ELSEVIER Discrete Mathematics 158 (1996) 77–85

DISCRETE MATHEMATICS

# An umbral relation between pattern and commutation in strings

I.P. Goulden, D.M. Jackson*, C.M. Springer

*Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*

Received 7 March 1994; revised 21 October 1994

**Abstract**

Two settings for string enumeration are considered in which string statistics can be constructed such that the generating series for the set of all strings have the form $(F^{-1} \circ a)^{-1}$ in both cases, where $F$ is a formal power series and $a$ is a sequence. The two settings are qualitatively different, one involving pattern, which is locally testable, and the other involving commutation in strings, which is not locally testable. Evidence for a common generalization of these two settings is considered.

## 1. Introduction

If $G(z) = g_0 + g_1 z + g_2 z^2 + \cdots$ is an arbitrary power series in $z$ and $a = (a_1, a_2, \ldots)$ is an arbitrary sequence, then their *umbral composition* is given by $G \circ a = g_0 + g_1 a_1 + g_2 a_2 + \cdots$, whenever this sum is defined. For the alphabet $\mathcal{N}$ of positive integers, we consider strings in $\mathcal{N}^*$; the empty string, of length zero, contained in $\mathcal{N}^*$, is denoted by $\varepsilon$.

In Section 2 we deal with the *pattern* of a string, and factorization into maximal $\pi_1$-strings. The enumerative result is the maximal decomposition theorem (see e.g. [2]) for strings and is given as Theorem 2.2. In Section 3 we deal with *commutation* in a string and factorization into commutation subsets. The enumerative result is the theorem for partial commutation monoids, and is given as Theorem 3.5. The combinatorial information that is captured in these two situations is qualitatively different. The factorization associated with patterns is obtained through locally testing the string, and sweeping from left to right. On the other hand, the factorization into commutation subsets cannot be obtained by local testing in general, and requires repeated sweeps through the string.

* Corresponding author.


*SSDI* 0012-365X(95)00038-0

It is therefore unexpected that for these two qualitatively different combinatorial faactorizations the generating series for strings in $\mathcal{N}^*$ have the common umbral form

$$(F^{-1} \circ a)^{-1},$$

where $F(z) = 1 + f_1 z + f_2 z^2 + \cdots$ is an arbitrary series with $f_i$ marking, for each string factor, a combinatorial statistic of strings that evaluates to $i$. In both cases, the sequence $a$ records information about the constitution of these factors. The proof of Theorem 2.1 given here is a new one, and is strikingly similar to those of Section 3. The proofs are based on first counting canonical configurations, and then 'lifting' them to the main result with a compound alphabet, and with the introduction of a combinatorial statistic. The special case in which all string factors associated with commutation have size one, obtained by setting $F(z) = 1 + z$, is the theorem of Cartier and Foata [1] for the partial commutation monoid.

In Section 4, we present the evidence that we have for a natural combinatorial statistic for strings that serves as common generalization of these two results. The generalization involves the possibility that information about $\pi_1$-strings other than $<$-strings can be combined with information about commutation strings.

## 2. The maximal decomposition theorem

Let $\pi_1 \subseteq \mathcal{N} \times \mathcal{N}$ be an arbitrary binary relation on $\mathcal{N}$, and let $\pi_2 = \mathcal{N} \times \mathcal{N} - \pi_1$, the complementary relation. Each nonempty string $s = s_1 \ldots s_k \in \mathcal{N}^*$ has a unique *pattern* $P(s) = \pi_{i_1} \ldots \pi_{i_{k-1}} \in \{\pi_1, \pi_2\}^*$ determined by $(s_j, s_{j-1}) \in \pi_{i_j}$ for $j = 1, \ldots, k-1$. In this case, the length of $s$ is $k$, and is denoted by $|s| = k$. The string $s$ is a $\pi_1$-*string* if its pattern is in $\pi_1^*$, and is a $\pi_2$-*string* if its pattern is in $\pi_2^*$. If the pattern of a string is written in the form $P(s) = \pi_1^{l_1 - 1}\pi_2 \pi_1^{l_2 - 1}\pi_2 \ldots \pi_2 \pi_1^{l_m - 1}$, where $l_1, \ldots, l_m \geqslant 1$, then the maximal $\pi_1$-substrings of $s$ have lengths $l_1, \ldots, l_m$, respectively, from left to right, and we call the list $\rho_{\pi_1}(s) = (l_1, \ldots, l_m)$ the *maximal decomposition* of $s$.

