. Thus

$$H\left(\frac{\partial G}{\partial s_n}\right)=\sum_{\mathbf{i}\geq\mathbf{0}}c(\mathbf{i})\sum_{\mathbf{k}\geq\mathbf{0}}[\mathbf{x}^{\mathbf{k}}]E_n(\mathbf{x})\frac{\partial^{\mathbf{k}}}{\partial\mathbf{y}^{\mathbf{k}}}(\mathbf{y}^{\mathbf{i}}/\mathbf{i}!)$$
$$=\sum_{\mathbf{k}\geq\mathbf{0}}\frac{\partial^{\mathbf{k}}}{\partial\mathbf{y}^{\mathbf{k}}}[\mathbf{x}^{\mathbf{k}}]E_n(\mathbf{x})\sum_{\mathbf{i}\geq\mathbf{0}}c(\mathbf{i})\mathbf{y}^{\mathbf{i}}/\mathbf{i}!$$
$$=E_n\left(\frac{\partial}{\partial\mathbf{y}}\right)H(T)\quad\text{and 1) follows.}$$

2) Let $\boldsymbol{\delta}_n=(0,\cdots,0,1,0,\cdots)$, where the one appears in the nth position. Thus, by definition we have

$$s_nA_{\mathbf{i}}=\left(\sum_{\tau(\mathbf{l})=\boldsymbol{\delta}_n}x_1^{l_1}x_2^{l_2}\cdots\right)\sum_{\tau(\mathbf{j})=\mathbf{i}}x_1^{j_1}x_2^{j_2}\cdots$$
$$=\sum_{\tau(\mathbf{l})=\boldsymbol{\delta}_n}\sum_{\tau(\mathbf{j})=\mathbf{i}}x_1^{l_1+j_1}x_2^{l_2+j_2}\cdots,$$

where $\mathbf{l}=(l_1,l_2,\cdots)$ and $\mathbf{j}=(j_1,j_2,\cdots)$. Now the effect of $\mathbf{l}$ is to change a single kth power in $\mathbf{j}$ to a $(k+n)$th power, for some k, in all possible ways. Each of the resulting monomials may be obtained in $i_{n+k}+1$ ways. Thus, if $\mathbf{m}=(m_1,\cdots)$,

$$s_nA_{\mathbf{i}}=\sum_{k\geq 1}(1+i_{n+k})\sum_{\tau(\mathbf{m})=\mathbf{i}+\boldsymbol{\delta}_{n+k}-\boldsymbol{\delta}_k} x_1^{m_1}x_2^{m_2}\cdots+(1+i_n)\sum_{\tau(\mathbf{m})=\mathbf{i}+\boldsymbol{\delta}_n} x_1^{m_1}x_2^{m_2}\cdots$$

and

$$H\Big(s_n\sum_{\mathbf{i}\geq \mathbf{0}}c(\mathbf{i})A_{\mathbf{i}}\Big)=C_n\Big(\mathbf{y},\frac{\partial}{\partial \mathbf{y}}\Big)H(G),$$

where $C_n(\mathbf{y},\partial/\partial\mathbf{y})=y_n+\sum_{i\geq 1}y_{n+i}\,\partial/\partial y_i$. The result follows immediately. □

It follows from the H-series theorem that $H(s_1^{i_1}\cdots\partial^{j_1}/\partial s_1^{j_1}\cdots G)=C_1^{i_1}(\mathbf{y},\partial/\partial\mathbf{y})\cdots E_1^{j_1}(\partial/\partial\mathbf{y})\cdots H(G)$. Thus any differential equation for $G(\mathbf{s})$ may be translated, by means of the H-series theorem, into a differential equation for $H(G)$. We note that $C_n(\mathbf{y},\partial/\partial\mathbf{y})$ and E_n are reminiscent of Hammond operators for symmetric functions (MacMahon [3] and Hammond [2]).

3. Preliminary application. In this section and §§ 4 and 5 we consider the enumeration of p-regular labelled graphs and simple graphs. We now set up a system of differential equations for labelled simple graphs and demonstrate the use of the H-series theorem for the 2-regular case.

Let $T(\mathbf{t})$ be the ordinary generating function for simple labelled graphs, where t_j marks the degree of vertex j for $j\geq 1$. The generating function for the pair $\{i,j\}$ of distinct vertices is $1+t_it_j$ since if i and j are not joined by an edge there is no contribution to the degrees of i and j, while if they are then there is a contribution of 1 to each of the degrees of i and j. Thus

$$T(\mathbf{t})=\prod_{1\leq i<j}(1+t_it_j).$$

We next derive $G(\mathbf{s})$, where $G(\mathbf{s}(\mathbf{t}))=T(\mathbf{t})$. Now

$$\begin{aligned}T(\mathbf{t})&=\exp\log\prod_{1\leq i<j}(1+t_it_j)=\exp\sum_{k\geq 1}\frac{1}{k}(-1)^{k-1}\sum_{1\leq i<j}(t_it_j)^k\\&=\exp\sum_{k\geq 1}\frac{1}{2k}(-1)^{k-1}\{s_k^2(\mathbf{t})-s_{2k}(\mathbf{t})\},\end{aligned}$$

whence

$$G(\mathbf{s})=\exp\sum_{k\geq 1}\frac{1}{2k}(-1)^{k-1}\{s_k^2-s_{2k}\}.$$

The system of differential equations which $G(\mathbf{s})$ satisfies is

$$\begin{aligned}&\frac{\partial G}{\partial s_{2k+1}}=\frac{1}{2k+1}s_{2k+1}G\quad\text{for } k\geq 0,\\ &\frac{\partial G}{\partial s_{2k}}=\frac{1}{2k}\{(-1)^k-s_{2k}\}G\quad\text{for } k\geq 1.\end{aligned}\tag{3.1}$$

This is the general system of equations for labelled simple graphs. For the moment we confine our attention to 2-regular simple graphs.

