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The Number of Ramified Coverings of the Sphere by the Torus and Surfaces of Higher Genera*

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Abstract. We obtain an explicit expression for the number of ramified coverings of the sphere by the torus with given ramification type for a small number of ramification points, and conjecture this to be true for an arbitrary number of ramification points. In addition, the conjecture is proved for simple coverings of the sphere by the torus. We obtain corresponding expressions for surfaces of higher genera for small number of ramification points, and conjecture the general form for this number in terms of a symmetric polynomial that appears to be new. The approach involves the analysis of the action of a transposition to derive a system of linear partial differential equations that give the generating series for the desired numbers.

Keywords: ramified covering, Riemann surface, Hurwitz Problem, factorization into transpositions, generating series

1. Introduction

1.1. Background and New Results

Let $f\colon \mathsf{M} \to \mathsf{S}^2$ be a non-constant meromorphic function on a compact connected Riemann surface M of genus $g \geq 0$. Then there exists an integer $n \geq 1$, called the *degree* of f, such that $|f^{-1}(p)| = n$ for all but a finite number of points $p \in \mathsf{S}^2$ called *critical values*; f is called a *ramified n-fold covering* of S^2.

There are two definitions of topological equivalence of ramified coverings. In one of them, two coverings f_1 and f_2 are considered equivalent if there exist homeomorphisms $\pi\colon \mathsf{M} \to \mathsf{M}$ and $\rho\colon \mathsf{S}^2 \to \mathsf{S}^2$ such that $\rho f_1 = f_2\pi$. The equivalence classes under this definition correspond to the connected components of the space of meromorphic functions. For the description and enumeration of such equivalence classes see [14, 20].

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In the second, more restrictive definition, two coverings f_1 and f_2 are considered equivalent if there exists a homeomorphism $\pi\colon \mathsf{M} \to \mathsf{M}$ such that $f_1 = f_2\pi$. In this paper we address the problem of determining the number of equivalence classes of ramified coverings under this more restrictive equivalence. This problem arose originally in the classic paper of Hurwitz [12] and is often called *Hurwitz's Enumeration Problem*.

Recall that a critical value p of f is called *simple* if $|f^{-1}(p)| = n-1$. A covering f is called *simple* if all of its critical values are simple, and *almost simple* if at most one critical value, which is assumed to be equal to ∞, is non-simple. The preimages of ∞ are called the *poles* of f. Let $\alpha_1 \geq \cdots \geq \alpha_m \geq 1$ be the orders of the poles; since $\alpha_1 + \cdots + \alpha_m = n$, we get a partition α of n, which is indicated by $\alpha \vdash n$. The number of parts is denoted by $l(\alpha)$. The specific question addressed in this paper is that of giving an explicit expression for the number $\mu_m^{(g)}(\alpha)$ of almost simple ramified n-fold coverings with a prescribed distribution α of ramification orders. We call α the *ramification type* of f.

A general answer to this question has been given by a number of authors (see, for example, [12, 19]). These answers allow us to determine $\mu_m^{(g)}(\alpha)$ in principle, but do not give explicit information, and are largely intractable because of the character sums they contain. For example, the following generating series (see [8]), in principle, gives complete information about the $\mu_m^{(g)}(\alpha)$. It gives $\mu_m^{(g)}(\alpha)/(n+m+2g-2)!$ as the coefficient of $x^n u^{n+m+2g-2} p_\alpha$ in

$$\log\left(\sum_{n\geq 0}\sum_{\theta\vdash n}\frac{x^n}{n!}\chi^{\theta}_{[1^n]}s_\theta e^{u\eta(\theta)}\right).$$

In this series p_k is the power sum symmetric function of degree k, $p_\alpha = p_{\alpha_1}p_{\alpha_2}\cdots$, χ^θ is the character of the ordinary irreducible representation of $\mathfrak{S}_n$ (the symmetric group on n symbols) indexed by θ,

$$s_\theta = \frac{1}{n!}\sum_{\alpha\vdash n}|\mathcal{C}_\alpha|\chi^\theta_\alpha p_\alpha,$$

the Schur function indexed by θ, $|\mathcal{C}_\alpha|$ is the size of the conjugacy class of $\mathfrak{S}_n$ indexed by α, and

$$\eta(\theta) = \sum_i \binom{\theta_i}{2} - \sum_i \binom{\tilde\theta_i}{2},$$

where $\theta = (\theta_1,\dots)$, $\tilde\theta = (\tilde\theta_1,\dots)$, is the conjugate of θ.

A different approach to this problem comes from the singularity theory and is based on the notion of the Lyashko–Looijenga map (see [1]). This map takes a meromorphic function f to the polynomial whose roots are the critical values of f. It was shown in [17] that, for $\mathsf{M} = \mathsf{S}^2$ and the generic polynomial f, the Lyashko–Looijenga map is a finite covering whose degree coincides with $\mu_1^{(0)}((n))$ up to a factor of n. This approach was extended to generic Laurent polynomials on the sphere in [2], and to generic meromorphic functions on the sphere in [11]. The case of polynomials on S^2 with several non-simple critical values is treated in [25].

A further approach to the problem comes from enumerative algebraic geometry. Recall that the characteristic number $R_d(a,b)$ (with $a+b = 3d-1$) is defined as the number of irreducible degree d rational curves passing through a fixed general points, and tangent to b fixed general lines. Kontsevich and Manin [15, 16] were the first to relate the characteristic numbers (in the case $b = 0$) to Chern classes of the moduli space of stable maps and to give recursions for these numbers. Their results were later extended in [21] to the case of arbitrary b. A further extension to the characteristic numbers $R_d(a,b,c)$ of curves that satisfy the above conditions and, in addition, are tangent to c fixed general lines at fixed general points of tangency can be found in [5]. The same approach allows the finding of recurrence equations of the number $\mu_n^{(g)}(1^n)$ of simple coverings, at least in the cases $g = 0$ and $g = 1$ (see [24]). The case of general almost simple coverings has not yet been studied from this point of view. The first steps in this direction have been carried out in [4], where $\mu_n^{(g)}(\alpha)$ is related to the top Segre class of a certain bundle over the moduli space of stable curves.

It is convenient to rescale $\mu_m^{(g)}(\alpha)$ by writing it in terms of a quantity, $f_m^{(g)}(\alpha)$, which we introduce in this paper, where

$$\mu_m^{(g)}(\alpha) = \frac{1}{n!}|\mathcal{C}_\alpha|\,(n+m+2g-2)!\prod_{j=1}^{m}\frac{\alpha_j^{\alpha_j}}{(\alpha_j-1)!}\,f_m^{(g)}(\alpha). \tag{1.1}$$

Much of this paper is concerned with determining explicit expressions and properties of $f_m^{(g)}(\alpha)$ defined by this rescaling. In terms of this rescaling, the present state of knowledge of $\mu_m^{(g)}(\alpha)$ is summarized as follows:

$$f_m^{(0)}(\alpha) \;=\; n^{m-3}, \quad \alpha \vdash n,\; l(\alpha) = m, \tag{1.2}$$

$$f_1^{(g)}((n)) \;=\; \frac{1}{2^{2g}}n^{2g-2}\left[x^{2g}\right]\left(\frac{\sinh x}{x}\right)^{n-1}, \quad n \geq 1, \tag{1.3}$$

$$f_2^{(1)}((n-r,r)) \;=\; \frac{1}{24}\left(n^2-(r+1)n+r^2\right), \quad r = 1,2,3,\; n-r \neq r, \tag{1.4}$$

where $[x^r]g(x)$ is the coefficient of x^r in $g(x)$.

