SIAM J. ALG. DISC. METH.
Vol. 7, No. 1, January 1986

007

LABELLED GRAPHS WITH SMALL VERTEX DEGREES AND P-RECURSIVENESS*

I. P. GOULDEN† AND D. M. JACKSON†

Abstract. We show that the number of labelled graphs with vertices of degrees 1, 2, 3 or 4 only satisfy linear recurrence equations, and are therefore P-recursive. We conjecture that the number of labelled graphs with vertices whose degrees belong to a given finite set is also P-recursive.

AMS(MOS) subject classifications. 05C30, 05A15

1. Introduction. A sequence $\{a_n \mid n \geqq 0\}$ is said to be *P-recursive* if it satisfies a homogeneous linear recurrence equation of finite order, with polynomial coefficients. Such sequences are of interest because the n-th term can be computed in time that is linear in n, and space that is independent of n. The formal power series $A(x) = \sum_{n\geqq 0} a_n x^n/n!$, called the exponential generating function for $\{a_n \mid n \geqq 0\}$, is said to be *D-finite* if A satisfies a linear homogeneous differential equation of finite order, whose coefficients are polynomials in x. Stanley [8] discusses the equivalence of the D-finiteness of A and the P-recursiveness of $\{a_n \mid n \geqq 0\}$, as well as showing that many combinatorially defined power series are D-finite.

For $\alpha \subset \{0, 1, \cdots\}$, let $\mathrm{G}_{0,\alpha}$ be the set of labelled graphs, each of whose vertex degrees lies in α, and let $\mathrm{G}_{1,\alpha}$ denote the set of simple graphs in $\mathrm{G}_{0,\alpha}$. Suppose that the number of graphs on n vertices in $\mathrm{G}_{i,\alpha}$ is denoted by $g_{i,\alpha}(n)$, and that the exponential generating function for $\mathrm{G}_{i,\alpha}$ with respect to vertices is $G_{i,\alpha}(x) = \sum_{n\geqq 0} g_{i,\alpha}(n) x^n/n!$, for $i = 0, 1$. A p-regular graph is one in which each vertex has degree p, and corresponds to the choice $\alpha = \{p\}$ above.

Read [5] has shown that $G_{1,\{3\}}$ is D-finite, and it is implicit in Read and Wormald [6] that $G_{1,\{4\}}$ is D-finite. Goulden, Jackson and Reilly [2] have shown that $G_{0,\{3\}}$ and $G_{0,\{4\}}$ are D-finite. Stanley [8] has asked whether $G_{i,\{p\}}$ is D-finite for all p. In this paper we consider sets α of vertex-degrees with more than a single element. Applying the methods developed in Goulden, Jackson and Reilly [2], we construct differential equations which demonstrate that $G_{i,\alpha}$ is D-finite for $i = 0, 1$ and all choices of α whose maximum element (denoted by $m(\alpha)$) is less than or equal to 4.

Throughout this paper we denote the coefficient of $x_1^{i_1} x_2^{i_2} \cdots$ in the formal power series $f(x_1, x_2, \cdots)$ by $[x_1^{i_1} x_2^{i_2} \cdots] f$. For details of the sum and product lemmas for labelled configurations see Goulden and Jackson [1].

2. Preliminary cases. Certain $G_{i,\alpha}$ can be obtained immediately by elementary combinatorial arguments, using only the sum and product lemmas for exponential generating functions. The first simplification is to note that $G_{i,\{0\}\cup\alpha} = e^x G_{i,\alpha}$, for $0 \notin \alpha$, $i = 0, 1$. Thus $G_{i,\{0\}\cup\alpha}$ is D-finite if and only if $G_{i,\alpha}$ is D-finite, and so it is enough to consider only the case $\alpha \subset \{1, 2, \cdots\}$ in the remainder of this paper.

For the case $m(\alpha) = 1$, we immediately have $G_{i,\{1\}} = \exp(x^2/2)$ for $i = 0, 1$ since, for labelled graphs with only vertices of degree 1, the connected components are single edges, each of which has generating function $x^2/2$.