Let $r_2(n)$ denote the number of 2-regular simple labelled graphs on n vertices. Then

$$r_2(n)=[t_1^2\cdots t_n^2]T(\mathbf{t})=\left[\frac{y_2^n}{n!}\right]U(y_1,y_2),$$

where $U(y_1, y_2) = H(G)(y_1, y_2, 0, \cdots)$. Thus, applying the H-series theorem to $T(\mathbf{t})$ and setting $y_3 = y_4 = \cdots = 0$, we have

$$\frac{\partial U}{\partial y_1} = y_1 U + y_2 \frac{\partial U}{\partial y_1}, \qquad 2\frac{\partial U}{\partial y_2} - \frac{\partial^2 U}{\partial y_1^2} = -U - y_2 U.$$

Eliminating $\partial U/\partial y_1$ and $\partial^2 U/\partial y_1^2$ and then setting $y_1 = 0$, we have $dV/dy_2 = \{(1-y_2)^{-1} - (1+y_2)\}V/2$ and $r_2(n) = [y_2^n/n!]V$, where $V(y_2) = U(0, y_2)$ and $V(0) = 1$. Since V is differentiably finite (or D-finite) it follows (Stanley [6, Thm. 1.5]) that $\{r_2(n) | n \geqq 0\}$ is P-recursive. Indeed, applying $[y_2^n/n!]$ to both sides of this ordinary differential equation for V we have

$$2r_2(n+1) - 2nr_2(n) - n(n-1)r_2(n-2) = 0,$$

where $r_2(0) = 1$ and $r_2(k) = 0$ for $k < 0$. We note that we may solve the differential equation to obtain

$$V(y_2) = (1-y_2)^{-1/2} \exp\left\{-\frac{y_2}{2} - \frac{y_2^2}{4}\right\},$$

the well-known generating function for the number of cycle covers of the complete graph on n vertices. This may, of course, be obtained by a direct argument, but its derivation here has illustrated the use of the H-series.

It is important to note that the ordinary generating function for labelled graphs is, by a similar argument,

$$T'(\mathbf{t}) = \prod_{1 \leqq i \leqq j} (1 - t_i t_j)^{-1} = G'(\mathbf{s}(\mathbf{t})),$$

where

$$G'(\mathbf{s}) = \exp \sum_{k \geqq 1} \frac{1}{2k} \{s_k^2(\mathbf{t}) + s_{2k}(\mathbf{t})\}.$$

The system of differential equations associated with $G'(\mathbf{s})$ is

$$\text{(3.2)} \qquad \begin{aligned} \frac{\partial G'}{\partial s_{2k+1}} &= \frac{1}{2k+1} s_{2k+1} G' \quad \text{for } k \geqq 0, \\ \frac{\partial G'}{\partial s_{2k}} &= \frac{1}{2k}\{1 + s_{2k}\}G' \quad \text{for } k \geqq 1. \end{aligned}$$

Systems (3.1) and (3.2) are strongly related to each other. Accordingly, in § 4 we shall give certain details for calculations with (3.1) but totally suppress the corresponding details for calculations with (3.2) since they are similar.

4. The P-recursiveness of the numbers of 3- and 4-regular labelled graphs and simple graphs on n vertices. Let $r_p(n)$ be the number of p-regular simple labelled graphs on n vertices. We apply the method of § 3 to the cases $p = 3$ and 4 to derive differential equations with polynomial coefficients for $\sum_{n \geqq 0} r_p(n)(x^n/n!)$. The calculations are of course more prolonged, and we have suppressed their details because they add nothing of conceptual importance to the argument.

We consider first the case $p=3$. Applying the H-series theorem to system (3.1) for $G(\mathbf{s})$ and putting $y_4=y_5=\cdots=0$, we have

$$\text{(4.1)}\quad \begin{aligned} &\text{a)} && \frac{\partial A^{(3)}}{\partial y_1}=y_1A^{(3)}+y_2\frac{\partial A^{(3)}}{\partial y_1}+y_3\frac{\partial A^{(3)}}{\partial y_2},\\ &\text{b)} && 2\frac{\partial A^{(3)}}{\partial y_2}-\frac{\partial^2 A^{(3)}}{\partial y_1^2}=-(1+y_2)A^{(3)}-y_3\frac{\partial A^{(3)}}{\partial y_1},\\ &\text{c)} && 3\frac{\partial A^{(3)}}{\partial y_3}-3\frac{\partial^2 A^{(3)}}{\partial y_1\,\partial y_2}+\frac{\partial^3 A^{(3)}}{\partial y_1 3}=y_3A^{(3)}, \end{aligned}$$

where $A^{(3)}(y_1, y_2, y_3)=H(G)(y_1, y_2, y_3, 0, \cdots)$ and $r_3(n)=[y_3^n/n!]A^{(3)}$.