Although it has a strikingly simple form, result (1.2), which is due to Hurwitz [12] (see also [8, 11, 23]), is a remarkable one. The case $\alpha = [1^n]$ was rediscovered recently by Crescimanno and Taylor [3]. Result (1.3) was obtained by Shapiro, Shapiro and Vainshtein [22], who also showed that it could be obtained from a result of Jackson [13] on ordered factorizations of permutations into transpositions. Result (1.4) was obtained by Shapiro, Shapiro, and Vainshtein [22].

Thus, for the *sphere*, complete information is known; for the *torus* information is known only for ramification types (n) and $(n-r,r)$, $r = 1,2,3$ and $n-r > r$; and for *all surfaces*, explicit information is known only for one ramification point.

In this paper we determine several new explicit results for $\mu_m^{(g)}(\alpha)$. First, we find the number of simple coverings of a sphere by a torus, thus generalizing Hurwitz's result for the sphere.

Theorem 1.1. *For $n \geq 1$,*

$$f_n^{(1)}(1^n) = \frac{1}{24}\left(n^n - n^{n-1} - \sum_{i=2}^{n}\binom{n}{i}(i-2)!\,n^{n-i}\right).$$

Next, we consider almost simple coverings of the sphere by a torus. We prove that result (1.4) remains valid for arbitrary coverings with two ramification points.

Theorem 1.2. *For $n \geq 1$ and $0 < r < n$,*

$$f_2^{(1)}((n-r,r)) = \frac{1}{24}\left(n^2 - (r+1)n + r^2\right).$$

Moreover, we find explicit expressions for the number of almost simple coverings of a sphere by a torus with up to six ramification points (see Appendix). These expressions, together with Theorem 1.1, allow us to conjecture the general explicit result that gives complete information for the torus, and can therefore be regarded as an extension of result (1.2).

Conjecture 1.3. *For $m \geq 1$,*

$$f_m^{(1)}(\alpha) = \frac{1}{24}\left(n^m - n^{m-1} - \sum_{i=2}^{m}(i-2)!e_i n^{m-i}\right), \tag{1.5}$$

where e_i is the ith elementary symmetric function in $\alpha_1, \dots, \alpha_m$, and $e_1 = n = \alpha_1 + \cdots + \alpha_m$.

Furthermore, we determine new explicit results for $\mu_m^{(g)}(\alpha)$ in the cases $m+g \leq 6$, $m,g \geq 2$. These results consist of explicit expressions for the corresponding $f_m^{(g)}(\alpha)$ and are collected together in the Appendix. Each result $f_m^{(g)}(\alpha)$ is a symmetric polynomial in $\alpha_1, \dots, \alpha_m$. We have found that these polynomials are expressed more compactly in terms of the elementary symmetric functions e_k, $k = 1, \dots, m$, and, for this reason, this is the presentation of $f_m^{(g)}(\alpha)$ that is given in the Appendix (cf. Conjecture 1.3). In all cases we also observe that the total degree of $f_m^{(g)}(\alpha)$ is $m+3g-3$ and we therefore make the following conjecture.

Conjecture 1.4. *For $g = 0, m \geq 3$, and $m \geq 1$, $g \geq 1$, $f_m^{(g)}(\alpha_1, \dots, \alpha_m)$ is a symmetric polynomial in $\alpha_1, \dots, \alpha_m$ of total degree $m+3g-3$.*

Conjecture 1.4 is also in agreement with the previously known results (1.2), (1.3), and (1.4). Note that, for $\alpha \vdash n$, we have $n = e_1$, so Hurwitz's result (1.2) is rewritten in these terms as $f_m^{(0)} = e_1^{m-3}$, which is a (symmetric) polynomial for $m \geq 3$. For result (1.3), we have

$$\begin{aligned}
\left[x^{2g}\right]\left(\frac{\sinh x}{x}\right)^{n-1} &= [t^g]\left(\sum_{i \geq 0}\frac{t^i}{(2i+1)!}\right)^{n-1} \\
&= \sum_{\substack{i_1,\dots,i_g \geq 0 \\ i_1+2i_2+\cdots+gi_g = g}} (n-1)_{i_1+\cdots+i_g}\prod_{j=1}^{g}\frac{1}{i_j!(2j+1)!^{i_j}},
\end{aligned}$$

where $(u)_i = u(u-1)\cdots(u-i+1)$ for a non-negative integer i. Thus, $f_1^{(g)}$ is a polynomial of degree $3g-2$ in $n = e_1$, in agreement with Conjecture 1.4 for $g \geq 1$.

Although we have been unable to use this approach to prove Conjectures 1.3 and 1.4, we believe that the symmetric functions $f_m^{(g)}$ are of considerable interest and would reward further study. Explicit expressions for a few of them are given in the Appendix in terms of the elementary symmetric functions. It may be that there is a more natural basis for the $f_m^{(g)}$, but we have been unable to determine it.

1.2. New Methods and the Organization of the Paper

Our approach uses Hurwitz reduction of the question to the ordered factorization of permutations in $\mathfrak{S}_n$ into transpositions that generate $\mathfrak{S}_n$. We obtain a differential equation for the generating series by an analysis that lies at the center of Hurwitz's approach. Our point of departure from Hurwitz's approach is to show that the symmetrization of this series satisfies a differential equation in a new set of variables, and that its solution is a rational function in the transformed variables. This enables us not only to obtain additional explicit results that extend what is currently known but also to make conjectures about the general form of $\mu_m^{(g)}(\alpha)$.

In Section 2 the determination of $\mu_m^{(g)}(\alpha)$ is expressed in terms of ordered factorizations of permutations in $\mathfrak{S}_n$ into transpositions such that the transpositions generate $\mathfrak{S}_n$. A differential equation for the generating series for the number $c_g(\alpha)$ of such factorizations is given in Section 3 by combinatorially analyzing the action of transpositions. Section 4 gives a partial differential equation induced by symmetrizing the portion of the generating series relating to given genus g and a prescribed number m of ramification points. A general form for the solution of this system is conjectured in Section 5. Section 6 gives brief details about the computation of the results in the Appendix and a proof of Theorem 1.2. Section 7 gives a proof of Theorem 1.1 and makes use of one of the new variables that was introduced in Section 4.

2. Transitive Ordered Factorizations

The question of determining the number $\mu_m^{(g)}(\alpha)$ of almost simple ramified n-fold coverings, by a surface of genus g, with a prescribed ramification type α, can be translated into one concerning products of transpositions, and the construction is given by Hurwitz [12]. Let $c_g(\alpha)$ be the number of j-tuples of permutations $(\sigma_1,\dots,\sigma_j)$ such that, for an arbitrary but fixed $\pi\in\mathcal{C}_\alpha$, the conjugacy class of $\mathfrak{S}_n$ indexed by $\alpha\vdash n$, the following conditions hold:

(1) $\pi=\sigma_1\cdots\sigma_j$;
(2) $\sigma_1,\dots,\sigma_j\in\mathcal{C}_{[2\,1^{n-2}]}$;
(3) $\sigma_1,\dots,\sigma_j$ generate $\mathfrak{S}_n$;
(4) $j=\mu(\alpha)+2g$, where $\mu(\alpha)=n+l(\alpha)-2$.

We note that $\mu(\alpha)$ is the minimum value of j for which conditions (1)–(3) hold. We call $(\sigma_1,\dots,\sigma_r)$ an *ordered transitive factorization* of π into transpositions, which is said to be *minimal* if $g=0$. The numbers $\mu_m^{(g)}(\alpha)$ and $c_g(\alpha)$ are related by

$$\mu_m^{(g)}(\alpha)=\frac{1}{n!}|\mathcal{C}_\alpha|c_g(\alpha). \tag{2.1}$$

Condition (1) ensures that the ramification type is $\alpha = [1^{a_1}, 2^{a_2}, \ldots]$, where a_k is the number of ramification points of order k. Condition (2) ensures that the ramification points are simple. Condition (3) ensures that the surface is connected. Condition (4) ensures that the sphere is covered by a surface of genus g.

