THE GROMOV–WITTEN POTENTIAL OF A POINT, HURWITZ NUMBERS, AND HODGE INTEGRALS

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[Received 17 November 1999; revised 5 January 2001]

1. *Introduction*

The moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed genus g curves, with stability condition

$$2g-2+n>0 \tag{1}$$

has dimension

$$3g-3+n. \tag{2}$$

It is the Deligne–Mumford compactification of the moduli space $\mathcal{M}_{g,n}$ of smooth n-pointed genus g curves. It has n natural line bundles $\mathbb{L}_i$ (roughly, the cotangent space to the ith marked point) and a natural rank g vector bundle $\mathbb{E}$ (the Hodge bundle; its fibers correspond to global differentials on the curve). Let $\psi_i = c_1(\mathbb{L}_i)$ and $\lambda_k = c_k(\mathbb{E})$, where c_j is the jth Chern class; intersections of ψ-classes are called *descendant integrals*, and intersections of ψ-classes and λ-classes are called *Hodge integrals* (see [**13**] for fuller information).

The Gromov–Witten potential F of a point (Witten's total free energy of two-dimensional gravity) is a generating series for all descendant integrals. Witten's conjecture (Kontsevich's theorem, [**29**]) and the Virasoro conjecture for a point can be expressed as the fact that e^F is annihilated by certain differential operators (see [**17**] for example). We define G as a generalization of F (§ 2), a generating series for all intersections of ψ-classes and (up to) one 'λ-class'. (This is part of the very large phase space of [**34**].) Then F can easily be recovered from G.

Hurwitz numbers enumerate covers of the projective line by smooth connected curves of specified degree and genus, with specified branching above one point, simple branching over other specified points, and no other branching. Equivalently, they are purely combinatorial objects counting factorizations of permutations into transpositions that generate a group which acts transitively on the sheets. Hurwitz numbers have long been of interest (see, for example, [**26**, **39**] for more recent references, and [**2**] for the relation to mathematical physics). Let H be a generating series for Hurwitz numbers (defined precisely in § 2).

It is straightforward (if tedious) to produce expressions for Hurwitz numbers for any given degree (see [**26**, **7**] for degrees up to 6), but geometrical arguments are required for obtaining expressions for fixed genus and it is the latter that we consider.

The research of the first two authors was supported by research grants from the Natural Sciences and Engineering Research Council of Canada. That of the third author was partially supported by research grant DMS 9970101 from the National Science Foundation.

2000 *Mathematics Subject Classification*: primary 14H10, 81T40; secondary 05C30, 58D29.

Proc. London Math. Soc. (3) 83 (2001) 563–581.

1.1. *Recursions and Gromov–Witten theory*

One proof of the power of the theory of stable maps is the large number of striking recursions it has produced for solutions to classical problems in enumerative geometry, often as consequences of 'topological recursion relations'. The original example was Kontsevich and Manin's remarkable recursion for rational plane curves [**30**, Claim 5.2.1]. Eguchi, Hori and Xiong [**6**] used the Virasoro conjecture to find a recursion for genus 1 plane curves (proved in [**36**, Theorem 2]; see also [**5**] for the genus 1 Virasoro conjecture in the semisimple case). Similar recursive structure also underlies characteristic numbers in low genus [**11**, **38**, **23**].

There are strong analogies between plane curves and covers of the projective line. Similar techniques in Gromov–Witten theory have produced recursions for Hurwitz numbers (see [**15**, pp. 17–18] or [**38**, § 5.11] for a summary), including a genus 2 relation conjectured by Graber and Pandharipande and proved in [**20**]. Ionel (in a personal communication) has produced recursions using topological recursion relations and the Virasoro conjecture. Some geometers (including Fantechi and Pandharipande, see Example 4.1, as well as the third author) have thought that recursions among Hurwitz numbers should be rare, and should not occur in high genus. Philosophically, § 4 shows that in fact recursions are 'thick on the ground', and that there is an algorithm for producing (and verifying) them. It is expected that only a few will have straightforward (and enlightening) geometric explanations. (It would be interesting to reverse the Gromov–Witten approach and, for example, to produce relations in the cohomology of $\overline{\mathscr{M}}_{g,n}$ using recursions, but this does not seem to be tractable.)