Let $B^{(3)}(y_1, y_3)=A^{(3)}(y_1, 0, y_3)$. By inspection we may express $\partial B^{(3)}/\partial y_3$ and $\partial^2 B^{(3)}/\partial y_3^2$ solely in terms of $\partial B^{(3)}/\partial y_1$. We therefore have a system of two simultaneous linear equations for the unknown $\partial B^{(3)}/\partial y_1$. Eliminating $\partial B^{(3)}/\partial y_1$ between these equations and setting $y_1=0$, we obtain a second order linear ordinary differential equation in y_3 for $B^{(3)}(0, y_3)$, with polynomial coefficients, so $\{r_3(n)|n\geqq 0\}$ is P-recursive. To simplify this equation we note that $r_3(2n+1)=0$ for $n\geqq 0$ since the sum of the degrees in a graph is even. Thus $B^{(3)}(0, y_3)$ is a power series $R_3(x)$ in $y_3^2=x$. Now

$$y_3\frac{\partial B^{(3)}}{\partial y_3}=2x\frac{dR_3}{dx} \quad\text{and}\quad \frac{\partial^2 B^{(3)}}{\partial y_3^2}=2\frac{\partial R_3}{\partial x}+4x\frac{\partial^2 R_3}{\partial x^2}.$$

Thus $R_3(x)=\sum_{n\geqq 0} r_3(2n)x^n/(2n)!$ satisfies the differential equation given in Table 4.1(i). This agrees with Read [4]. A similar argument applied to system (3.2) gives the ordinary differential equation for $Q_3(x)=\sum_{n\geqq 0} q_3(2n)x^n/(2n)!$, where $q_3(2n)$ is the number of labelled 3-regular graphs on $2n$ vertices. This is given in Table 4.1(ii).

TABLE 4.1(i)
The differential equation for the number of 3-regular simple labelled graphs.

i	$\phi_i(x)$
0	$x(-x^2-2x+2)^2$
1	$-6(x^5+6x^4+6x^3-32x+8)$
2	$36x^2(-x^2-2x+2)$

$$R_3(x)=\sum_{n\geqq 0} r_3(2n)\frac{x^n}{(2n)!}:\ \phi_2(x)\frac{d^2R_3(x)}{dx^2}+\phi_1(x)\frac{dR_3(x)}{dx^2}+\phi_0(x)R_3(x)=0$$

TABLE 4.1(ii)
The differential equation for the number of 3-regular labelled graphs.

i	$\phi_i(x)$
0	$x^5-10x^4+24x^3-4x^2-44x-48$
1	$-6(x^5-6x^4+6x^3+24x^2+16x-8)$
2	$36x^2(x^2-2x-2)$

$$Q_3(x)=\sum_{n\geqq 0} q_3(2n)\frac{x^n}{(2n)!}:\ \phi_2(x)\frac{d^2Q_3}{dx^2}(x)+\phi_1(x)\frac{dQ_3}{dx}(x)+\phi_0(x)Q_3(x)=0$$

The case $p=4$ may be treated in a similar way. Applying the H-series theorem to system (3.1) for $G(\mathbf{s})$ and putting $y_5=y_6=\cdots=0$, we have

$$
\begin{aligned}
&\frac{\partial A^{(4)}}{\partial y_1}=y_1A^{(4)}+y_2\frac{\partial A^{(4)}}{\partial y_1}+y_3\frac{\partial A^{(4)}}{\partial y_2}+y_4\frac{\partial A^{(4)}}{\partial y_3},\\
&2\frac{\partial A^{(4)}}{\partial y_2}-\frac{\partial^2 A^{(4)}}{\partial y_1^2}=-(1+y_2)A^{(4)}-y_3\frac{\partial A^{(4)}}{\partial y_1}-y_4\frac{\partial A^{(4)}}{\partial y_2},\\
&3\frac{\partial A^{(4)}}{\partial y_3}-3\frac{\partial^2 A^{(4)}}{\partial y_1\,\partial y_2}+\frac{\partial^3 A^{(4)}}{\partial y_1^3}=y_3A^{(4)}+y_4\frac{\partial A^{(4)}}{\partial y_1},\\
&4\frac{\partial A^{(4)}}{\partial y_4}-4\frac{\partial^2 A^{(4)}}{\partial y_1\,\partial y_3}-2\frac{\partial^2 A^{(4)}}{\partial y_2^2}+4\frac{\partial^3 A^{(4)}}{\partial y_1^2\,\partial y_2}-\frac{\partial^4 A^{(4)}}{\partial y_1^4}=(1-y_4)A^{(4)},
\end{aligned}
\tag{4.2}
$$

where $A^{(4)}(y_1,y_2,y_3,y_4)=H(G)(y_1,y_2,y_3,y_4,0,\cdots)$ and $r_4(n)=[y_4^n/n!]A^{(4)}$.

Let $B^{(4)}(y_1,y_4)=A^{(4)}(y_1,0,0,y_4)$. By inspection, we may express $\partial^m B^{(4)}/\partial y_4^m$ linearly in terms of $B^{(4)}$, $\partial B^{(4)}/\partial y_1$, $\partial^2 B^{(4)}/\partial y_1^2$ alone for $m\geqq 1$. In fact, when we carry this out for $m=1,2$ (using the symbolic algebra system VAXIMA, as described in § 8) and set $y_1=0$, we find that the coefficient of $\partial B^{(4)}/\partial y_1$ at $y_1=0$ is 0 in both equations. Eliminating $\partial^2 B^{(4)}/\partial y_1^2$ at $y_1=0$ between these two equations, we obtain a second order differential equation for $R_4(x)=\sum_{n\geqq 0} r_4(n)(x^n/n!)$, where $R_4(x)=B^{(4)}(0,x)$. This differential equation is given in Table 4.2(i) and demonstrates that $R_4(x)$ is D-finite so $\{r_4(n)|n\geqq 0\}$ is P-recursive. The corresponding differential equation for $Q_4(x)=\sum_{n\geqq 0} q_4(n)(x^n/n!)$, where $q_4(n)$ is the number of 4-regular labelled graphs, is deduced in a similar way from system (3.2) and is given in Table 4.2(ii).

We have therefore established the following result.