For example, if $\pi_1 = \{(i,j)\colon 1 \leqslant i < j\}$, so we may write $\pi_1 = <$, then the maximal $\pi_1$-strings of 2 3 5 5 4 6 7 8 1 2 are 2 3 5, 5, 4, 6 7 8, 1 2, so in this case the string has maximal decomposition (3, 1, 1, 3, 2).

For $s = s_1 \ldots s_k \in \mathcal{N}^*$, let $x_s = x_{s_1} \ldots x_{s_k}$, let $x_\varepsilon = 1$, and let

$$\gamma_k = \sum_{P(s) = \pi_1^{k-1}} x_s$$

be the generating series for $\pi_1$-strings of length $k$, $k \geqslant 1$. We begin with a duality result, expressing the generating series for $\pi_2$-strings in terms of the $\gamma$'s by means of a sign-reversing involution (see Lemma 3.11 of [4] for a matrix algebra proof).

**Theorem 2.1.** *The generating series for* $\pi_2$-*strings in* $\mathcal{N}^*$ *is*

$$1 + \sum_{\substack{s \in \mathcal{N}^* \\ P(s) \in \pi_2^*}} x_s = \{1 - \gamma_1 + \gamma_2 - \cdots\}^{-1}.$$

**Proof.** Let $\mathscr{S} = \{\varepsilon\} \cup \{s: P(s) \in \pi_1^*\}$, $\mathscr{T} = \{\varepsilon\} \cup \{t: P(t) \in \pi_2^*\}$, and $\mathscr{R} = \mathscr{S} \times \mathscr{T} - \{(\varepsilon, \varepsilon)\}$. For $(s, t) = (s_1 \ldots s_m, t_1 \ldots t_n) \in \mathscr{R}$, define $\xi(s, t) = (s', t')$ as follows: if $(s_m, t_1) \in \pi_1$ or $s = \varepsilon$, then $s' = st_1$, $t' = t_2 \ldots t_k$; otherwise, if $(s_m, t_1) \in \pi_2$ or $t = \varepsilon$, then $s' = s_1 \ldots s_{k-1}$, $t' = s_k t$.

Clearly, $\xi$ is an involution without fixed points on $\mathscr{R}$, and if $\mathrm{wt}(s, t) = (-1)^{|s|} x_s x_t$, we have $\mathrm{wt}(s', t') = -\mathrm{wt}(s, t)$, so we conclude that

$$\sum_{(s,\, t) \in \mathscr{R}} \mathrm{wt}(s, t) = 0.$$

But the left-hand side can be rewritten to give

$$\sum_{s \in \mathscr{S}} (-1)^{|s|} x_s \sum_{t \in \mathscr{T}} x_t - 1 = 0$$

and the result follows on adding 1 to both sides and dividing by

$$\sum_{s \in \mathscr{S}} (-1)^{|s|} x_s = 1 - \gamma_1 + \gamma_2 - \cdots \qquad \square$$

This result works noncommutatively, since there is no reordering of symbols in the above proof. Next we deduce the maximal decomposition theorem [3], for enumerating strings with respect to maximal decompositions, by 'lifting' the above result using a different alphabet.

For $\rho_{\pi_1}(s) = (l_1, \ldots, l_m)$, let $f_{\rho_{\pi_1}(s)} = f_{l_1} \ldots f_{l_m}$. The result involves the sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ of $\pi_1$-string generating series. This is the first of the pair of generating series of the form $(F^{-1} \circ a)^{-1}$ for strings in $\mathscr{N}^*$.