It is this reformulation of the original question that we now consider in detail. Goulden [6] showed that such problems can be considered by analyzing the combinatorial action of a transposition on a permutation through the use of differential operators acting on monomials that encode the cycle structure of a permutation. These techniques were extended in [7] to factorizations into cycles of length k alone. We refer to this technique as the *cut-and-join* analysis of the action of a k-cycle. The cut-and-join analysis is applied to a graphical encoding of transitive ordered factorizations as connected edge-labeled and vertex-labeled graphs. These appear in the work of Arnold [2], and independently in [6, 7], and are called the *monodromy graph* of the ordered factorization. For the question addressed in this paper, it is necessary to relax the condition of minimality in the cut-and-join analysis.

3. A Partial Differential Equation

Let $p_1, p_2, \ldots$ be indeterminates and $\mathbf{p} = (p_1, p_2, \ldots)$. For $\alpha = (\alpha_1, \ldots, \alpha_m)$ where $\alpha_1, \ldots, \alpha_m$ are positive integers, let $p_\alpha = p_{\alpha_1} \cdots p_{\alpha_m}$. Now consider the generating series

$$\Phi(u, z, \mathbf{p}) = \sum_{\substack{n,m\geq 1\\ g\geq 0}} \sum_{\substack{\alpha\vdash n\\ l(\alpha)=m}} \frac{|\mathcal{C}_\alpha|}{n!} c_g(\alpha) \frac{u^{n+m+2(g-1)}}{(n+m+2(g-1))!} z^g p_\alpha \tag{3.1}$$

for ordered transitive factorizations. In Lemma 3.1, we show that Φ satisfies a second order partial differential equation of the second degree.

The core of the derivation is a cut-and-join analysis. Suppose $\sigma \in \mathfrak{S}_n$ is the transposition that interchanges s and t, and that $\pi \in \mathfrak{S}_n$. Then there are two cases for the action of σ on π in the product $\pi\sigma$: (i) if s, t are on the same cycle of π, then the cycle is cut into two cycles in $\pi\sigma$, with s, t on different cycles (here σ is called a *cut* for π); (ii) if s, t are on different cycles of π, then these cycles are joined to form a single cycle in $\pi\sigma$, which contains both s and t (here σ is called a *join* for π). This type of analysis has been previously used in [7] and [8], to obtain partial differential equations for other generating series for ordered factorizations into transpositions. In the proof of Lemma 3.1, some of the details are suppressed because of the similarity to this previous work.

Lemma 3.1.

$$\frac{\partial\Phi}{\partial u} = \frac{1}{2}\sum_{i,j\geq 1}\left(ij p_{i+j} z \frac{\partial^2\Phi}{\partial p_i \partial p_j} + ij\, p_{i+j} \frac{\partial\Phi}{\partial p_i}\frac{\partial\Phi}{\partial p_j} + (i+j) p_i p_j \frac{\partial\Phi}{\partial p_{i+j}}\right). \tag{3.2}$$

Proof. Consider an ordered transitive factorization $(\sigma_1, \ldots, \sigma_r)$ of π that satisfies conditions (1)–(4), for any $n \geq 1$. Remove σ_r but leave π unchanged. This is an ordered transitive factorization of π with its rightmost factor deleted. Then the generating series for the set of such objects is

$$\frac{\partial \Phi}{\partial u},$$

since Φ is an exponential generating series in u. This gives the left-hand side of the partial differential equation.

To obtain the right-hand side of the partial differential equation, we now show that ordered transitive factorizations of π with the rightmost factor deleted can be determined in another way. Let $G = G_{(\sigma_1,\ldots,\sigma_r)}$ be the monodromy graph of $(\sigma_1,\ldots,\sigma_r)$: the graph with vertex labels $\{1,\ldots,n\}$, and edge labels $\{1,\ldots,r\}$, in which the edge labeled i joins vertices interchanged by σ_i. Then condition (3) implies that G is a connected graph. Let T be the spanning tree of G constructed by considering the edges in increasing order of their labels, and selecting the edge labelled i if and only if its incident vertices are in different components of $G_{(\sigma_1,\ldots,\sigma_{i-1})}$ (this is Kruskal's Algorithm). Then, by construction, the transpositions corresponding to the edges of T are joins. Let $\sigma_r = (s,t)$. There are three cases.

(i) σ_r corresponds to an edge of G not in T, and σ_r is a cut for $\pi\sigma_r$. Thus $(\sigma_1,\ldots,\sigma_{r-1})$ is an ordered transitive factorization of $\pi\sigma_r$, with s and t on the same cycle of $\pi\sigma_r$. The contribution from this case is

$$\frac{1}{2}\sum_{i,j\geq 1}(i+j)p_i p_j \frac{\partial \Phi}{\partial p_{i+j}},$$

since the effect of multiplying by σ_r is to cut a $i+j$-cycle into an i-cycle and a j-cycle.

(ii) σ_r corresponds to an edge of G not in T, and σ_r is a join for $\pi\sigma_r$. Thus $G_{(\sigma_1,\ldots,\sigma_{r-1})}$ is an ordered transitive factorization of $\pi\sigma_r$ with s and t on different cycles of $\pi\sigma_r$. The contribution from this case is

$$\frac{1}{2}\sum_{i,j\geq 1} ij p_{i+j} z \frac{\partial^2 \Phi}{\partial p_i \partial p_j},$$

since the effect of multiplying by σ_r is to join an i-cycle and a j-cycle to create an $i+j$-cycle.

(iii) σ_r corresponds to an edge of T. Then σ_r corresponds to a join for $\pi\sigma_r$ and $G_{(\sigma_1,\ldots,\sigma_{r-1})}$ has exactly two components, with s and t on different components. The contribution from this case is

$$\frac{1}{2}\sum_{i,j\geq 1} ij p_{i+j} \frac{\partial \Phi}{\partial p_i}\frac{\partial \Phi}{\partial p_j},$$

where here the i-cycle and j-cycle that have been joined come from each of two ordered transitive factorizations (one for each component). Note that Φ is an exponential generating series in both u and vertices, the latter being indicated by the division by $n!$ in the definition of Φ. So the product $ij(\partial\Phi/\partial p_i)(\partial\Phi/\partial p_j)$ gives the correct cardinalities through the product of exponential generating series. The result follows by combining these cases and equating the two expressions. ∎

In [8] the special case of Φ corresponding to minimal ordered transitive factorizations was considered. This is obtained by considering only $g = 0$ or, equivalently, by setting $z = 0$ in Φ. The partial differential equation obtained by setting $z = 0$ in (3.2) was derived in [8] by a cut-and-join analysis, and an explicit solution was obtained. Although we have been unable to obtain explicit solutions to (3.2) itself, we have been able to obtain results for various positive values of g by considering a symmetrized form of (3.2), and the rest of the paper is devoted to this analysis.

4. The Symmetrical Form of Φ

We begin by defining a symmetrical form of Φ. For this purpose, let $\mathbf{x}^\alpha = x_1^{\alpha_1}\cdots x_m^{\alpha_m}$ and let $\omega\alpha = (\alpha_{\omega(1)},\dots,\alpha_{\omega(m)})$ where $\omega \in \mathfrak{S}_m$. Let

$$\Xi^{\{1,\dots,m\}} p_\alpha = \begin{cases} \sum_{\pi\in\mathfrak{S}_m} \mathbf{x}^{\pi(\alpha)}, & \text{if } l(\alpha) = m, \\ 0, & \text{otherwise,} \end{cases}$$

extended linearly to all series in the p_i's. Let

$$\Psi_m^{(g)}(\mathbf{x}) = [z^g]\Xi^{\{1,\dots,m\}}\Phi(u,z,\mathbf{p})\Big|_{u=1}. \tag{4.1}$$

Thus $\Psi_m^{(g)}(\mathbf{x})$ is obtained by taking the term of degree g in z and total degree m in the p_i's from Φ, then setting $u = 1$ and symmetrizing the monomials in the p_i's, replacing the subscripts of p_i's by exponents of x's when we do so. Of course, $c_g(\alpha)$ is recoverable from this series in a straightforward way.