COROLLARY. *$\{r_p(n)|n\geqq 0\}$ and $\{q_p(n)|n\geqq 0\}$ are P-recursive for $p=2,3,4$.*

TABLE 4.2(i)

The differential equation for the number of 4-regular simple labelled graphs.

i	$\phi_i(x)$
0	$-x^4(x^5+2x^4+2x^2+8x-4)^2$
1	$-4(x^{13}+4x^{12}-16x^{10}-10x^9-36x^8-220x^7-348x^6-48x^5+200x^4-336x^3-240x^2+416x-96)$
2	$16x^2(x-1)^2(x^5+2x^4+2x^2+8x-4)(x+2)^2$

$$R_4(x)=\sum_{n\geqq 0} r_4(n)\frac{x^n}{n!}:\ \phi_2(x)\frac{d^2R_4(x)}{dx^2}+\phi_1(x)\frac{dR_4(x)}{dx}+\phi_0(x)R_4(x)=0$$

TABLE 4.2(ii)

The differential equation for the number of 4-regular labelled graphs.

i	$\phi_i(x)$
0	$x^{14}-4x^{13}-8x^{12}+44x^{11}-8x^{10}-40x^9-244x^8+288x^7+192x^6+1056x^5-944x^4-2688x^3$ $+448x^2+1408x+384$
1	$-4(x^{13}-4x^{12}+8x^{10}+22x^9-20x^8-92x^7-36x^6+48x^5+760x^4-464x^3-400x^2+160x+96)$
2	$16x^2(x+1)^2(x-2)^2(x^5-2x^4-2x^2+8x+4)$

$$Q_4(x)=\sum_{n\geqq 0} q_4(n)\frac{x^n}{n!}:\ \phi_2(x)\frac{d^2Q_4}{dx^2}+\phi_1(x)\frac{dQ_4}{dx}+\phi_0(x)Q_4=0$$

Using the recurrence equations implied by the above differential equations for R_3, R_4, Q_3, Q_4, we have calculated $r_p(n)$ and $q_p(n)$ for $p=3,4$ and $n \leqq 20$. These numbers are displayed in Tables A and B of the Appendix.

Read [4] has already given the differential equation associated with $\{r_3(n)|n \geqq 0\}$. Read and Wormald [5] have given a system of simultaneous recurrence equations for $\{r_4(n)|n \geqq 0\}$ and an inspection of these indicates that the P-recursiveness of $\{r_4(n)|n \geqq 0\}$ may be deduced quite easily. The differential equations for $\{q_p(n)|n \geqq 0\}$ for $p=3$ and $p=4$ appear to be new. We draw the reader's attention to the fact that the H-series theorem enables us to write down the system of partial differential equations for the H-series for arbitrary p without difficulty. However, the reduction of this system to a single ordinary differential equation in y_p is a technical task which we are unable to carry out for the general case.

5. A combinatorial construction. The differential equations for the H-series associated with p-regular simple labelled graphs may be given a direct combinatorial interpretation. This is achieved by distinguishing precisely k monovalent vertices for $k=1, \cdots, p$. This clearly involves a difficult case analysis, which is long even for the case $p=3$. It is noteworthy that in this instance the H-series theorem carries out this case analysis automatically. In this section we give a combinatorial interpretation of system (4.1) for simple labelled 3-regular graphs.

Let $\mathscr{A}$ be the set of simple labelled graphs whose vertices have degree at most 3. Then the power series $A^{(3)}(y_1, y_2, y_3)$ of § 4 is the exponential generating function for the elements of $\mathscr{A}$ with y_i marking vertices of degree i for $i=1,2,3$. Thus if $a(i_1, i_2, i_3)$ is the number of graphs in $\mathscr{A}$ with i_j vertices of degree $j=1,2,3$, then we have

$$A^{(3)}(y_1, y_2, y_3) = \sum_{i_1, i_2, i_3 \geqq 0} a(i_1, i_2, i_3) \frac{y_1^{i_1}}{i_1!} \frac{y_2^{i_2}}{i_2!} \frac{y_3^{i_3}}{i_3!}.$$

The combinatorial derivations of (4.1a, b, c) are now given. To obtain these we count the graphs in $\mathscr{A}$ once for each set of i distinct monovalent vertices for $i=1,2,3$. For this purpose the i-set is regarded as being distinguished.

Equation (4.1a). Distinguish exactly one monovalent vertex in each element in $\mathscr{A}$. The generating function for this is $y_1\, \partial A^{(3)}/\partial y_1$. We now derive this in another way.

1) The distinguished monovalent vertex is adjacent to a vertex of degree one, forming a component consisting of a single edge joining two vertices. The generating function for this is $y_1^2 A^{(3)}$.

2) The distinguished monovalent vertex is adjacent to a vertex of degree two. We may construct such graphs by distinguishing a monovalent vertex, v, and then connecting this by an edge to a new monovalent vertex u. Now u is the distinguished monovalent vertex adjacent to a bivalent vertex v. The generating function for this is $y_1 y_2\, \partial A^{(3)}/\partial y_1$. We note that the operator $y_2\, \partial/\partial y_1$ arises because a monovalent vertex is first distinguished and then connected to another vertex, making the former bivalent.

3) The distinguished monovalent vertex is adjacent to a vertex of degree three. Following 2), the generating function for this is $y_1 y_3\, \partial A^{(3)}/\partial y_2$.

It follows that

$$\frac{\partial A^{(3)}}{\partial y_1} = y_1 A^{(3)} + y_2 \frac{\partial A^{(3)}}{\partial y_1} + y_3 \frac{\partial A^{(3)}}{\partial y_2},$$

and we have derived (4.1a) combinatorially.