* Received by the editors August 25, 1983, and in revised form September 17, 1984. This work was supported by the Natural Sciences and Engineering Research Council of Canada under grants U0073 and A8235.

† Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.

For the case $m(\alpha)=2$, we consider labelled graphs whose connected components are paths or cycles. Thus

$$G_{0,\{2\}}=(1-x)^{-1/2}\exp\left(\frac{x}{2}+\frac{x^2}{4}\right),\qquad G_{1,\{2\}}=(1-x)^{-1/2}\exp\left(-\frac{x}{2}-\frac{x^2}{4}\right),$$

$$G_{0,\{1,2\}}=(1-x)^{-1/2}\exp\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^2}{2(1-x)}\right),$$

$$G_{1,\{1,2\}}=(1-x)^{-1/2}\exp\left(-\frac{x}{2}-\frac{x^2}{4}+\frac{x^2}{2(1-x)}\right),$$

so for $m(\alpha)\leqq 2$ and $i=0,1$, we have directly obtained an expression for $G_{i,\alpha}$. Differentiating these expressions once, we immediately obtain the first order differential equation $\phi_1(d/dx)G_{i,\alpha}+\phi_0G_{i,\alpha}=0$, where ϕ_1 and ϕ_0 are given explicitly for each such i and α in Table 1.

TABLE 1
Differential equations for $G_{i,\alpha}(x)$ with $m(\alpha)\leqq 2$.

α	i	ϕ_1	ϕ_0
$\{1\}$	0	1	$-x$
$\{1\}$	1	1	$-x$
$\{2\}$	0	$2(1-x)$	x^2-2
$\{2\}$	1	$2(1-x)$	$-x^2$
$\{1,2\}$	0	$2(1-x)^2$	$-x^3+2x^2-2$
$\{1,2\}$	1	$2(1-x)^2$	$x(x^2-2)$

For the cases $m(\alpha)=3$ and $m(\alpha)=4$, we have no explicit expression for $G_{i,\alpha}(x)$, so we cannot proceed as we have in the previous cases $m(\alpha)=1,2$. Instead, we follow the indirect procedure given in the next section.

3. Symmetric multivariate generating functions for $m(\alpha)=3, 4$. Suppose that we are interested in the sequence $\{c_p(n)\,|\,n\geqq 0\}$ where $c_p(n)=[t_1^p\cdots t_n^p]T(\mathbf{t})$, and $T(\mathbf{t})$ is a symmetric function in the indeterminates $\mathbf{t}=(t_1,t_2,\cdots)$. We say that $c_p(n)$ is a *regular coefficient* of $T(\mathbf{t})$. Further suppose that $T(\mathbf{t})$ is expressed in terms of the power sum symmetric functions $s_i=\sum_{j\geqq 1}t_j^i$ as $T(\mathbf{t})=E(\mathbf{s})$, where $\mathbf{s}=(s_1,s_2,\cdots)$. Then $c_p(n)=[y_p^n/n!]V(y_1,\cdots,y_p)$, by the H-series theorem (Goulden, Jackson and Reilly [2]) where $V(=H(E)$, the *H-series* of $E)$ is the solution to a system of p partial differential equations derived from a system of partial differential equations for E itself. If these equations for V can be manipulated in a way that eliminates all differentiation with respect to $y_1,\cdots,y_{p-1}$, we can then set $y_1=\cdots=y_{p-1}=0$ to obtain an ordinary differential equation for $V(0,\cdots,0,y_p)=\sum_{n\geqq 0}c_p(n)y_p^n/n!$, and hence deduce the D-finiteness of $V(0,\cdots,0,y_p)$. This procedure has been followed for 3- and 4-regular graphs in [2]. The following result enables us to carry it out for sets α with more than a single element.