In this section we determine a partial differential equation for $\Psi_m^{(g)}(\mathbf{x})$ that is induced by the partial differential equation for Φ, given in Lemma 3.1. It is convenient to express the equations in terms of $x_1,\dots,x_m$ and, in addition, $w_1,\dots,w_m$, where

$$w_i = x_i e^{w_i} \tag{4.2}$$

has a unique solution $w_i \equiv w_i(x_i)$ as a power series in x_i. Then $w_1,\dots,x_1,\dots$ are algebraically independent, and it is clear, by differentiating the functional Equation (4.2), that

$$x_i\frac{\partial}{\partial x_i} = \frac{w_i}{1-w_i}\frac{\partial}{\partial w_i}. \tag{4.3}$$

The reason for introducing the w_i's is that in [7] and [8] we showed that Hurwitz's result (1.2) for $g = 0$ can be expressed as

$$\Psi_m^{(0)}(\mathbf{x}) = \left(x_1\frac{\partial}{\partial x_1} + \cdots + x_m\frac{\partial}{\partial x_m}\right)^{m-3} V_m(\mathbf{w}) \tag{4.4}$$

in terms of the new indeterminates, where

$$V_m(\mathbf{w}) = \prod_{i=1}^m \frac{w_i}{1-w_i} \tag{4.5}$$

and $\mathbf{w} = (w_1, \ldots, w_m)$. This expression will be used as an initial condition in the partial differential equation for $\Psi_m^{(g)}(\mathbf{x})$.

The following mapping is needed in the statement of the differential equation, for expressing the action of $\Xi^{\{1,\ldots,m\}}$ on terms involving the p_i's. Let f be a series in $x_1, \ldots, x_m$, and $0 \le i \le m-1$. Then the mapping Θ_i is defined by

$$\Theta_i f(x_1, \ldots, x_m) = \sum_{\mathcal{R}, \mathcal{S}, \mathcal{T}} f(\mathbf{x}_{\mathcal{R}}, \mathbf{x}_{\mathcal{S}}, \mathbf{x}_{\mathcal{T}}),$$

where the sum is over all ordered partitions $(\mathcal{R}, \mathcal{S}, \mathcal{T})$ of $\{1, \ldots, m\}$ with $|\mathcal{R}| = 1$, $|\mathcal{S}| = i$, $|\mathcal{T}| = m - i - 1$, and where

$$(\mathbf{x}_{\mathcal{R}}, \mathbf{x}_{\mathcal{S}}, \mathbf{x}_{\mathcal{T}}) = (x_{r_1}, x_{s_1}, \ldots, x_{s_i}, x_{t_1}, \ldots, x_{t_{m-i-1}}),$$

in which $s_1 < \cdots < s_i$, and $t_1 < \cdots < t_{m-i-1}$.

The next three lemmas are quite technical, which will be used for determining the action of $\Xi^{\{1,\ldots,m\}}$ on products of the p_i's. For this purpose, let

$$\Xi^{\{a_1,\ldots,a_m\}} p_\alpha = \Xi^{\{1,\ldots,m\}} p_\alpha \Big|_{x_i \mapsto x_{a_i},\, i=1,\ldots,m},$$

where $a_1 < \cdots < a_m$.

Lemma 4.1. *Let* α, β *be partitions with* $l(\alpha) = k$ *and* $l(\beta) = m$. *Then*

$$\Xi^{\{1,\ldots,m+k\}} p_\alpha p_\beta = \sum_{(\mathcal{A},\mathcal{B})} \left(\Xi^{\mathcal{A}} p_\alpha\right)\left(\Xi^{\mathcal{B}} p_\beta\right),$$

where the sum is over all ordered partitions $(\mathcal{A},\mathcal{B})$ *of* $\{1, \ldots, m+k\}$ *with* $|\mathcal{A}| = k$ *and* $|\mathcal{B}| = m$.

Proof. Immediate. ∎

Lemma 4.2. *Let* α *be a partition,* $l(\alpha) = m$, *and* $1 \le l \le m$. *Then*

$$\sum_{i \ge 1} x_l^i \Xi^{\{1,\ldots,m\}-\{l\}} i \frac{\partial p_\alpha}{\partial p_i} = x_l \frac{\partial}{\partial x_l} \Xi^{\{1,\ldots,m\}} p_\alpha.$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$. It is convenient to regard p_α as the word $p_{\alpha_1} \cdots p_{\alpha_m}$ in which each symbol is written with exponent equal to one. Now apply the operator $\sum_{i\ge1} x_l^i i\partial/\partial p_i$ to this product of m p_i's. Its effect, by the product rule, is to create a sum of m terms, in which each of the m p_j's in the word p_α is replaced, in turn, by jx_l^j. But $jx_l^j = x_l x_l^j \partial/\partial x_l$, and the result follows immediately. ∎

Lemma 4.3. *Let* α *be a partition,* $l(\alpha) = m+1$, *and* $1 \le l \le m$. *Then*

$$\sum_{i,j\ge1} x_l^{i+j} \Xi^{\{1,\ldots,m+1\}-\{l,m+1\}} ij \frac{\partial^2}{\partial p_i \partial p_j} p_\alpha = x_l \frac{\partial}{\partial x_l} x_{m+1} \frac{\partial}{\partial x_{m+1}} \Xi^{\{1,\ldots,m+1\}} p_\alpha \Big|_{x_{m+1}=x_l}.$$

Proof. Similar to the proof of Lemma 4.2. ∎

The partial differential equation for the symmetrized form of Φ is given in the following result.

Theorem 4.4. *The series $\Psi_m^{(g)}$, for $g, m = 1, 2, \dots$, satisfy the partial differential equation*

$$\left(w_1\frac{\partial}{\partial w_1} + \cdots + w_m\frac{\partial}{\partial w_m} + m + 2(g-1)\right)\Psi_m^{(g)}(x_1,\dots,x_m) = T_1 + \cdots + T_4, \quad (4.6)$$

where

$$\begin{aligned}
T_1 &= \frac{1}{2}\sum_{i=1}^{m}\left(x_i\frac{\partial}{\partial x_i}x_{m+1}\frac{\partial}{\partial x_{m+1}}\Psi_{m+1}^{(g-1)}(x_1,\dots,x_{m+1})\right)\Bigg|_{x_{m+1}=x_i},\\
T_2 &= \Theta_1\frac{w_2}{1-w_1}\frac{1}{w_1-w_2}x_1\frac{\partial}{\partial x_1}\Psi_{m-1}^{(g)}(x_1,x_3,\dots,x_m),\\
T_3 &= \sum_{k=3}^{m}\Theta_{k-1}\left(x_1\frac{\partial}{\partial x_1}\Psi_k^{(0)}(x_1,\dots,x_k)\right)\left(x_1\frac{\partial}{\partial x_1}\Psi_{m-k+1}^{(g)}(x_1,x_{k+1},\dots,x_m)\right),\\
T_4 &= \frac{1}{2}\sum_{\substack{1\le k\le m\\ 1\le a\le g-1}}\Theta_{k-1}\left(x_1\frac{\partial}{\partial x_1}\Psi_k^{(a)}(x_1,\dots,x_k)\right)\left(x_1\frac{\partial}{\partial x_1}\Psi_{m-k+1}^{(g-a)}(x_1,x_{k+1},\dots,x_m)\right)
\end{aligned}$$

with initial condition (4.4), where $\Psi_0^{(g)} = 0$ for $g = 1, 2, \dots$.