Equation (4.1b). Distinguish two distinct monovalent vertices in each element in $\mathscr{A}$. The generating function for this is $(y_1^2/2!)\,\partial^2 A^{(3)}/\partial y_1^2$. We now derive this in another way.

1) The two distinguished vertices are connected by a path of edge-length 1, forming a component consisting of one edge. The generating function for this is $(y_1^2/2!)A^{(3)}$.

2) The two distinguished vertices are connected by a path of edge-length 2. There are two subcases.

i) The path contains exactly one bivalent vertex. The generating function for this is $(y_1^2/2!)y_2A^{(3)}$, since $(y_1^2/2!)y_2$ is the generating function for a component consisting of a path of edge-length two.

ii) The path contains exactly one trivalent vertex. Such graphs may be obtained by joining a distinguished monovalent vertex in an element of $\mathscr{A}$ to two new monovalent vertices, which are themselves the distinguished monovalent vertices in the resulting graph. The generating function for this is $(y_1^2/2!)y_3\,\partial A^{(3)}/\partial y_1$.

The generating function for these two cases is therefore

$$\frac{1}{2!}y_1^2y_2A^{(3)}+\frac{1}{2!}y_1^2y_3\frac{\partial A^{(3)}}{\partial y_1}.$$

3) We may obtain the remaining such graphs by deleting from a graph in $\mathscr{A}$ a vertex, u, of degree 2 connected to distinct vertices a and b and connecting a distinguished isolated vertex a' to a and a distinguished isolated vertex b' to b. The vertices a' and b' are the distinguished monovalent vertices and are not connected by a path of edge-length 1 or 2. The generating function for this is

$$2\frac{y_1^2}{2!}\frac{\partial A^{(3)}}{\partial y_2}$$

since a', b' may be labelled in two ways.

It follows that

$$\frac{\partial^2 A^{(3)}}{\partial y_1^2}=2\frac{\partial A^{(3)}}{\partial y_2}+A^{(3)}+y_2A^{(3)}+y_3\frac{\partial A^{(3)}}{\partial y_1},$$

and we have derived Equation (4.1b) combinatorially.

Equation (4.1c). Distinguish three distinct monovalent vertices in each element in $\mathscr{A}$. The generating function for this is $(y_1^3/3!)\,\partial^3 A^{(3)}/\partial y_1^3$. We now derive this in another way.

1) Exactly two of the distinguished vertices are joined by a path of edge-length one. We may construct such graphs by joining two isolated vertices, u and v, by an edge and by distinguishing one monovalent vertex w in a graph in $\mathscr{A}$. The generating function for this is $(y_1^2/2!)y_1\,\partial A^{(3)}/\partial y_1$.

2) At least one pair of distinguished vertices are joined by a path of edge-length two. There are three subcases.

i) All three distinguished vertices are joined by paths of edge-length exactly two. Thus the distinguished vertices are the monovalent vertices of a component whose remaining vertex has degree three. The generating function for this is therefore $(y_1^3/3!)y_3A^{(3)}$, since the component may be adjoined to any element in $\mathscr{A}$.

ii) Exactly two of the distinguished vertices are joined by a path of edge-length equal to two. There are two subcases.

a) The path contains a bivalent vertex. Such graphs may be constructed from a path of edge-length two joining two distinguished vertices and a graph in $\mathcal{A}$ with exactly one distinguished monovalent vertex. The generating function for this is $y_2(y_1^2/2!)y_1\,\partial A^{(3)}/\partial y_1$.

b) The path contains a vertex of degree three. We may construct such graphs by considering a path uvw, of edge-length two, and a graph in $\mathcal{A}$ with exactly two distinct distinguished monovalent vertices a and b, separated by more than one edge. The vertices v and a are now identified, and u, w and b are the distinct distinguished vertices of the resulting graph. The generating function due to all graphs in $\mathcal{A}$ treated in this way is $(y_1^2/2!)y_1y_3\,\partial^2A^{(3)}/\partial y_1^2$. But this set includes graphs in which two distinguished monovalent vertices are separated by a single edge, and hence form a component, enumerated by y_1^2, adjoined to an element of $\mathcal{A}$, enumerated by $A^{(3)}$. When this is treated in the above manner, the generating function is $(y_1^2/2!)y_3y_1A^{(3)}$, so the contribution of this case is

$$\frac{1}{2!}y_1^3y_3\left(\frac{\partial^2A^{(3)}}{\partial y_1^2}-A^{(3)}\right).$$

3) No pairs of the distinguished monovalent vertices are joined by paths of edge-length at least one or two. We may construct such graphs by deleting from a graph in $\mathcal{A}$ a vertex of degree three connected to vertices a, b and c and connecting a to a', b to b' and c to c', where a', b', c' are isolated vertices. In the resulting graph, a', b', c' are the distinguished vertices. The generating function for this is $y_1^3\,\partial A^{(3)}/\partial y_3$ since a', b', c' may be labelled in $3!$ ways.

It follows that

$$\begin{aligned}\frac{\partial^3A^{(3)}}{\partial y_1^3}&=3(1+y_2)\frac{\partial A^{(3)}}{\partial y_1}+3y_3\frac{\partial^2A^{(3)}}{\partial y_1^2}-2y_3A^{(3)}+6\frac{\partial A^{(3)}}{\partial y_3}\\&=\frac{\partial}{\partial y_1}\left\{3(1+y_2)A^{(3)}+3y_3\frac{\partial A^{(3)}}{\partial y_1}\right\}-2y_3A^{(3)}+6\frac{\partial A^{(3)}}{\partial y_3}\\&=-3\frac{\partial}{\partial y_1}\left\{2\frac{\partial A^{(3)}}{\partial y_2}-\frac{\partial^2A^{(3)}}{\partial y_1^2}\right\}-2y_3A^{(3)}+6\frac{\partial A^{(3)}}{\partial y_3}\end{aligned}$$

from (4.1b). Thus

$$\frac{\partial^3A^{(3)}}{\partial y_1^3}+3\frac{\partial A^{(3)}}{\partial y_3}=y_3A^{(3)}+3\frac{\partial^2A^{(3)}}{\partial y_1\,\partial y_2},$$

and we have combinatorially derived (4.1c). This completes the combinatorial treatment of system (4.1).