PROPOSITION 3.1.

$$g_{i,\alpha}(n)=[t_1^{m(\alpha)}\cdots t_n^{m(\alpha)}]\prod_{j\geqq 1}\left(\sum_{k\in\alpha}t_j^{m(\alpha)-k}\right)T_i\quad \textit{for } i=0,1$$

where

$$T_0=\prod_{1\leqq l\leqq j}(1-t_lt_j)^{-1},\qquad T_1=\prod_{1\leqq l<j}(1+t_lt_j).$$

Proof. $[t_1^{d_1}\cdots t_n^{d_n}]T_i$ is the number of labelled graphs in which the vertex with label k has degree d_k, for $k=1,\cdots,n$, when $i=0$. In the case $i=1$, we have the number of such graphs that are simple. Thus

$$\begin{aligned} g_{i,\alpha}(n) &= \sum_{d_1\in\alpha}\cdots\sum_{d_n\in\alpha}[t_1^{d_1}\cdots t_n^{d_n}]T_i \\ &= \sum_{d_1\in\alpha}\cdots\sum_{d_n\in\alpha}[t_1^{m(\alpha)}\cdots t_n^{m(\alpha)}]t_1^{m(\alpha)-d_1}\cdots t_n^{m(\alpha)-d_n}T_i \\ &= [t_1^{m(\alpha)}\cdots t_n^{m(\alpha)}]\prod_{j=1}^{n}\left(\sum_{k\in\alpha} t_j^{m(\alpha)-k}\right)T_i \end{aligned}$$

and the result follows, since $(\sum_{k\in\alpha} t_j^{m(\alpha)-k})|_{t_j=0}=1$. □

This result gives the required numbers of graphs as regular coefficients in symmetric power series. For each i and α, with $m(\alpha)=3$ or 4, we denote the expression for this symmetric power series in terms of $\mathbf{s}$ by $E_{i,\alpha}(\mathbf{s})$ and determine $E_{i,\alpha}(\mathbf{s})$ by applying exp log to the generating function in Proposition 3.1. For example,

$$\begin{aligned} g_{0,\{1,2,4\}}(n) &= [t_1^4\cdots t_n^4]\prod_{j\geq 1}(1+t_j^2+t_j^3)T_0 \\ &= [t_1^4\cdots t_n^4]\prod_{j\geq 1}(1+t_j^2)(1-t_j^3)^{-1}T_0 \\ &= [t_1^4\cdots t_n^4]\exp\left\{\sum_{j\geq 1}\log(1+t_j^2)+\log(1-t_j^3)^{-1}+\sum_{l\leq j}\log(1-t_lt_j)^{-1}\right\} \\ &= [t_1^4\cdots t_n^4]\exp\left\{\sum_{j\geq 1}\sum_{k\geq 1}\frac{1}{k}((-1)^{k-1}t_j^{2k}+t_j^{3k})+\sum_{l\leq j}\sum_{k\geq 1}\frac{1}{k}t_l^kt_j^k\right\}, \end{aligned}$$

so that

$$E_{0,\{1,2,4\}}(\mathbf{s})=\exp\left\{\sum_{k\geq 1}\frac{1}{k}(s_{3k}+(-1)^{k-1}s_{2k}+(s_k^2+s_{2k})/2)\right\}.$$

Similarly, for all α with $m(\alpha)=3$ or 4, $E_{i,\alpha}(\mathbf{s})=\exp\{a_i+b_\alpha\}$, where

$$a_0=\sum_{k\geq 1}(s_k^2+s_{2k})/2k,\qquad a_1=\sum_{k\geq 1}(-1)^{k-1}(s_k^2-s_{2k})/2k$$

and the b_α, for $m(\alpha)=3$ or 4, are given in Table 2.

TABLE 2
Power sum representations for $\log(G_{i,\alpha})-a_i$ *with* $m(\alpha)=3,4$.

α	b_α
{3}	1
{1, 3}	$\sum_{k\geq 1} s_{2k}/k$
{2, 3}	$\sum_{k\geq 1} (-1)^{k-1}s_k/k$
{1, 2, 3}	$\sum_{k\geq 1} (s_k-s_{3k})/k$
{4}	1
{1, 4}	$\sum_{k\geq 1} s_{3k}/k$
{2, 4}	$\sum_{k\geq 1} (-1)^{k-1}s_{2k}/k$
{3, 4}	$\sum_{k\geq 1} (-1)^{k-1}s_k/k$
{1, 2, 4}	$\sum_{k\geq 1} (s_{3k}+(-1)^{k-1}s_{2k})/k$
{1, 3, 4}	$\sum_{k\geq 1} (s_{3k}-s_{4k}+(-1)^{k-1}s_k)/k$
{2, 3, 4}	$\sum_{k\geq 1} (s_k-s_{3k})/k$
{1, 2, 3, 4}	$\sum_{k\geq 1} (s_k-s_{4k})/k$