Proof. Let E_1 denote the left-hand side of Equation (3.2), and let E_2, E_3, E_4 denote the three summations on the right-hand side (indexing them from left to right). We apply $[z^g]\Xi^{\{1,\dots,m\}}$, with $u = 1$, to each of these in turn, and then denote the results of this by U_2, U_3, and U_4, respectively.

For E_1: We consider, with $l(\alpha) = m, \alpha \vdash n$,

$$\begin{aligned}
\Xi^{\{1,\dots,m\}}(n+m+2g-2)p_\alpha &= \left(\sum_{j=1}^{n}\alpha_j + m + 2g - 2\right)\Xi^{\{1,\dots,m\}}p_\alpha\\
&= \left(\sum_{i=1}^{m}x_i\frac{\partial}{\partial x_i} + m + 2g - 2\right)\Xi^{\{1,\dots,m\}}p_\alpha,
\end{aligned}$$

so

$$[z^g]\Xi^{\{1,\dots,m\}}E_1\Big|_{u=1} = \left(x_1\frac{\partial}{\partial x_1} + \cdots + x_m\frac{\partial}{\partial x_m} + m + 2g - 2\right)\Psi_m^{(g)}(x_1,\dots,x_m).$$

For E_2: We consider, with $l(\alpha) = m+1$,

$$\begin{aligned}
\Xi^{\{1,\dots,m\}} \sum_{i,j\geq 1} p_{i+j} ij \frac{\partial^2 p_\alpha}{\partial p_i \partial p_j} &= \sum_{l=1}^{m} \sum_{i,j\geq 1} x_l^{i+j} \Xi^{\{1,\dots,m\}-\{l\}} ij \frac{\partial^2 p_\alpha}{\partial p_i \partial p_j}, \\
&= \sum_{l=1}^{m} \sum_{i,j\geq 1} x_l^{i+j} \Xi^{\{1,\dots,m+1\}-\{l,m+1\}} ij \frac{\partial^2 p_\alpha}{\partial p_i \partial p_j}, \\
&= \sum_{l=1}^{m} \left(x_l \frac{\partial}{\partial x_l} x_{m+1} \frac{\partial}{\partial x_{m+1}} \Xi^{\{1,\dots,m+1\}} p_\alpha \right) \Bigg|_{x_{m+1}=x_l}. \\
&\quad \text{(by Lemma 4.3)}
\end{aligned}$$

Thus

$$U_2 = \frac{1}{2} \sum_{l=1}^{m} \left(x_l \frac{\partial}{\partial x_l} x_{m+1} \frac{\partial}{\partial x_{m+1}} \Psi_{m+1}^{(g-1)}(x_1, \dots, x_{m+1}) \right) \Bigg|_{x_{m+1}=x_l}.$$

For E_3: We consider, with $l(\alpha) = k, l(\beta) = m-k+1$,

$$\begin{aligned}
\Xi^{\{1,\dots,m\}} \sum_{i,j\geq 1} p_{i+j} ij \frac{\partial p_\alpha}{\partial p_i} \frac{\partial p_\beta}{\partial p_j} &= \sum_{l=1}^{m} \sum_{i,j\geq 1} x_l^{i+j} \Xi^{\{1,\dots,m\}-\{l\}} \left(i \frac{\partial p_\alpha}{\partial p_i} \right) \left(j \frac{\partial p_\beta}{\partial p_j} \right) \\
&= \sum_{l=1}^{m} \sum_{\substack{\mathcal{A}\cup\mathcal{B}=\{1,\dots,m\}-\{l\} \\ |\mathcal{A}|=k-1, |\mathcal{B}|=m-k \\ \mathcal{A}\cap\mathcal{B}=\emptyset}} \left(\sum_{i\geq 1} x_l^i \Xi^{\mathcal{A}}\, i \frac{\partial p_\alpha}{\partial p_i} \right) \left(\sum_{j\geq 1} x_l^j \Xi^{\mathcal{B}}\, j \frac{\partial p_\beta}{\partial p_j} \right) \\
&\quad \text{(by Lemma 4.1)} \\
&= \sum_{l=1}^{m} \sum_{\substack{\mathcal{A}\cup\mathcal{B}=\{1,\dots,m\}-\{l\} \\ |\mathcal{A}|=k-1, |\mathcal{B}|=m-k \\ \mathcal{A}\cap\mathcal{B}=\emptyset}} \left(x_l \frac{\partial}{\partial x_l} \Xi^{\mathcal{A}\cup\{l\}} p_\alpha \right) \left(x_l \frac{\partial}{\partial x_l} \Xi^{\mathcal{B}\cup\{l\}} p_\beta \right) \\
&\quad \text{(by Lemma 4.2)} \\
&= \Theta_{k-1} \left(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1,\dots,k\}} p_\alpha \right) \left(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1,k+1,\dots,m\}} p_\beta \right),
\end{aligned}$$

whence

$$U_3 = \frac{1}{2} \sum_{\substack{1\leq k\leq m \\ 0\leq a\leq g}} \Theta_{k-1} \left(x_1 \frac{\partial}{\partial x_1} \Psi_k^{(a)}(x_1, \dots, x_k) \right) \left(x_1 \frac{\partial}{\partial x_1} \Psi_{m-k+1}^{(g-a)}(x_1, x_{k+1}, \dots, x_m) \right).$$

For E_4: We consider, with $l(\alpha)=m-1$,

$$\begin{aligned}
\Xi^{\{1,\dots,m\}} \sum_{i,j\geq 1}(i+j)p_ip_j\frac{\partial p_\alpha}{\partial p_{i+j}} &= 2\sum_{1\leq l<k\leq m}\sum_{i,j\geq 1} x_l^i x_k^j \Xi^{\{1,\dots,m\}-\{l,k\}}(i+j)\frac{\partial p_\alpha}{\partial p_{i+j}}\\
&= 2\sum_{1\leq l<k\leq m}\sum_{r\geq 1}\frac{x_kx_l^r-x_lx_k^r}{x_k-x_l}\Xi^{\{1,\dots,m\}-\{l,k\}} r\frac{\partial p_\alpha}{\partial p_r}\\
&= 2\sum_{1\leq l\neq k\leq m}\frac{x_k}{x_l-x_k}x_l\frac{\partial}{\partial x_l}\Xi^{\{1,\dots,m\}-\{k\}}p_\alpha\\
&\quad \text{(by Lemma 4.2)}\\
&= 2\Theta_1\frac{x_2}{x_1-x_2}x_1\frac{\partial}{\partial x_1}\Xi^{\{1,3,4\dots,m\}}p_\alpha.
\end{aligned}$$

Thus

$$U_4=\Theta_1\frac{x_2}{x_1-x_2}x_1\frac{\partial}{\partial x_1}\Psi^{(g)}_{m-1}(x_1,x_3,x_4,\dots,x_m).$$

Collecting these cases and combining them, we have

$$\left(\sum_{i=1}^m x_i\frac{\partial}{\partial x_i}+m+2(g-1)\right)\Psi^{(g)}_m(x_1,\dots,x_m)=U_2+U_3+U_4. \tag{4.7}$$

Now U_2 gives T_1 in (4.6). Also, from (4.4), $(x_1(\partial/\partial x_1))^2\,\Psi_1^{(0)}=w_1/(1-w_1)$. So from (4.3), we obtain

$$x_1\frac{\partial}{\partial x_1}\Psi_1^{(0)}(x_1)=w_1.$$

This allows us to simplify the terms with $a=0$, $k=1$, and $a=g$, $k=m$ in U_3, giving

$$\sum_{i=1}^m w_ix_i\frac{\partial}{\partial x_i}\Psi^{(g)}_m(x_1,\dots,x_m).$$

This summation is then moved to the left-hand side of Equation (4.7), and the left-hand side of Equation (4.6) follows immediately from (4.3).