Recurrences can be obtained in the more general setting of ramified coverings of surfaces of higher genera. These were considered by Hurwitz [**26**]. When his approach is carried out by means of a cut-and-join analysis, the resulting partial differential equation (for example, see § 4.2) is, of course, identical to the one for the sphere, although the initial conditions are different. It is then a straightforward matter to write down the recurrence for arbitrary ramification over infinity. Li, Zhao and Zheng [**32**] have obtained such a recurrence by other methods, although boundary conditions were not included (see also [**32**, Theorem B; **22**, Lemma 3.1]).

As we expect this paper also to be of interest to combinatorialists, we have tried to make it as self-contained as possible, including reviewing some results and definitions well known in algebraic and symplectic geometry, and mathematical physics.

1.2. *Organization of the paper*

We first show that, after a non-trivial change of variables (denoted by Ξ), $G = H$ in positive genus (Theorem 2.5). Hence the Gromov–Witten potential of a point is a purely combinatorial object seen in a new way. The proof uses a remarkable formula of Ekedahl, Lando, Shapiro and Vainshtein [**8**, Theorem 1.1] expressing Hurwitz numbers in terms of Hodge integrals. In some sense this addresses an obstacle to dealing with descendant integrals, the fact that they 'do not admit so easily of an enumerative interpretation' [**17**, p. 1]. (Of course, Kontsevich's original formula [**29**, p. 10] is also combinatorial, and much more useful). However, the awkwardness of the change of variables makes it difficult to transpose results between 'the world of H' (involving Hurwitz numbers) and 'the world of G' (involving the moduli space of curves).

Second, we prove a generalization (Theorem 3.1) of an ansatz of Itzykson and

Zuber ([**28**, (5.32)], hereinafter called the 'IZ genus expansion ansatz'). The philosophy behind the IZ genus expansion ansatz is that, for a fixed genus, starting from a finite number of descendant integrals (involving those monomials in the ψ where each ψ-class appears with multiplicity at least 2), one can calculate any descendant integral using only the string equation and the dilaton equation. The IZ genus expansion ansatz algebraically encodes this fact.

Thirdly, we use this to prove a conjecture of Gouldon and Jackson on Hurwitz numbers (Theorem 3.2, [**20**, Conjecture 1.2]), revealing it as a 'genus expansion ansatz for Hurwitz numbers'. The erstwhile mysterious combinatorial constants in the conjecture are actually single Hodge integrals.

As an application, we observe that there are trivial combinatorial recurrences on H, which lead to new conditions satisfied by G (and hence F). It would be desirable to give a new proof of Witten's conjecture using the combinatorics of covers of the projective line. Such a proof has recently been announced by Okounkov and Pandharipande [**35**]. As a second application, Theorem 3.2 provides an algorithm for proving and producing recursions for Hurwitz numbers. We produce simple (and surprising) new recursions in genus up to 3 as examples of the algorithm's effectiveness. Theorem 3.2 also yields explicit formulas for Hurwitz numbers of any given genus; we give an example (28) in genus 3.

1.3. *For combinatorialists*

Conjecture 1.2 of [**20**] came from a combinatorial approach to Hurwitz's encoding of ramified covers, and the proof given here suggests that further combinatorial questions of substance remain to be investigated (for example, the combinatorialization of Hodge integrals). Therefore, to make this paper more accessible to combinatorialists, we specify the essential results that are taken without proof from algebraic and differential geometry. These are the stability condition (1) and dimension condition (2) for $\overline{\mathscr{M}}_{g,n}$, $\lambda_k = 0$ unless $0 \leqslant k \leqslant g$, the convention $\lambda_0 = 1$, the genus condition (4) for the non-vanishing of Hodge integrals, the evaluation (6) of the base values $\langle \tau_0^3 \rangle_0$, $\langle \tau_1 \rangle_1$ and $\langle \lambda_1 \rangle_1$, the string (8) and dilaton (10) equations for Hodge integrals, the Riemann–Hurwitz formula (12) for the genus of a ramified cover, and the result (13) of Ekedahl, Lando, Shapiro and Vainshtein relating Hurwitz numbers to Hodge integrals. References are given to sources where the proofs of these are to be found. All of our work with Hodge integrals is through the dilaton and string equations which, in a real sense, remove the need to use the primary definition (3) of Hodge integrals.