Of course, $g_{i,\alpha}(n)=[t_1^3\cdots t_n^3]E_{i,\alpha}(\mathbf{s})$ for $m(\alpha)=3$, and $g_{i,\alpha}(n)=[t_1^4\cdots t_n^4]E_{i,\alpha}(\mathbf{s})$ for $m(\alpha)=4$.

4. Univariate generating functions for $m(\alpha)=3$, 4. It is now a straightforward matter to obtain a system of partial differential equations for $E_{i,\alpha}(\mathbf{s})$. For example

$$k\frac{\partial}{\partial s_k}E_{0,\{1,2,4\}}=\begin{cases}(4-2(-1)^{k/2}+s_k)E_{0,\{1,2,4\}}, & k=0\ (\mathrm{mod}\ 6)\\ (3+s_k)E_{0,\{1,2,4\}}, & k=3\ (\mathrm{mod}\ 6)\\ (1-2(-1)^{k/2}+s_k)E_{0,\{1,2,4\}}, & k=2,4\ (\mathrm{mod}\ 6)\\ s_kE_{0,\{1,2,4\}}, & k=1,5\ (\mathrm{mod}\ 6).\end{cases}$$

Carrying this out for all α with $m(\alpha)=3$, we find that the H-series $V(y_1,y_2,y_3)=H(E_{i,\alpha})$ satisfies the system

$$\begin{aligned}V_1&=(c+y_1)V+y_2V_1+y_3V_2,\\ 2V_2-V_{11}&=(d+fy_2)V+fy_3V_1,\\ 3V_3-3V_{12}+V_{111}&=(e+y_3)V,\end{aligned}\tag{1}$$

where $V_{ij\cdots}$ denotes $\partial/\partial y_i\,\partial/\partial y_j\cdots V$, and the values of c, d, e, f corresponding to each (i,α) are given in Table 3.

TABLE 3
Parameter values for system (1).

α	c	d	e	i	f
{3}	0	f	0	0	1
{1, 3}	0	$2+f$	0	1	−1
{2, 3}	1	$-1+f$	1		
{1, 2, 3}	1	$1+f$	2		

For $m(\alpha)=4$, the H-series $V(y_1,y_2,y_3,y_4)$ satisfies the system

$$\begin{aligned}V_1&=(c+y_1)V+y_2V_1+y_3V_2+y_4V_3,\\ 2V_2-V_{11}&=(d+gy_2)V+gy_3V_1+gy_4V_2,\\ 3V_3-3V_{12}+V_{111}&=(e+y_3)V+y_4V_1,\\ 4V_4-4V_{13}-2V_{22}+4V_{112}-V_{1111}&=(f+gy_4)V,\end{aligned}\tag{2}$$

where the values of c, d, e, f, g corresponding to each (i,α) are given in Table 4.

TABLE 4
Parameter values for system (2).

α	c	d	e	f	i	g
{4}	0	g	0	1	0	1
{1, 4}	0	g	3	1	1	−1
{2, 4}	0	$2+g$	0	−1		
{3, 4}	1	$-1+g$	1	0		
{1, 2, 4}	0	$2+g$	3	−1		
{1, 3, 4}	1	$-1+g$	4	−4		
{2, 3, 4}	1	$1+g$	−2	2		
{1, 2, 3, 4}	1	$1+g$	1	−2		

The two special cases of system (1) corresponding to 3-regular graphs and simple graphs have been given in [2]. If we remove all partial derivatives with respect to y_1 and y_2 from system (1) by means of the elimination scheme given in [2], and then set $y_1 = y_2 = 0$, we obtain a second order differential equation for $G_{i,\alpha}(x) = V(0, 0, x)$. If this equation is denoted by

$$\phi_2(x)\frac{d^2}{dx^2}G_{i,\alpha}(x) + \phi_1(x)\frac{d}{dx}G_{i,\alpha}(x) + \phi_0(x)G_{i,\alpha}(x) = 0,$$

then the values of ϕ_0, ϕ_1, ϕ_2 for each (i, α) with $m(\alpha) = 3$ are given in Table A of the Appendix. The values of $g_{i,\alpha}(n)$, $n = 0, \cdots, 10$, deduced from the differential equations are given in Table B, for checking purposes.