Also, from (4.4), $(x_1(\partial/\partial x_1)+x_2(\partial/\partial x_2))\,\Psi_2^{(0)}=w_1w_2/(1-w_1)(1-w_2)$, and it can be verified that $\Psi_2^{(0)}(x_1,x_2)=\log((w_1-w_2)/(x_1-x_2))-w_1-w_2$, whence

$$\left(x_1\frac{\partial}{\partial x_1}\right)\Psi_2^{(0)}(x_1,x_2)=\frac{w_2}{(1-w_1)(w_1-w_2)}-\frac{x_2}{x_1-x_2}. \tag{4.8}$$

This allows us to simplify the terms with $a=0$, $k=2$, and $a=g$, $k=m-1$ in U_3, giving

$$\Theta_1\left(\frac{w_2}{(1-w_1)(w_1-w_2)}-\frac{x_2}{x_1-x_2}\right)x_1\frac{\partial}{\partial x_1}\Psi^{(g)}_{m-1}(x_1,x_3,\dots,x_m).$$

This expression is then combined with U_4 to give T_2 in (4.6). The terms with $a=0$, $k=3,\dots,m$ and $a=g$, $k=1,\dots,m-2$ in U_3 combine to give T_3 in (4.6). Finally, the remaining terms in U_3 give T_4 in (4.6), and the result follows from (4.7). ∎

5. Determining the Symmetric Generating Series

We now consider how to use the transformed partial differential Equation (4.6) to determine $\Psi_m^{(g)}$. We begin with the following result that establishes in general that this series is a rational series in $w_1, \ldots, w_m$.

Theorem 5.1. *For $g \geq 1$ and for $g = 0, m \geq 3$, $\Psi_m^{(g)}$ is a symmetric polynomial in $(1-w_1)^{-1}, \ldots, (1-w_m)^{-1}$ of total degree less than or equal to $2m+6g-5$.*

Proof. The proof is by induction. The result is true for $g = 0, m \geq 3$, from the initial condition (4.4), using (4.2) to express $\Psi_m^{(0)}$ in terms of the w_i's for $i = 1, \ldots, m$, where $m \geq 3$.

Now, for $g = m = 1$, the right-hand side of Equation (4.6) comes from T_1 alone, and is given by

$$\lim_{x_2 \to x_1} \frac{1}{2} x_1 \frac{\partial}{\partial x_1} x_2 \frac{\partial}{\partial x_2} \Psi_2^{(0)}(x_1, x_2),$$

which can be evaluated straightforwardly from (4.8), to give

$$\frac{1}{24} w_1^2 \frac{w_1^2 - 4w_1 + 6}{(1-w_1)^4}.$$

Thus, for $g = m = 1$, Equation (4.6) contains only rational functions of w_1, and the result is true for $g = m = 1$.

To prove the result for $g \geq 1, m \geq 1$, $(g, m) \neq (1, 1)$, note that Equation (4.6) contains no explicit occurrences of the x_i's except through the operator $x_i \partial/\partial x_i$, and that (4.3) allows us to replace these by $(w_i/(1-w_i))\partial/\partial w_i$, for $i = 1, \ldots, m$. Moreover, for these values of g, m, the right-hand side of (4.6) does not contain $\Psi_1^{(0)}$ or $\Psi_2^{(0)}$. Thus, if the induction hypothesis is that the result is true for $\Psi_j^{(i)}$, where $0 \leq i \leq g$, and $0 \leq j \leq m+g-i$, with $(i, j) \neq (0, 1), (0, 2), (g, m)$, then Equation (4.6) is of the form

$$\left(w_1 \frac{\partial}{\partial w_1} + \cdots + w_m \frac{\partial}{\partial w_m} + m + 2(g-1) \right) \Psi_m^{(g)}(x_1, \ldots, x_m) = K_m^{(g)}(\mathbf{w}), \qquad (5.1)$$

where $K_m^{(g)}(\mathbf{w})$ depends only on those $\Psi_j^{(i)}$ to which we have applied the induction hypothesis. Thus $K_m^{(g)}(\mathbf{w})$ is a rational function of the w_i's and $w_1 - w_2$ perfectly divides by symmetry, so $K_m^{(g)}(\mathbf{w})$ is a polynomial in $(1-w_1)^{-1}, \ldots, (1-w_m)^{-1}$. Moreover, from the induction hypothesis, we can easily bound the total degree of $K_m^{(g)}(\mathbf{w}) = T_1 + \cdots + T_4$. First note that $x_i \partial/\partial x_i$ increases the degree of a power of $(1-w_i)^{-1}$ by 2. Thus, from the induction hypothesis, the terms arising from T_1 have total degree less than or equal to $2m+6g-5$, and the terms arising from T_2, T_3, and T_4 have total degree less than or equal to $2m+6g-4$.

But $w_i \partial/\partial w_i$ on the left-hand side of (4.6) increases the degree of a power of $(1-w_i)^{-1}$ by 1. Thus, by induction, the result follows that $\Psi_m^{(g)}$ has total degree less that or equal to $2m+6g-5$, for $g \geq 2$ and for $g = 1, m \geq 2$. The proof is complete. ∎

Now we return to the question of determining $\Psi_m^{(g)}$ from Equation (4.6). If we determine the $\Psi_m^{(g)}$ in the order specified in the induction above, Equation (4.6) becomes (5.1), where $K_m^{(g)}(\mathbf{w})$ is a known rational function of $\mathbf{w}$, and this partial differential equation is trivial, as we show next.

Theorem 5.2. *In the notation of (5.1),*

$$\Psi_m^{(g)} = \int_0^1 K_m^{(g)}(t\mathbf{w})t^{m+2g-3}dt.$$

Proof. In Equation (5.1), substitute tw_i for w_i for $i = 1,\dots,m$, so $\sum_{i=1}^m w_i\partial/\partial w_i$ becomes td/dt. Now multiply both sides by t^{m+2g-3}, giving

$$\frac{d}{dt}\left(t^{m+2g-2}\Psi_m^{(g)}\Big|_{\mathbf{w}\mapsto t\mathbf{w}}\right) = t^{m+2g-3}K_m^{(g)}(t\mathbf{w}),$$

and the result follows. ∎

In principle, Theorem 5.2 provides an iterative procedure for explicitly determining the $\Psi_m^{(g)}$. We have used Maple in this way to determine $\Psi_m^{(g)}$ for $g = 1$, $m = 1,\dots,6$ and $g = 2,\dots,5$, $m = 1,\dots,6-g$. Because of certain technical issues that arise in using Maple for this purpose, we in fact modified the procedure in practice, and sketch the details of this in Section 6.

In each case, where we have made computations, $\Psi_m^{(g)}$ can be written in the form

$$\Psi_m^{(g)}(\mathbf{x}) = f_m^{(g)}\left(x_1\frac{\partial}{\partial x_1},\dots,x_m\frac{\partial}{\partial x_m}\right)V_m(\mathbf{w}),$$

where $f_m^{(g)}(x_1\partial/\partial x_1,\dots,x_m\partial/\partial x_m)$ is a symmetric polynomial in $x_1\partial/\partial x_1,\dots,x_m\partial/\partial x_m$. We have found that these polynomials are expressed most compactly in terms of the elementary symmetric functions $e_k \equiv e_k(x_1\partial/\partial x_1,\dots,x_m\partial/\partial x_m)$, for $k = 1,\dots,m$ and, for this reason, the Appendix gives the values of the $f_m^{(g)}$, that are defined by this process, as polynomials in $e_1,\dots,e_m$. We conjecture next that this is always the case, and that the total degree of the polynomial is $m+3g-3$ in general, as can be observed in all cases in the Appendix.

Conjecture 5.3. *For $g = 0$, $m \ge 3$ and for $g \ge 1$, $m \ge 1$,*

$$\Psi_m^{(g)}(\mathbf{x}) = f_m^{(g)}\left(x_1\frac{\partial}{\partial x_1},\dots,x_m\frac{\partial}{\partial x_m}\right)V_m(\mathbf{w}), \tag{5.2}$$

for a unique symmetric polynomial $f_m^{(g)}(x_1,\dots,x_m)$ of total degree $m+3g-3$ in $x_1,\dots,$ x_m.