6. Bisymmetric *H*-series. In the final part of this paper we consider the extension of the H-series theorem to the bisymmetric case. As an application of this extension we enumerate $n\times n$ matrices over the nonnegative integers with line sum p (each row sum and column sum equals p) for $p=2,3$.

Let $\mathbf{r}=(r_1,r_2,\cdots)$ and $\mathbf{c}=(c_1,c_2,\cdots)$ be sets of indeterminates, and let

$$T(\mathbf{r},\mathbf{c})=\sum_{\mathbf{i},\mathbf{j}\geqq\mathbf{0}}c(\mathbf{i},\mathbf{j})A_{\mathbf{i}}(\mathbf{r})A_{\mathbf{j}}(\mathbf{c}),$$

where $A_{\mathbf{i}}$ is the monomial symmetric function defined in § 2. Then the H-series of T is

$$(H(T))(\mathbf{x},\mathbf{y})=\sum_{\mathbf{i},\mathbf{j}\geqq\mathbf{0}}c(\mathbf{i},\mathbf{j})(\tau(\mathbf{i})!\,\tau(\mathbf{j})!)^{-1}\mathbf{x}^{\tau(\mathbf{i})}\mathbf{y}^{\tau(\mathbf{j})}.$$

Let T be the ordinary generating function for nonnegative integer matrices, with r_i and c_i marking the sums of the elements in row i and column i respectively. Now a k in row i and column j contributes k to the ith row sum and the jth column sum so the generating function for the (i, j)-element is $1+(r_ic_j)+(r_ic_j)^2+\cdots$ whence

$$T(\mathbf{r}, \mathbf{c}) = \prod_{i,j\geqq 1} (1-r_ic_j)^{-1}.$$

Clearly T is bisymmetric since it is symmetric in $\mathbf{r}$ and in $\mathbf{c}$. Let $s_k = r_1^k + r_2^k + \cdots$ and $t_k = c_1^k + c_2^k + \cdots$ for $k \geqq 1$, the power sum symmetric functions for $\mathbf{r}$ and $\mathbf{c}$, respectively. Now

$$T(\mathbf{r}, \mathbf{c}) = \exp \sum_{i,j\geqq 1} \log (1-r_ic_j)^{-1} = \exp \sum_{k\geqq 1} \frac{1}{k} \sum_{i,j\geqq 1} r_i^k c_j^k,$$

so

$$G(\mathbf{s}, \mathbf{t}) = \exp \sum_{k\geqq 1} \frac{1}{k} s_k t_k.$$

The system of differential equations satisfied by $G(\mathbf{s}, \mathbf{t})$ is

$$\frac{\partial G}{\partial s_k} = \frac{1}{k} t_k G \quad \text{for } k \geqq 1, \qquad \frac{\partial G}{\partial t_k} = \frac{1}{k} s_k G \quad \text{for } k \geqq 1. \tag{6.1}$$

We illustrate the use of the H-series theorem in the bisymmetric case by applying it to nonnegative integer matrices with line sum two in this section and line sum three in § 7. Let $l_p(n)$ be the number of $n \times n$ nonnegative integer matrices with line sum p. Then

$$l_2(n) = \left[\frac{x_2^n}{n!}\frac{y_2^n}{n!}\right] D^{(2)}(x_1, x_2, y_1, y_2),$$

where $D^{(2)}(x_1, x_2, y_1, y_2) = (H(T))(x_1, x_2, 0, \cdots, y_1, y_2, 0, \cdots)$. Applying the H-series theorem to system (6.1) and setting $x_3 = x_4 = \cdots = 0$ and $y_3 = y_4 = \cdots = 0$ we have the following system of equations for $D^{(2)}(x_1, x_2, y_1, y_2)$.

$$\text{(6.2)}\quad \begin{aligned} &\text{a)}\ \frac{\partial D^{(2)}}{\partial x_1} = y_1 D^{(2)} + y_2 \frac{\partial D^{(2)}}{\partial y_1}, &&\text{a)}'\ \frac{\partial D^{(2)}}{\partial y_1} = x_1 D^{(2)} + x_2 \frac{\partial D^{(2)}}{\partial x_1},\\ &\text{b)}\ \frac{\partial D^{(2)}}{\partial x_2} - \frac{1}{2}\frac{\partial^2 D^{(2)}}{\partial x_1^2} = \frac{1}{2} y_2 D^{(2)}, &&\text{b)}'\ \frac{\partial D^{(2)}}{\partial y_2} - \frac{1}{2}\frac{\partial^2 D^{(2)}}{\partial y_1^2} = \frac{1}{2} x_2 D^{(2)}. \end{aligned}$$

From (6.2a) and (6.2a)′ we obtain

$$\frac{\partial D^{(2)}}{\partial x_1} = (y_1 + y_2 x_1)(1 - x_2 y_2)^{-1} D^{(2)}. \tag{6.3}$$

Differentiating (6.2b)′ partially with respect to x_2 we have

$$2\frac{\partial^2 D^{(2)}}{\partial x_2\, \partial y_2} = \frac{\partial^3 D^{(2)}}{\partial y_1^2\, \partial x_2} + x_2 \frac{\partial D^{(2)}}{\partial x_2} + D^{(2)}.$$