It is hoped that, for the most part, the remainder of the paper can be read without recourse to algebraic or differential geometry.

2. *Background*

We begin with the necessary background on the generating series F, G and H that are central to the subject of this paper.

2.1. *Algebraic notation*

Suppose α is the composition $d = \alpha_1 + \ldots + \alpha_m$ where the α_i are non-negative integers. Set $l(\alpha) = m$, the *length* of α, and let $\#\mathrm{Aut}(\alpha)$ be the number of automorphisms of the multiset $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ (so if β_j of the α_i are j, then $\#\mathrm{Aut}(\alpha) = \beta_0!\beta_1!\dots$). If the α_i are positive and non-decreasing, we write $\alpha \vdash d$, and say that α is a *partition*. If, furthermore, all α_i are equal to at least 2, we write $\alpha \vDash d$.

Throughout, $t = (t_0, t_1, \dots)$ and $p = (p_1, p_2, \dots)$ where $t_0, t_1, \dots$ and $p_1, p_2, \dots$ are indeterminates. Thus, for example, $\mathbb{Q}[[t]] = \mathbb{Q}[[t_0, t_1, \dots]]$ and $\mathbb{Q}[[x, p]] = \mathbb{Q}[[x, p_0, p_1, \dots]]$. If Z is a polynomial in t, let $[(t_0^{k_0}/k_0!)\dots(t_i^{k_i}/k_i!)]Z$ be the coefficient of $(t_0^{k_0}/k_0!)\dots(t_i^{k_i}/k_i!)$ in Z.

Functional equations of the form $v = xg(v)$, where $v \in \mathbb{Q}[[x]]$ and $g(0) \neq 0$, have a unique solution $v(x)$ in $\mathbb{Q}[[x]]$, and an explicit expression for $f(v)$, where f is an arbitrary series, may be obtained by Lagrange inversion (see, for example, [**21**, § 1.2]; also known as Lagrange's Implicit Function Theorem). We will invoke Lagrange inversion a number of times, particularly when deriving explicit expressions for certain Hurwitz numbers.

2.2. *The Gromov–Witten and enriched Gromov–Witten potentials F and G of a point*

Recall that ψ_i and λ_k are Chow classes on $\overline{\mathscr{M}}_{g,n}$ of codimension 1 and k respectively, where $1 \leqslant i \leqslant n$ and $0 \leqslant k \leqslant g$ with $\lambda_0 = 1$. For non-negative integers $\theta_1, \dots, \theta_n$ define

$$\langle \tau_{\theta_1} \dots \tau_{\theta_n} \lambda_k \rangle_g = \int_{\overline{\mathscr{M}}_{g,n}} \psi_1^{\theta_1} \dots \psi_n^{\theta_n} \lambda_k \tag{3}$$

if

$$3g - 3 + n = \sum \theta_i + k \tag{4}$$

and $2g - 2 + n > 0$, and define $\langle \tau_{\theta_1} \dots \tau_{\theta_n} \lambda_k \rangle_g = 0$ otherwise. (Condition (4) arises because non-zero intersections can only occur when the sum of the codimensions of the classes intersected equals the dimension $3g - 3 + n$ of the space $\overline{\mathscr{M}}_{g,n}$.) The condition equivalent to (4) for $\langle \tau_0^{b_0} \tau_1^{b_1} \dots \lambda_k \rangle_g$ is

$$k = \sum (1 - i) b_i + 3g - 3. \tag{5}$$

In sums involving Hodge integrals it is convenient to include k as a summation index, but then to recall that the condition (either (4) or (5)) on k is implicit. When $k = 0$, this agrees with the usual definition. In particular,

$$\langle \tau_0^3 \rangle_0 = 1, \qquad \langle \tau_1 \rangle_1 = \langle \lambda_1 \rangle_1 = \tfrac{1}{24}. \tag{6}$$

Definition 2.1. Let $g \geqslant 0$. The *genus g Gromov–Witten potential* of a point is

$$F_g(t) = \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\theta_1, \dots, \theta_n \geqslant 0} t_{\theta_1} \dots t_{\theta_n} \langle \tau_{\theta_1} \dots \tau_{\theta_n} \rangle_g,$$

where the sum is constrained by (4) with $k = 0$.