**Theorem 2.2.** *The generating series for strings in $\mathscr{N}^*$ with respect to $\rho_{\pi_1}$ and itself is*

$$\sum_{s \in \mathscr{N}^*} f_{\rho_{\pi_1}(s)} x_s = (F^{-1} \circ \gamma)^{-1}.$$

**Proof.** Consider strings on the alphabet $\mathscr{A}$ of $\pi_1$-strings, with binary relations

$$\pi_1^{(1)} = \{(s_1 \ldots s_k, t_1 \ldots t_m): (s_k, t_1) \in \pi_1\}$$

and its complement $\pi_2^{(1)}$. For $s$ in $\mathscr{A}$, mark it by $f_{|s|} x_s$. With these replacements, we apply Theorem 2.1; the left-hand side of the theorem becomes

$$1 + \sum_{\substack{\sigma_1 \sigma_2 \cdots \in \mathscr{A}^* \\ P(\sigma_1 \sigma_2 \cdots) \in \pi_2^{(1)*}}} f_{|\sigma_1|} x_{\sigma_1} f_{|\sigma_2|} x_{\sigma_2} \cdots = \sum_{s \in \mathscr{A}} f_{\rho_{\pi_1}(s)} x_s,$$

since every string $s$ in $\mathscr{A}$ can be written uniquely as a string $\sigma_1 \sigma_2 \ldots$ in $\mathscr{A}^*$ whose constituent $\pi_1^{(1)}$-strings $\sigma_1, \sigma_2, \ldots$ are the maximal $\pi_1$-strings of $s$. The right-hand side of Theorem 2.1 becomes

$$\left(1+\sum_{k\geq 1}(-1)^k \sum_{\substack{\sigma_1\cdots\sigma_k\in\mathscr{A}^* \\ P(\sigma_1\cdots\sigma_k)\in\pi_2^{(1)*}}} f_{|\sigma_1|}x_{\sigma_1}\cdots f_{|\sigma_k|}x_{\sigma_k}\right)^{-1}.$$

But each $\sigma_1 \ldots \sigma_k$ in the above sum, regarded as a string in $\mathscr{N}^*$, is a $\pi_1$-string of length $|\sigma_1| + \cdots + |\sigma_k|$. Thus the right-hand side becomes, with $|\sigma_1| = i_1, \ldots, |\sigma_k| = i_k$,

$$\left(1+\sum_{m\geq 1}\left\{\sum_{k\geq 0}\sum_{\substack{i_1+\cdots+i_k=m \\ i_1\cdots i_k\geq 1}}(-f_{i_1})\cdots(-f_{i_k})\right\}\gamma_m\right)^{-1}$$

$$=\left(1+\sum_{m\geq 1}\left\{[z^m]\sum_{k\geq 0}(1-F(z))^k\right\}\gamma_m\right)^{-1}=\left(1+\sum_{m\geq 1}\{[z^m]F(z)^{-1}\}\gamma_m\right)^{-1}$$

$$=(F^{-1}\circ\gamma)^{-1}$$

as required. □

This result also works noncommutatively in the $f_i$'s.

Note that although Theorem 2.2 has been obtained from Theorem 2.1 (by changing the alphabet), we can also obtain Theorem 2.1 as the special case $f_1 = 1$, $f_2 = f_3 = \cdots = 0$ of Theorem 2.2 (on the same alphabet), so these results are equivalent. This result has many applications to string enumeration (see e.g. [2]) by appropriately specializing the series $F$ and the sequence $\gamma$.