Similarly, two special cases of system (2) have been given in [2]. The elimination scheme which was used in [2] to obtain a second order differential equation for $G_{i,\alpha}(x) = V(0, 0, 0, x)$ will only work in 4 of the 16 cases that arise from $m(\alpha) = 4$ (including the two cases reported in [2]). This is because our elimination scheme involved finding linear equations in derivatives with respect to y_1 and y_4. For 4 sets of values of c, d, e, f, g, the two equations given in [2] involve only V_{44}, V_4, V, V_{11}, so V_{11} is eliminated to yield a second order ordinary differential equation. For the other 12 sets of parameter values, the two equations involve V_{44}, V_4, V, V_{11}, V_1. Thus we derive a third equation from these, involving V_{444}, V_{44}, V_4, V, V_{11}, V_1, and eliminate V_{11}, V_1 between these three equations to yield a third order differential equation.

Since these third order differential equations have large polynomials as coefficients, we do not give them here. The four cases with second order differential equations are $i = 0$, 1 and $\alpha = \{4\}$, $\{2, 4\}$. The cases with $\alpha = \{4\}$ have been reported in [2], so we omit them, and give the values of ϕ_0, ϕ_1, ϕ_2, for the differential equation

$$\phi_2(x)\frac{d^2}{dx^2}G_{i,\alpha}(x) + \phi_1(x)\frac{d}{dx}G_{i,\alpha}(x) + \phi_0(x)G_{i,\alpha}(x) = 0$$

with $\alpha = \{2, 4\}$ in Table C of the Appendix. The values of $g_{i,\{2,4\}}(n)$ for $n = 0, \cdots, 10$ are given in Table D.

5. A conjecture. In general, for any α, it is routine to derive a system of $m(\alpha)$ partial differential equations for $V(y_1, y_2, \cdots, y_{m(\alpha)})$. These can, of course, be transformed into a system of simultaneous recurrence equations in $m(\alpha)$ dimensions, which can be used to give the required number, $g_{i,\alpha}(n) = [y^n_{m(\alpha)}/n!]V$, in time which is of order $n^{m(\alpha)}$. To enable us to calculate $g_{i,\alpha}(n)$ in time which is linear in n, we must first reduce the system of partial differential equations for $V(y_1, \cdots, y_{m(\alpha)})$ to a single ordinary differential equation for $V(0, \cdots 0, y_{m(\alpha)})$, as we have done in the previous section when $m(\alpha) = 3, 4$. When $m(\alpha) \geqq 5$, we can find elimination schemes to perform this reduction, but the computation becomes very lengthy. For example, for the 5-regular simple graphs, with $i = 1$, $\alpha = \{5\}$, we have carried out the very time-consuming elimination, and have obtained a differential equation for $G_{1,\{5\}}(x)$. Unfortunately, it is of sixth order, and the degrees of the polynomial coefficients exceed 100. The first 20 values of $g_{1,\{5\}}(n)$, deduced from this equation, agree with the results of McKay [4]. This differential equation demonstrates that $G_{1,\{5\}}(x)$ is D-finite, but there is certainly no guarantee that it is the lowest-order ordinary differential equation with polynomial coefficients which can be found for $G_{1,\{5\}}(x)$.

The differential equations that we have obtained lead us to make the following conjecture.

CONJECTURE 5.1. *The numbers* $g_{0,\alpha}(n)$ *and* $g_{1,\alpha}(n)$, *of labelled graphs and simple labelled graphs, respectively, with n vertices, each with degree in* α, *are P-recursive for any finite* α.