Conjecture 5.3 is also in agreement with the previous known results, as given in Subsection 1.1. For example, Hurwitz's result (1.2) for $g = 0$, as restated in (4.4), gives $f_m^{(0)} = e_1^{m-3}$ for $m \ge 3$. The form of Conjecture 5.3 might seem surprising, compared with the form of the solution established in Theorem 5.1. However, as we discuss in Section 6, Conjecture 5.3 is only a slight strengthening of Theorem 5.1.

Conjecture 5.3 is also especially useful for determining the ramification numbers $\mu_m^{(g)}(\alpha)$, since, as we show in the next result, $f_m^{(g)}$ evaluated at the argument α actually gives $\mu_m^{(g)}(\alpha)$ up to a known scaling factor.

Lemma 5.4. *If (5.2) holds, then for* $l(\alpha) = m$, $\alpha = (\alpha_1, \ldots, \alpha_m)$,

$$\mu_m^{(g)}(\alpha) = \frac{1}{n!}|\mathcal{C}_\alpha|\,(n+m+2g-2)!\left(\prod_{j=1}^{m} \frac{\alpha_j^{\alpha_j}}{(\alpha_j-1)!}\right) f_m^{(g)}(\alpha).$$

Proof. The solution of (4.2) is

$$w_r = \sum_{j\geq 1} \frac{j^{j-1}}{j!} x_r^j,$$

by Lagrange's Theorem. So from (4.5),

$$[x_r^{\alpha_r}]\left(x_r \frac{\partial}{\partial x_r}\right)^{k_r} \frac{w_r}{1-w_r} = [x_r^{\alpha_r}]\left(x_r \frac{\partial}{\partial x_r}\right)^{k_r+1} w_r = \frac{\alpha_r^{\alpha_r+k_r}}{\alpha_r!}.$$

Thus, if $\alpha = (\alpha_1, \ldots, \alpha_m)$, then

$$[\mathbf{x}^\alpha]\prod_{r=1}^{m}\left(x_r \frac{\partial}{\partial x_r}\right)^{k_r} V_m(\mathbf{w}) = \prod_{r=1}^{m} \alpha_r^{k_r} \frac{\alpha^\alpha}{\alpha!}.$$

So by linearity, we have

$$[\mathbf{x}^\alpha]\ f_m^{(g)}\left(x_1 \frac{\partial}{\partial x_1}, \ldots, x_m \frac{\partial}{\partial x_m}\right) V_m(\mathbf{w}) = f_m^{(g)}(\alpha)\frac{\alpha^\alpha}{\alpha!}. \tag{5.3}$$

Now we look at $\Psi_m^{(g)}(\mathbf{x})$ as defined in (4.1) and (3.1). First, by counting how often each monomial appears, we have

$$\frac{|\mathcal{C}_\alpha|}{n!}\sum_{\omega\in\mathfrak{S}_m} \mathbf{x}^{\omega(\alpha)} = \frac{1}{\prod_{j=1}^m \alpha_j} m_\alpha(\mathbf{x}),$$

where m_α is a monomial symmetric function. Thus $\Psi_m^{(g)}(\mathbf{x})$ can be re-expressed in the form:

$$\Psi_m^{(g)}(\mathbf{x}) = \sum_{n\geq 1}\sum_{\substack{\alpha\vdash n\\ l(\alpha)=m}} \frac{1}{(n+m+2(g-1))!}\frac{1}{\prod_{j=1}^m \alpha_j} c_g(\alpha) m_\alpha(\mathbf{x}).$$

Thus

$$[\mathbf{x}^\alpha]\,\Psi_m^{(g)}(\mathbf{x}) = \frac{1}{(n+m+2g-2)!}\prod_{j=1}^{m}\alpha_j c_g(\alpha),$$

and the result follows by comparing this with (2.1), (5.3), and (5.2). ∎

In Section 1, we have recast Conjecture 5.3 and Lemma 5.4 to make them refer as immediately as possible to the ramification numbers $\mu_m^{(g)}$: in Section 1, we have *defined* $f_m^{(g)}$ in (1.1) as the rescaling of $\mu_m^{(g)}$ given in the statement of Lemma 5.4; in Section 1 we give Conjecture 1.4, that $f_m^{(g)}$ thus defined is a symmetric polynomial of total degree $m+3g-3$. Of course, this means that the symmetric polynomials $f_m^{(g)}$ given in the Appendix thus have two equivalent interpretations. One interpretation is in the arguments $x_1\partial/\partial x_1,\ldots,x_m\partial/\partial x_m$, giving a partial differential operator as considered in Conjecture 5.3. The second interpretation is in the arguments $\alpha_1,\ldots,\alpha_m$, giving a scaled expression for $\mu_m^{(g)}$ as considered in (1.1).

Finally, as a further strengthening of Theorem 5.1, for $g=1$, we are able to conjecture a closed form from the data in the Appendix as follows. Note the similarity to Hurwitz's result that $f_m^{(0)}=e_1^{m-3}$.

Conjecture 5.5. *For* $m\geq 1$,

$$f_m^{(1)}=\frac{1}{24}\left(e_1^m-e_1^{m-1}-\sum_{i=2}^{m}(i-2)!e_ie_1^{m-i}\right).$$

As stated above, the data we have obtained in the Appendix, for $g=1$, $m=1,\ldots,6$ is in agreement with Conjecture 5.5. This is restated as Conjecture 1.3 because of its strikingly simple form.

6. Computational Comments

We first address the relationship between the forms for $\Psi_m^{(g)}$ proved in Theorem 5.1 and conjectured in Conjecture 5.3.

Note that, for $j\geq 1$,

$$\left(x_i\frac{\partial}{\partial x_i}\right)^j\frac{w_i}{1-w_i}=\sum_{k=j+1}^{2j+1}c(j,k)\frac{1}{(1-w_i)^k}, \tag{6.1}$$

(for $j=0$, the lower limit of the summation becomes j), for some integers $c(j,k)$. Also, since $w_i\partial/\partial w_i=(1-w_i)x_i\partial/\partial x_i$, we have for $j\geq 1$, from (4.3),

$$w_i\frac{\partial}{\partial w_i}\left(x_i\frac{\partial}{\partial x_i}\right)^{j-1}\frac{w_i}{1-w_i}=\sum_{k=j}^{2j}c(j,k)\frac{1}{(1-w_i)^k}. \tag{6.2}$$

This triangular system of linear equations can be inverted to express $((1-w_i)^{-1})^k$ as a linear combination of

$$\left(x_i\frac{\partial}{\partial x_i}\right)^j \quad\text{and}\quad w_i\frac{\partial}{\partial w_i}\left(x_i\frac{\partial}{\partial x_i}\right)^j.$$

Consequently, since, from (4.5),

$$V_m(\mathbf{w})=\prod_{i=1}^{m}\left(\frac{1}{1-w_i}-1\right),$$

then Theorem 5.1 implies that, in general, $\Psi_m^{(g)}$ can be written as $f_m^{(g)} V_m(\mathbf{w})$, where $f_m^{(g)}$ is a differential operator which is a symmetric sum of monomials in $x_i\partial/\partial x_i$, $i = 1, \ldots, m$ plus a sum of a single $w_j\partial/\partial w_j$ multiplied by symmetric sums in the $x_i\partial/\partial x_i$'s. Thus Conjecture 5.3 strengthens Theorem 5.1 concerning the form of $\Psi_m^{(g)}$ only in asserting that the terms involving $w_j\partial/\partial w_j$, for $j = 1, \ldots, m$ do not appear. This is in agreement with all known values for $\Psi_m^{(g)}$, although we have been unable to prove that this holds in general. Also in the conjecture, the degree of this symmetric differential operator follows from (6.1) and the upper bound given in Theorem 5.1, and it is part of the conjecture that the bound is actually attained.