## 3. Partial commutation in strings

Let $\binom{\mathscr{N}}{2}$ denote the set of all unordered pairs of distinct elements from $\mathscr{N}$, and let $\mathscr{C}$ be an arbitrary subset of $\binom{\mathscr{N}}{2}$. Equivalently, in the context of the previous section, we can regard $\mathscr{C}$ as a symmetric, irreflexive relation on $\mathscr{N} \times \mathscr{N}$. Suppose that any pair of symbols in $\mathscr{C}$ are allowed to commute when they appear in adjacent positions in a string, and that two strings in $\mathscr{N}^*$ are *equivalent* if one can be transformed into the other by allowable such commutations. A string in $\mathscr{N}^*$ is said to be *canonical* if it is lexicographically largest (with respect to the usual total order on $\mathscr{N}$) among all strings to which it is equivalent. Let $\langle \mathscr{N}^* \rangle$ denote the canonical strings in $\mathscr{N}^*$.

For example, when $\mathscr{C} = \binom{\mathscr{N}}{2} - \{\{2, 4\}\}$, the strings 4 3 3 2 4 1 1 1 and 3 2 2 4 4 4 are canonical, but 3 3 4 3 2 is not.

The generating series for canonical strings is due to Cartier and Foata [1] and is given below in an adapted form. The following notation is needed. A *commutation subset* is a nonempty subset of $\mathscr{N}$, each pair of which belong to $\mathscr{C}$; sets of size one are commutation subsets. Let com($\mathscr{C}$) denote the set of all commutation subsets associated with $\mathscr{C}$. For $\alpha = \{i_1, \ldots, i_m\} \in \text{com}(\mathscr{C})$, when $m \geqslant 1$, let $x_\alpha = x_{i_1} \ldots x_{i_m}$ and

$$c_m = \sum_{\substack{\alpha \in \mathscr{C} \\ |\alpha| = m}} x_\alpha. \tag{1}$$

**Theorem 3.1.** *The generating series for canonical strings in* $\mathscr{N}^*$ *is*

$$\sum_{s \in \langle \mathscr{N}^* \rangle} x_s = (1 - c_1 + c_2 - \cdots)^{-1}.$$

**Proof.** We introduce the sets $\mathscr{V} = \{\emptyset\} \cup \mathscr{C}$, $\mathscr{W} = \langle \mathscr{N}^* \rangle$, and $\mathscr{U} = \mathscr{V} \times \mathscr{W} - \{(\emptyset, \varepsilon)\}$. For $(\alpha, s) = (\{\alpha_1, \ldots, \alpha_m\}, s_1 \ldots s_n) \in \mathscr{U}$, with $\alpha_1 < \cdots < \alpha_m$, define $\zeta(\alpha, s) = (\alpha', s')$ as follows: let $s_i$ be the largest symbol in $s$ that commutes with everything in $\alpha$ and for which $s$ is equivalent to a string with $s_i$ in the left-most position. Then, if $\alpha = \emptyset$ or $\alpha_m < s_i$ then $\alpha' = \alpha \cup \{s_i\}$, $s' = u$, where $s = s_i u$ with $u \in \langle \mathscr{N}^* \rangle$. Otherwise, if $s = \varepsilon$ or $\alpha_m \geqslant s_i$ then $\alpha' = \alpha - \{\alpha_m\}$, $s' = \alpha_m s$, canonically reordered.

Clearly, $\zeta$ is an involution without fixed point on $\mathscr{U}$, and if $\text{wt}(\alpha, s) = (-1)^{|\alpha|} x_\alpha x_s$, we have $\text{wt}(\alpha', s') = -\text{wt}(\alpha, s)$, and the result follows as in Theorem 2.1. □

This is a duality result similar in form to Theorem 2.1. This theorem does not work noncommutatively in general; indeed the sign-reversing involution $\zeta$ given above (from [6]) in general requires reordering of symbols while the involution $\xi$ given in the proof of Theorem 2.1 does not. However, Theorem 3.1 is true up to equivalence; that is, by allowing $x_i x_j = x_j x_i$ for $\{i, j\} \in \mathscr{C}$.