From the results of this paper, it seems that k-regular graphs are computationally equivalent to graphs whose vertex-degrees lie in α, where α has maximum element k. It might be that certain choices of α, say $\alpha=\{0,1,\cdots,k\}$ would be more convenient to work with, in proving P-recursiveness, than k-regular graphs because of more "freedom" in constructions, while yielding equivalent results.

6. Plane partitions. If $p(i_1,\cdots,i_n)$ is the number of plane partitions with i_j copies of j for $j=1,\cdots,n$, then

$$\begin{aligned} p(i_1,\cdots,i_n)&=[t_1^{i_1}\cdots t_n^{i_n}]\prod_{j\geqq 1}(1-t_j)^{-1}\prod_{l<j}(1-t_lt_j)^{-1}\\ &=[t_1^{i_1}\cdots t_n^{i_n}]\prod_{j\geqq 1}(1+t_j)\prod_{l\leqq j}(1-t_lt_j)^{-1}, \end{aligned}$$

from Stanley [7] or Macdonald [3]. Thus if $q_m(n)$ is the number of plane partitions with m copies of each of $1,2,\cdots,n$, then

$$q_m(n)=g_{0,\{m-1,m\}}(n).$$

Thus, we have demonstrated that $\{q_m(n)\mid n\geqq 0\}$ is P-recursive for $m\leqq 4$, and conjecture that it is P-recursive for all m.

Appendix.

TABLE A

Polynomial coefficients in ordinary differential equations for $G_{i,\alpha}(x)$ when $m(\alpha)=3$.

i	α	j	ϕ_j
0	$\{3\}$	0	$x(x^{10}-10x^8+24x^6-4x^4-44x^2-48)$
		1	$-3(x^{10}-6x^8+9x^6+18x^4+10x^2-8)$
		2	$9x^3(x^4-2x^2-2)$
0	$\{1,3\}$	0	$x(x^{10}-18x^8+120x^6-272x^4-324x^2-120)$
		1	$-3(x^{10}-14x^8+41x^6+36x^4+2x^2-8)$
		2	$9x^3(x^4-4x^2-2)$
0	$\{2,3\}$	0	$x^{11}+x^{10}-6x^9-4x^8+11x^7-15x^6+8x^5-2x^3+12x^2-24x-24$
		1	$-3(x^{10}-2x^8+2x^6-6x^5+8x^4+2x^3+8x^2+16x-8)$
		2	$9x^3(x^4-x^2+x-2)$
0	$\{1,2,3\}$	0	$x^{11}-2x^{10}-14x^9+24x^8+74x^7-61x^6-99x^5$ $-55x^4-180x^3-48x^2-96x-24$
		1	$-3(x^{10}-10x^8-6x^7+22x^6+8x^5+20x^4+26x^3+16x-8)$
		2	$9x^3(x+2)(x^3-2x^2+x-1)$
1	$\{3\}$	0	$-x^3(x^4+2x^2-2)^2$
		1	$3(x^{10}+6x^8+3x^6-6x^4-26x^2+8)$
		2	$9x^3(x^4+2x^2-2)$
1	$\{1,3\}$	0	$-x(x^4-4x^2+2)(x^6-2x^2+12)$
		1	$3(x^{10}-2x^8-5x^6-18x^2+8)$
		2	$9x^3(x^4-2)$
1	$\{2,3\}$	0	$-x^2(x^9+x^8+8x^7+14x^6+15x^5+9x^4-24x^3-22x^2+16x+12)$
		1	$3(x^{10}+10x^8-4x^7+16x^6-2x^5-14x^4+34x^3-24x^2-16x+8)$
		2	$9x^3(x^4+3x^2+x-2)$
1	$\{1,2,3\}$	0	$-x(x^{10}-2x^9-6x^7-12x^6+x^5-x^4+39x^3-10x^2+24)$
		1	$3(x^{10}+2x^8+2x^7-4x^6+8x^5-2x^4+10x^3-16x^2-16x+8)$
		2	$9x^3(x^4+x^2+x-2)$

TABLE B
Initial values for $g_{i,\alpha}(n)$ when $m(\alpha)=3$.