In implementing Theorem 5.2 (using Maple) to determine explicitly the $\Psi_m^{(g)}$ given in the Appendix, we have actually used (6.1) and (6.2) and their inverses to transform the process into one involving only polynomial instead of more awkward rational series. In particular, this process gives a proof of Theorem 1.2

7. The Proof of Theorem 1.1

We conclude with a proof of Theorem 1.1. The proof uses the series w_1 that has been introduced for the symmetrization of Φ, and is the solution of the functional Equation (4.2).

According to [24], $\mu_n^{(1)} \equiv \mu_n^{(1)}(1^n)$ satisfies the following recurrence equation that was discovered by Pandharipande and Graber:

$$\mu_n^{(1)} = \frac{n}{6}\binom{n}{2}(2n-1)\mu_n^{(0)} + 2(2n-1)\sum_{j=1}^{n-2}(n-j)\,j^2\binom{2n-2}{2j-2}\mu_j^{(0)}\mu_{n-j}^{(1)},$$

where $\mu_n^{(0)} \equiv \mu_n^{(0)}(1^n)$. Since $\mu_n^{(0)} = (2n-2)!n^{n-3}/n!$ and $\mu_n^{(1)} = (2n)!\,f_n^{(1)}/n!$ from (1.2) and (1.1), respectively, where $f_n^{(1)} \equiv f_n^{(1)}(1^n)$, then

$$\frac{(2n)!}{n!}f_n^{(1)} = \frac{n}{6}\binom{n}{2}\frac{(2n-1)!}{n!}n^{n-3} + 2(2n-1)!\sum_{j=1}^{n-2}\frac{n-j}{j!(n-j)!}\,j^{j-1}f_{n-j}^{(1)}.$$

Then a_n, defined by $a_n = 24n f_n^{(1)}$, satisfies the recurrence equation

$$a_n = (n-1)n^{n-1} + \sum_{j=1}^{n-2}\binom{n}{j}j^{j-1}a_{n-j}.$$

Let $w \equiv w_1$. Then, by Lagrange's Theorem (see, for example, [10]),

$$w = \sum_{n\geq 1}\frac{n^{n-1}}{n!}x^n \quad \text{and} \quad \frac{w}{1-w} = \sum_{n\geq 1}\frac{n^n}{n!}x^n.$$

Let

$$A(x) = \sum_{n\geq 2} a_n \frac{x^n}{n!}.$$

It follows from the recurrence equation for a_n that

$$A(x) = \frac{w}{1-w} - w + wA(x)$$

so

$$A(x) = \frac{w^2}{(1-w)^2}.$$

Now note that $A(x)$ can be re-expressed in the form:

$$\begin{aligned} A(x) &= \frac{w}{(1-w)^3} - \frac{w}{1-w} - \frac{w^2}{(1-w)^3} \\ &= \frac{w}{(1-w)^3} - \frac{w}{1-w} - \sum_{i\geq 2} \frac{1}{i(i-1)} \frac{iw^i - (i-1)w^{i+1}}{(1-w)^3}. \end{aligned}$$

But, again by Lagrange's Theorem,

$$\frac{w}{(1-w)^3} = \sum_{n\geq 1} \frac{n^{n+1}}{n!} x^n \quad \text{and} \quad \frac{iw^i - (i-1)w^{i+1}}{(1-w)^3} = \sum_{j\geq 0} \frac{(i+j)^{j+1}}{j!} x^{i+j}.$$

The second of these is obtained by considering the expansion of $w^i/(1-w)$ and $(xd/dx)(w^i/(1-w))$, having observed that $w/(1-w) = xdw/dx$, from the functional equation for w. Each term in the expression for $A(x)$ has now been given a series expansion, and the result follows.

Appendix. Explicit Results

The following gives explicit expressions for $f_m^{(g)}$. The results can be expressed in a compact form in terms of weighted divided differences $\Delta_m^{(g)}$ as follows:

Let $\alpha \vdash n$ and $l(\alpha) = m$. Let $d_1 = 3!2^2$, $d_2 = 6!2^3$, $d_3 = 9!2^3$, and $d_4 = 12!2^5$. Then, for $g = 1$, $m = 1, \ldots, 6$; $g = 2, 3, 4$, $m = 1, \ldots, 6-g$,

$$d_g f_m^{(g)} = e_1 d_g f_{m-1}^{(g)} + e_m \Delta_m^{(g)},$$

for $m \geq 1$, where $f_0^{(g)} \equiv 0$, $g \geq 1$. Thus

$$f_m^{(g)} = \frac{1}{d_g} \left(e_1^m \Delta_1^{(g)} + e_1^{m-2} e_2 \Delta_2^{(g)} + e_1^{m-3} e_3 \Delta_3^{(g)} + \cdots + e_m \Delta_m^{(g)} \right).$$

Genus 1:
$$\left.\begin{aligned}\Delta_1^{(1)} &= 1-\tfrac{1}{e_1},\\ \Delta_2^{(1)} &= -0!,\\ \Delta_3^{(1)} &= -1!,\\ \Delta_4^{(1)} &= -2!,\\ \Delta_5^{(1)} &= -3!,\\ \Delta_6^{(1)} &= -4!.\end{aligned}\right\} \tag{A.1}$$

Genus 2:
$$\left.\begin{aligned}\Delta_1^{(2)} &= \quad 5e_1^3-12e_1^2+7e_1,\\ \Delta_2^{(2)} &= \left(-10e_1^3+9e_1e_2\right)+\left(12e_1^2-2e_2\right),\\ \Delta_3^{(2)} &= \left(-18e_1^3+18e_1e_2-3e_3\right)+\left(16e_1^2-6e_2\right),\\ \Delta_4^{(2)} &= \left(-36e_1^3+60e_1e_2-12e_3\right)+\left(38e_1^2-24e_2\right).\end{aligned}\right\} \tag{A.2}$$

Genus 3:
$$\left.\begin{aligned}\Delta_1^{(3)} &= 35e_1^6-147e_1^5+205e_1^4-93e_1^3,\\ \Delta_2^{(3)} &= \left(-105e_1^6+189e_1^4e_2-135e_1^2e_2^2\right)+\left(294e_1^5-321e_1^3e_2+90e_1e_2^2\right)\\ &\quad+\left(-205e_1^4+74e_1^2e_2-16e_2^2\right),\\ \Delta_3^{(3)} &= \left(-273e_1^6+594e_1^4e_2+153e_1^3e_3-405e_1^2e_2^2+135e_1e_2e_3-27e_3^2\right)\\ &\quad+\left(642e_1^5-912e_1^3e_2-111e_1^2e_3+360e_1e_2^2-66e_2e_3\right)\\ &\quad+\left(-353e_1^4+270e_1^2e_2+64e_1e_3-80e_2^2\right).\end{aligned}\right\} \tag{A.3}$$

Genus 4:
$$\left.\begin{aligned}\Delta_1^{(4)} &= 1925e_1^9-12320e_1^8+29854e_1^7-32032e_1^6+12573e_1^5,\\ \Delta_2^{(4)} &= \left(-7700e_1^9+20790e_1^7e_2-29700e_1^5e_2^2+17325e_1^3e_2^3\right)\\ &\quad+\left(36960e_1^8-74316e_1^6e_2+72600e_1^4e_2^2-23100e_1^2e_2^3\right)\\ &\quad+\left(-59708e_1^7+77814e_1^5e_2-44880e_1^3e_2^2+10780e_1e_2^3\right)\\ &\quad+\left(32032e_1^6-182e_1^4e_2+8800e_1^2e_2^2-1584e_2^3\right).\end{aligned}\right\} \tag{A.4}$$

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