Eliminating $\partial D^{(2)}/\partial x_2$ from the right-hand side of this equation by means of (6.2b) we have

$$4\frac{\partial^2 D^{(2)}}{\partial x_2\, \partial y_2} = \frac{\partial^4 D^{(2)}}{\partial x_1^2\, \partial y_1^2} + y_2 \frac{\partial^2 D^{(2)}}{\partial y_1^2} + x_2 \frac{\partial^2 D^{(2)}}{\partial x_1^2} + x_2 y_2 D^{(2)} + 2D^{(2)}. \tag{6.4}$$

We wish to eliminate x_1 and y_1. Thus let $E^{(2)}(x_2, y_2) = D^{(2)}(0, x_2, 0, y_2)$ so that, from (6.3),

$$\left.\frac{\partial^2 D^{(2)}}{\partial x_1^2}\right|_{x_1=y_1=0} = y_2(1-x_2y_2)^{-1}E^{(2)}.$$

From (6.2a)′ we have

$$\left.\frac{\partial^4 D^{(2)}}{\partial x_1^2\,\partial y_1^2}\right|_{x_1=y_1=0} = 2(1+x_2y_2)(1-x_2y_2)^{-2}E^{(2)}$$

and

$$\left.\frac{\partial^2 D^{(2)}}{\partial y_1^2}\right|_{x_1=y_1=0} = x_2(1-x_2y_2)^{-1}E^{(2)}.$$

Substituting these expressions into (6.4) and simplifying we obtain

$$(4-8x_2y_2+4x_2^2y_2^2)\frac{\partial^2 E^{(2)}}{\partial x_2\,\partial y_2} = (4-2x_2^2y_2^2+x_2^3y_2^3)E^{(2)}.$$

But a matrix with line sum 2 must be square, so $E^{(2)}(x_2, y_2) = M^{(2)}(x_2y_2)$, where $M^{(2)}(z) = \sum_{n\geqq 0} l_2(n)\, z^n/(n!)^2$.

Thus $M^{(2)}(z)$ satisfies the differential equation

$$4z(1-z)^2\frac{d^2}{dz^2}M^{(2)}(z) + 4(1-z)^2\frac{d}{dz}M^{(2)}(z) - (4-2z^2+z^3)M^{(2)}(z) = 0,$$

so $M^{(2)}(z)$ is D-finite and $\{l_2(n) | n \geqq 0\}$ is P-recursive. By inspection this equation may be rewritten as

$$\left\{2z(1-z)\frac{d}{dz} + 2 + 2z - z^2\right\}G(z) = 0, \quad \text{where } G(z) = \left\{2(1-z)\frac{d}{dz} - (2-z)\right\}M^{(2)}(z).$$

But $G(z)$ is a formal power series with no negative exponents so $G(z) = 0$, yielding the recurrence equation

$$l_2(n+1) = (n+1)^2 l_2(n) - \tfrac{1}{2}n^2(n+1)l_2(n-1)$$

for $n \geqq 0$, where $l_2(0) = 1$, $l_2(-1) = 0$. This simplifies the recurrence equation given by Anand, Dumir and Gupta [1]. The differential equation may be solved to give

$$G(z) = (1-z)^{-1/2}\exp\left(\frac{z}{2}\right),$$

which may be obtained immediately by a combinatorial construction involving cycles.

7. Nonnegative integer matrices with line sum 3. Now

$$l_3(n) = \left[\frac{x_3^n}{n!}\frac{y_3^n}{n!}\right] D^{(3)}(x_1, x_2, x_3, y_1, y_2, y_3),$$

where $D^{(3)}(x_1, x_2, x_3, y_1, y_2, y_3) = (H(T))(x_1, x_2, x_3, 0, \cdots, y_1, y_2, y_3, 0, \cdots)$, and T is given in § 6. Following § 6 we apply the H-series theorem and set $x_4 = x_5 = \cdots = 0$

and $y_4=y_5=\cdots=0$ to obtain the following system of equations for $D^{(3)}$.

$$
(7.1)\quad
\begin{aligned}
&\text{a)} && \frac{\partial D^{(3)}}{\partial x_1}=y_1D^{(3)}+y_2\frac{\partial D^{(3)}}{\partial y_1}+y_3\frac{\partial D^{(3)}}{\partial y_2},\\
&\text{a)}' && \frac{\partial D^{(3)}}{\partial y_1}=x_1D^{(3)}+x_2\frac{\partial D^{(3)}}{\partial x_1}+x_3\frac{\partial D^{(3)}}{\partial x_2},\\
&\text{b)} && \frac{\partial D^{(3)}}{\partial x_2}-\frac{1}{2}\frac{\partial^2 D^{(3)}}{\partial x_1^2}=\frac{1}{2}y_2D^{(3)}+\frac{1}{2}y_3\frac{\partial D^{(3)}}{\partial y_1},\\
&\text{b)}' && \frac{\partial D^{(3)}}{\partial y_2}-\frac{1}{2}\frac{\partial^2 D^{(3)}}{\partial y_1^2}=\frac{1}{2}x_2D^{(3)}+\frac{1}{2}x_3\frac{\partial D^{(3)}}{\partial x_1},\\
&\text{c)} && \frac{\partial D^{(3)}}{\partial x_3}-\frac{\partial^2 D^{(3)}}{\partial x_1\,\partial x_2}+\frac{1}{3}\frac{\partial^3 D^{(3)}}{\partial x_1^3}=\frac{1}{3}y_3D^{(3)},\\
&\text{c)}' && \frac{\partial D^{(3)}}{\partial y_3}-\frac{\partial^2 D^{(3)}}{\partial y_1\,\partial y_2}+\frac{1}{3}\frac{\partial^3 D^{(3)}}{\partial y_1^3}=\frac{1}{3}x_3D^{(3)}.
\end{aligned}
$$