We now consider strings in com($\mathscr{C}^*$), that is, strings of commutation subsets. We define a partial order on com($\mathscr{C}$) such that $\alpha \prec \beta$ whenever all elements of $\alpha$ are smaller than all elements of $\beta$. Suppose that any pair $\alpha$, $\beta$ of subsets in com($\mathscr{C}$) are allowed to commute when they are comparable and each element of $\alpha$ commutes (with respect to $\mathscr{C}$ itself) with each element of $\beta$. Two strings of commutation subsets are *equivalent* if one can be transformed into the other by allowable such commutations. A string in com($\mathscr{C}$)* is said to be *canonical* if it is lexicographically largest (with respect to the above partial order) among all the strings to which it is equivalent. Let $\langle \text{com}(\mathscr{C})^* \rangle$ denote the canonical strings of com($\mathscr{C}$)*.

For example, when $\mathscr{C} = \binom{[\,]}{2} - \{\{2, 4\}\}$ then $\{1, 3\}$ and $\{5, 6, 8\}$ commute, $\{2, 5\}$ and $\{7\}$ commute, but $\{1, 4\}$ and $\{3\}$, $\{2, 5\}$ and $\{4, 9\}$, $\{2, 3\}$ and $\{4, 6, 7\}$ do not commute. The string $\{4, 6, 7\}\ \{1, 3\}\ \{2, 5\}$, is canonical and equivalent to $\{1, 3\}\ \{4, 6, 7\}\ \{2, 5\}$.

The following result, giving the generating series for canonical strings of commutation subsets, follows from Theorem 3.1 by lifting to the alphabet of commutation subsets. For $\sigma = \alpha_1 \cdots \alpha_k \in \text{com}(\mathscr{C})^*$, let $L(\sigma) = (|\alpha_1|, \ldots, |\alpha_k|)$ and $x_\sigma = x_{\alpha_1} \cdots x_{\alpha_k}$.

**Theorem 3.2.** *The generating series for canonical strings in* com($\mathscr{C}$)* *is*

$$\sum_{\sigma \in \langle \text{com}(\mathscr{C})^* \rangle} f_{L(\sigma)} x_\sigma = (F^{-1} \circ c)^{-1}.$$

**Proof.** Consider the strings in $\operatorname{com}(\mathscr{C})^*$ with the commutation defined above, marking $\alpha \in \operatorname{com}(\mathscr{C})$ by $f_{|\alpha|} x_\alpha$. Now we can apply Theorem 3.1 with these replacements; the left-hand side becomes

$$\sum_{\sigma \in \langle \operatorname{com}(\mathscr{C})^* \rangle} f_{L(\sigma)} x_\sigma .$$

For the right-hand side we use the fact that $\{\alpha_1, \dots, \alpha_k\}$ is a commutation subset on the alphabet $\operatorname{com}(\mathscr{C})$ whenever $\alpha_1 \dot{\cup} \cdots \dot{\cup} \alpha_k$ is a commutation subset on the alphabet $\mathscr{C}$, of size $|\alpha_1| + \cdots + |\alpha_k|$, with its elements totally ordered from left to right when partitioned into $\alpha_1, \dots, \alpha_k$. This gives, with $|\alpha_1| = i_1, \dots, |\alpha_k| = i_k$,

$$\left(1 + \sum_{m \geqslant 1} \left\{ \sum_{k \geqslant 0} \sum_{\substack{i_1 + \cdots + i_k = m \\ i_1 \cdots i_k \geqslant 1}} (-f_{i_1}) \cdots (-f_{i_k}) \right\} c_m \right)^{-1} .$$

This reduces to $(F^{-1} \circ c)^{-1}$, as in Theorem 2.2. □

Again, note that although Theorem 3.2 has been obtained from Theorem 3.1 (by changing the alphabet), we can also obtain Theorem 3.1 as the special case $f_1 = 1$, $f_2 = f_3 = \cdots = 0$ of Theorem 3.2, so these results are equivalent.