i	α	$\{g_{i,\alpha}(n) \mid 0 \leqq n \leqq 10\}$
0	{3}	1, 0, 2, 0, 47, 0, 4720, 0, 1256395, 0, 699971370
0	{1, 3}	1, 0, 5, 0, 186, 0, 22960, 0, 6831650, 0, 4071581010
0	{2, 3}	1, 1, 4, 23, 214, 2698, 44288, 902962, 22262244, 68446612, 21940389584
0	{1, 2, 3}	1, 1, 7, 47, 521, 7233, 129443, 2811701, 73203561, 2229207953, 78389689559
1	{3}	1, 0, 0, 0, 1, 0, 70, 0, 19355, 0, 11180820
1	{1, 3}	1, 0, 1, 0, 8, 0, 730, 0, 188790, 0, 102737670
1	{2, 3}	1, 0, 0, 1, 10, 112, 1760, 35150, 848932, 24243520, 805036704
1	{1, 2, 3}	1, 0, 1, 4, 41, 512, 8285, 166582, 4054953, 116797432, 3912076929

TABLE C
Polynomial coefficients in ordinary differential equations for $G_{i,\{2,4\}}(x)$, $i=0, 1$.

i	j	ϕ_j
0	0	$(-x^{14}+6x^{13}+2x^{12}-76^{11}+112x^{10}+96x^9+356x^8-1320x^7$ $-568x^6+768x^5+9248x^4+12224x^3-2496x^2-3968x-768)$
	1	$4(x^{13}-4x^{12}-6x^{11}+36x^{10}-6x^9+24x^8-352x^7+380x^6$ $+152x^5+2104x^4-1472x^3-688x^2+256x+96)$
	2	$-16(x-2)^2x^2(x+1)^2(x^5-2x^4+2x^3-2x^2+12x+4)$
1	0	$x^2(x^{12}+6x^{11}+14x^{10}+12x^9-16x^8+24x^7+116x^6-184x^5$ $-456x^4+480x^3+512x^2-704x+192)$
	1	$4(x^{13}+4x^{12}-2x^{11}-20x^{10}+2x^9+40x^8-104x^7-204x^6$ $+200x^5+328x^4-288x^3-208x^2+320x-96)$
	2	$-16(x-1)^2x^2(x+2)^2(x^2+2x-2)(x^3+2)$

TABLE D
Initial values for $g_{0,\{2,4\}}(n)$ and $g_{1,\{2,4\}}(n)$.

i	$\{g_{i,\{2,4\}}(n) \mid 0 \leqq n \leqq 10\}$
0	1, 2, 9, 65, 751, 13044, 320803, 10609256, 453774440, 24375801464, 1607240682376
1	1, 0, 0, 1, 3, 38, 730, 20670, 781578, 37885204, 2289786624

Acknowledgments. The calculations were carried out by the symbolic algebra system called VAXIMA at the University of Waterloo. VAXIMA is based on the MACSYMA system developed at the Massachusetts Institute of Technology.

REFERENCES

[1] I. P. GOULDEN AND D. M. JACKSON, *Combinatorial Enumeration*, John Wiley, New York, 1983.
[2] I. P. GOULDEN, D. M. JACKSON AND J. W. REILLY, *The Hammond series of a symmetric function and its application to P-recursiveness*, this Journal, 4 (1983), 179–193.
[3] I. G. MACDONALD, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, New York, 1979.
[4] B. D. MCKAY, *Applications of a technique for labelled enumeration*, preprint.
[5] R. C. READ, *The enumeration of locally restricted graphs* (II), J. Lond. Math. Soc., 35 (1960), pp. 334–351.
[6] R. C. READ AND N. C. WORMALD, *Number of labelled 4-regular graphs*, J. Graph Theory, 4 (1980), pp. 203–212.
[7] R. P. STANLEY, *Theory and applications of plane partitions: Parts* I, II, Studies Appl. Math., 50 (1971), pp. 167–188, pp. 259–279.
[8] ———, *Differentiably finite power series*, European J. Comb., 1 (1980), pp. 175–188.