By inspection we may express $\partial^{2i}D^{(3)}/\partial x_3^i\,\partial y_2^i$ at $x_2=0$, $y_2=0$ linearly in terms of $D^{(3)}$, $\partial D^{(3)}/\partial x_1$, $\partial D^{(3)}/\partial y_1$, $\partial^2D^{(3)}/\partial x_1\,\partial y_1$ at $x_2=y_2=0$ for $i\geqq 1$. Moreover, when we carry this out for $i=1,2$ (again using VAXIMA) and set $x_1=y_1=0$, we discover that the coefficients of $\partial D^{(3)}/\partial x_1$ and $\partial D^{(3)}/\partial y_1$ are 0 in both equations. Eliminating $\partial^2D^{(3)}/\partial x_1\,\partial y_1$ at $x_1=x_2=y_1=y_2=0$ between these two equations, we get a linear equation involving $\partial^4D^{(3)}/\partial x_3^2\,\partial y_3^2$, $\partial^2D^{(3)}/\partial x_3\,\partial y_3$ and $D^{(3)}$, all at $x_1=x_2=y_1=y_2=0$. But $D^{(3)}(0,0,x_3,0,0,y_3)=E^{(3)}(x_3y_3)$, where

$$E^{(3)}(z)=\sum_{n\geqq 0} l_3(n)\frac{z^n}{(n!)^2}.$$

Finally, this partial differential equation for $E^{(3)}(x_3y_3)$ can be transformed to a fourth-order ordinary differential equation for $E^{(3)}(x)$, with polynomial coefficients in x, by making the substitution $x=x_3y_3$. This differential equation is displayed in Table 7.1.

We therefore have the following result.

COROLLARY.

$\{l_3(n)\mid n\geqq 0\}$ is P-recursive.

TABLE 7.1
The differential equation for the number of nonnegative integer matrices with line sum 3.

i	ϕ_i
0	$x^{11}-7x^{10}+30x^9-16x^8-43x^7+51x^6+238x^5+630x^4+36x^3-1944x^2-1152x+576$
1	$-9(x^{10}-4x^9+22x^8-8x^7-4x^6+8x^5+88x^4+252x^3+120x^2-320x+64)$
2	$-9(x^{10}-4x^9+22x^8-8x^7-22x^6+8x^5+106x^4+234x^3+48x^2-320x+64)x$
3	$324x^4(x^4-x^2+x+4)$
4	$81x^5(x^4-x^2+x+4)$

$$E^{(3)}(x)=\sum_{n\geqq 0} l_3(n)\frac{x^n}{(n!)^2}:\ \phi_4(x)\frac{d^4E^{(3)}}{dx^4}+\phi_3(x)\frac{d^3E^{(3)}}{dx^3}+\phi_2(x)\frac{d^2E^{(3)}}{dx^2}+\phi_1(x)\frac{dE^{(3)}}{dx}+\phi_0(x)E^{(3)}=0$$

This appears to be a new result (see Stanley [6, p. 186]). The recurrence for $\{l_3(n)|n \geqq 0\}$ which follows from the equation in Table 7.1 has been used to compute $l_3(n)$ for $n \leqq 15$. These numbers are given in Table C of the Appendix.

8. Concluding comments. Each of the differential equations displayed in tables in this paper was obtained by using the symbolic algebra system called VAXIMA. The elimination procedures for R_4, Q_4 and $E^{(3)}$ were so substantial that we could not have carried them out by hand. Each of the tables given in the Appendix was computed from the corresponding differential equation by means of VAXIMA. The computer calculations were carried out at the University of Waterloo. VAXIMA is based on the MACSYMA system developed at the Massachusetts Institute of Technology.

Appendix.

TABLE A

Numbers of 3-regular simple labelled graphs (i) *and labelled graphs* (ii).

n	$r_3(n)$	$q_3(n)$
0	1	1
2	0	2
4	1	47
6	70	4720
8	19355	1256395
10	11180820	699971370
12	11555272575	706862729265
14	19506631814670	1173744972139740
16	50262958713792825	2987338986043236825
18	187747837889699887800	11052457379522093985450
20	976273961160363172131825	57035105822801295375685775
	(i)	(ii)

TABLE B

Numbers of 4-regular simple labelled graphs (i) *and labelled graphs* (ii).

n	$r_4(n)$	$q_4(n)$
0	1	1
1	0	1
2	0	3
3	0	15
4	0	138
5	1	2021
6	15	43581
7	465	1295493
8	19355	50752145
9	1024380	2533755933
10	66462606	157055247261
11	5188453830	11836611005031
12	480413921130	1066129321651668
13	52113376310985	113117849882149725
14	6551246596501035	13965580274228976213
15	945313907253606891	1985189312618723797371
16	155243722248524067795	321932406123733248625851
17	28797220460586826422720	59079829666712346141491403
18	5993002310427150494060340	12182062872168618012045410805
19	1390759561507559001823665540	2804416350168401031334025488653
20	357920518512934324278467820756	716675823235860386364568072658826
	(i)	(ii)

TABLE C

Numbers of n × n nonnegative integer matrices with line sum 3.

n	$l_3(n)$
0	1
1	1
2	4
3	55
4	2008
5	153040
6	20933840
7	4662857360
8	1579060246400
9	772200774683520
10	523853880779443200
11	477360556805016931200
12	569060910292172349004800
13	868071731152923490921728000
14	1663043727673392444887284377600
15	3937477620391471128913917360384000

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