Thus, we see that both $(F^{-1} \circ c)^{-1}$ and $(F^{-1} \circ \gamma)^{-1}$ are combinatorial generating series. The connection between them can be made more striking by a simple combinatorial operation.

**Lemma 3.3.** *For $a \in \mathscr{N}$ and $\sigma \in \langle \operatorname{com}(\mathscr{C})^* \rangle$, construct $\psi(a, \sigma) = \sigma'$ as follows.*

*Case 1: If $\{a\}\sigma \in \langle \operatorname{com}(\mathscr{C})^* \rangle$ then $\sigma' = \{a\}\sigma$.*

*Case 2: Otherwise, there is a string equivalent to $\sigma$ with a left-most element that commutes with $\{a\}$. Let $\beta$ be the largest such element and suppose that $\sigma$ is equivalent to $\beta w$. Then $\sigma'$ is the element of $\langle \operatorname{com}(\mathscr{C})^* \rangle$ that is equivalent to $(\{a\} \cup \beta) w$.*

*Then $\psi$ is a bijection between $\mathscr{N} \times \langle \operatorname{com}(\mathscr{C})^* \rangle$ and $\langle \operatorname{com}(\mathscr{C})^* \rangle - \{\varepsilon\}$.*

**Proof.** By construction, $\sigma' \in \langle \operatorname{com}(\mathscr{C})^* \rangle$ in both Cases 1 and 2. Moreover, we can see that the procedure is reversible as follows. Consider an arbitrary nonempty $\sigma'$ in $\langle \operatorname{com}(\mathscr{C})^* \rangle$. If the left-most set in $\sigma'$ has a single element, Case 1 must have been used in the construction, so that element is $a$, and the remaining sets form a canonical string, which is $\sigma$. Otherwise, if the left-most set in $\sigma'$ has more than one element, then Case 2 must have been used in the construction, so the smallest element must be $a$, the remaining elements form a commutation subset, giving $\beta$, and the remaining sets form the string $w$. But $\beta w$ need not be canonical, so reorder $\beta w$ to get the canonical equivalent string $\sigma$. The result follows. □

We now define a statistic for strings.

**Definition 3.4** (*A string statistic*). For $s = i_1 \dots i_k \in \mathcal{N}^*$, the statistic $\rho_{\mathscr{C}}$ is defined by $\rho_{\mathscr{C}}(s) = L(\phi_{\mathscr{C}}(s))$, where

$$\phi_{\mathscr{C}}(\varepsilon) = \varepsilon,$$

$$\phi_{\mathscr{C}}(i_j \dots i_k) = \psi(i_j, \phi_{\mathscr{C}}(i_{j+1} \dots i_k)), \quad j = k, k-1, \dots, 1.$$

For example, to calculate $\phi_{\mathscr{C}}(4\,1\,2\,4\,6\,3\,3)$ when $\mathscr{C} = \binom{\mathcal{N}}{2} - \{\{2,4\}\}$, we successively create $\phi_{\mathscr{C}}(3)$, $\phi_{\mathscr{C}}(33)$, ... as follows:

$$\{3\},\ \{3\}\{3\},\ \{6\}\ \{3\}\{3\},\ \{4,6\}\ \{3\}\{3\},\ \{2,3\}\{4,6\}\{3\},\ \{1,2,3\}\{4,6\}\{3\},$$

and $\phi_{\mathscr{C}}(4\,1\,2\,4\,6\,3\,3) = \{4\}\{1,2,3\}\{4,6\}\{3\}$, so $\phi_{\mathscr{C}}(4\,1\,2\,4\,6\,3\,3) = (1,3,2,1)$.

As a further example, for this choice of $\mathscr{C}$, the values of $\phi_{\mathscr{C}}$ and $\rho_{\mathscr{C}}$ for the 24 permutations of $\{1,2,3,4\}$ are summarized in Table 1.

Therefore, in the presence of partial commutation, we obtain the second of the pair of generating series for strings in $\mathcal{N}^*$ of the form $(F^{-1} \circ a)^{-1}$.

**Theorem 3.5.** *The generating series for strings s in $\mathcal{N}^*$ with respect to $\rho_{\mathscr{C}}$ and s itself is*

$$\sum_{s \in \mathcal{N}^*} f_{\rho_{\mathscr{C}}(s)} x_s = (F^{-1} \circ c)^{-1}.$$

**Proof.** Suppose that $\phi_{\mathscr{C}}(s) = \sigma$ for $s \in \mathcal{N}^*$, $\sigma \in \langle \mathrm{com}(\mathscr{C})^* \rangle$. Then from the above description of $\rho_{\mathscr{C}}$, we immediately have $x_s = x_\sigma$ and $f_{\rho_{\mathscr{C}}(s)} = f_{L(\sigma)}$. But $\psi$ is a bijection, so $\phi_{\mathscr{C}}$ is a bijection between $\mathcal{N}^*$ and $\langle \mathrm{com}(\mathscr{C})^* \rangle$. Thus,

$$\sum_{s \in \mathcal{N}^*} f_{\rho_{\mathscr{C}}(s)} x_s = \sum_{\sigma \in \langle \mathrm{com}(\mathscr{C})^* \rangle} f_{L(\sigma)} x_\sigma.$$

Table 1

| $s$ | $\phi_{\mathscr{C}}(s)$ | $\rho_{\mathscr{C}}(s)$ | $s$ | $\phi_{\mathscr{C}}(s)$ | $\rho_{\mathscr{C}}(s)$ |
|---|---|---|---|---|---|
| 3241 | {3}{2}{4}{1} | (1,1,1,1) | 2134 | {2}{134} | (1,3) |
| 4321 | {4}{3}{2}{1} | (1,1,1,1) | 4123 | {4}{123} | (1,3) |
| 2413 | {2}{4}{13} | (1,1,2) | 1324 | {13}{2}{4} | (2,1,1) |
| 3214 | {3}{2}{14} | (1,1,2) | 1432 | {14}{3}{2} | (2,1,1) |
| 4213 | {4}{2}{13} | (1,1,2) | 2431 | {23}{4}{1} | (2,1,1) |
| 4312 | {4}{3}{12} | (1,1,2) | 3421 | {34}{2}{1} | (2,1,1) |
| 2143 | {2}{14}{3} | (1,2,1) | 1234 | {12}{34} | (2,2) |
| 2341 | {2}{34}{1} | (1,2,1) | 1423 | {14}{23} | (2,2) |
| 3124 | {3}{12}{4} | (1,2,1) | 2314 | {23}{14} | (2,2) |
| 3142 | {3}{14}{2} | (1,2,1) | 3412 | {34}{12} | (2,2) |
| 4132 | {4}{13}{2} | (1,2,1) | 1243 | {123}{4} | (3,1) |
| 4231 | {4}{23}{1} | (1,2,1) | 1342 | {134}{2} | (3,1) |

The result follows by identifying the sum on the right-hand side by means of Theorem 3.2. □

Note that, for $s \in \langle \mathcal{N}^* \rangle$, $\rho_{\mathscr{C}}(s)$ consists entirely of 1's since, in this case, $\phi_{\mathscr{C}}(s)$ consists entirely of singletons (at every stage we are in Case 1 of $\psi$). Thus when $f_1 = 1$, $f_2 = f_3 = \cdots = 0$. Theorem 3.5 reduces to Theorem 3.1.

## 4. The interrelation of the results

If $\mathscr{C} = \binom{\mathcal{N}}{2}$, then $\rho_{\mathscr{C}}(s) = \rho_{\pi_1}(s)$ where $\pi_1 = <$ and the commutation subsets for $\rho_{\mathscr{C}}$ are read as increasing strings for $\rho_{\pi_1}$, since no reordering is needed when implementing Case 2 of $\psi$ in Lemma 3.3 so, in this case $c_i = \gamma_i$, for $i \geqslant 1$, where $\gamma_i$ is the generating series for increasing strings. Moreover, $\gamma_i$ is the $i$th elementary symmetric function. Thus, when $\mathscr{C} = \binom{\mathcal{N}}{2}$ and $\pi_1 = <$, Theorem 3.5 and Theorem 2.2 agree.

The similarity of form of the generating series in Theorems 2.2 and 3.5, and indeed of the proofs of their underlying results Theorems 2.1 and 3.1, suggest that there might be a generalization containing all of these results as special cases. For example, if $w_k$ is the generating series for $\pi_1$-strings whose elements form a *commutation multiset*, a commutation subset with repetition, and $w = (w_1, w_2, \dots)$, then such a generalization might be a combinatorial string interpretation for the generating series $(F^{-1} \circ w)^{-1}$.

Of course, when $\pi_1 = <$, this is exactly what Theorem 3.5 provides, since in this case $w_i = c_i$, $i \geqslant 1$. If we explore this further with $\pi_1 = \leqslant$, then in this case $w_i = d_i$, $i \geqslant 1$, the generating function for commutation multisets of size $i$. Moreover, we find that the results of Section 3 all extend in this case, to yield a combinatorial string interpretation for the generating series $(F^{-1} \circ d)^{-1}$, with $d = (d_1, d_2, \dots)$. In this case the canonical strings of Theorem 3.1 exclude those in which there are adjacent repeated occurrences of a symbol. The description of all the mappings extend, with various occurrences of $<$ replaced by $\leqslant$, since $\leqslant$ is transitive.

However, this cannot be true in general without further conditions, for it fails at least for some choices of $\pi_1$ and $\mathscr{C}$. For example, if $\pi_1 = \{(1,2), (2,3), (3,2), (2,1)\}$ and $\mathscr{C} = \binom{\mathcal{N}}{2} - \{1, 3\}$, then $x_4 = x_5 = \cdots = 0$ gives $w_1 = x_1 + x_2 + x_3$, $w_2 = x_1x_2 + x_2x_3 + x_3x_2 + x_2x_1$, $w_3 = x_3x_2x_3 + x_2x_3x_2 + x_1x_2x_1 + x_2x_1x_2$, and, in commuting $x$'s,

$$[x_1x_2x_3]\{1 - w_1 + w_2 - w_3 + \cdots\}^{-1} = 6 - 8 = -2,$$

so there are negative terms in the expansion, denying a combinatorial interpretation.

For an alternative approach that generalizes Theorems 2.1 and 3.1, but not Theorems 2.2 and 3.2 see [5, Ch. 6, esp. Example 5, p. 111].

## Acknowledgements

This work was supported by grants from the Natural Sciences and Engineering Research Council of Canada.

## References

[1] P. Cartier and D. Foata, Problèmes combinatoires de commutation et réarrangements, Lecture Notes in Mathematics, Vol. 85 (Springer, Berlin, 1969).
[2] I.P. Goulden and D.M. Jackson, Combinatorial Enumeration (Wiley Interscience, New York, 1983).
[3] D.M. Jackson and R. Aleliunas, Decomposition based generating functions for sequences, Can. J. Math. 29 (1977) 181–187.
[4] D.M. Jackson and I.P. Goulden, A formal calculus for the enumerative systems of sequences – I. Combinatorial theorems, Studies. Appl. Math. 61 (1979) 141–178.
[5] P. Lalonde, Contribution à l'étude des empilements, in: Publications du LACIM, UQAM, Montréal, 1991.
[6] G.X. Viennot, Heap of pieces, I: Basic definitions and combinatorial lemmas, in: Combinatoire énumérative, Lecture Notes in Mathematics, Vol. 1234 (Springer, Berlin, 1986) 321–350.