The *Gromov–Witten potential* of a point is

$$F = \sum_{g \geqslant 0} y^{g-1} F_g.$$

The *genus g enriched Gromov–Witten potential* of a point is

$$G_g(t) = \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\theta_1,\dots,\theta_n \geqslant 0,\, 0 \leqslant k \leqslant g} (-1)^k t_{\theta_1} \dots t_{\theta_n} \langle \tau_{\theta_1} \dots \tau_{\theta_n} \lambda_k \rangle_g, \tag{7}$$

where the sum is constrained by (4).

The *enriched Gromov–Witten potential* of a point is

$$G = \sum_{g \geqslant 0} G_g y^{g-1}.$$

It will be convenient to use G_g in the form

$$G_g(t) = \sum_{a_1, a_2, \dots \geqslant 0,\, 0 \leqslant k \leqslant g} (-1)^k \langle \tau_0^{a_0} \tau_1^{a_1} \dots \lambda_k \rangle_g \frac{t_0^{a_0}}{a_0!} \frac{t_1^{a_1}}{a_1!} \dots,$$

where the sum is constrained by (5). (The $(-1)^k$ in the definition of G_g is included to make the change of variables simpler.) Note that $F_0 = G_0$. Note also that F can be recovered from G by substituting $v^{1-i} t_i$ for t_i, and $v^3 y$ for y, and letting $G^{\#}(t, y, v)$ be the resulting generating series in the t_i, y, and v. Then $F(t, y) = G^{\#}(t, y, 0)$ and $G(t, y) = G^{\#}(t, y, 1)$. Phrased differently, if t_i is given degree $1 - i$ and y is given degree 3, then G_g has terms only in degrees 0 to g, and F_g is the degree 0 part of G_g. Also,

$$\left[\frac{t_0^{l_0}}{l_0!} \dots \frac{t_i^{l_i}}{l_i!} v^k \right] G_g^{\#} = (-1)^k \langle \tau_0^{l_0} \dots \tau_i^{l_i} \lambda_k \rangle_g.$$

The following equations facilitate the systematic elimination of τ_0 and τ_1 from the Hodge integrals. Let $a_0, a_1, \dots$ be non-negative integers. The *string equation* (or puncture equation) is

$$\langle \tau_0^{a_0+1} \tau_1^{a_1} \dots \lambda_k \rangle_g = \sum_{i \geqslant 0} a_{i+1} \langle \tau_0^{a_0} \dots \tau_i^{a_i+1} \tau_{i+1}^{a_{i+1}-1} \dots \lambda_k \rangle_g, \tag{8}$$

unless $g = 0$, $k = 0$, $a_0 = 2$, and all other a_i are zero (in which case the left-hand side is $\langle \tau_0^3 \rangle_0 = 1$ by (6)). In genus 0, for example,

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{\theta_1} \dots \psi_n^{\theta_n} = \binom{n-3}{\theta_1, \dots, \theta_n} \tag{9}$$

by a trivial induction from the string equation (observe that one of the θ_i has to be zero, so the string equation may be applied) with $\langle \tau_0^3 \rangle_0 = 1$ as the base case.

The *dilaton equation* is

$$\langle \tau_0^{a_0} \tau_1^{a_1+1} \tau_2^{a_2} \dots \lambda_k \rangle_g = \left(2g - 2 + \sum_i a_i \right) \langle \tau_0^{a_0} \tau_1^{a_1} \tau_2^{a_2} \dots \lambda_k \rangle_g, \tag{10}$$

unless $g = 1$, $k = 0$, and the a_i are all zero (in which case the left-hand side is $\langle \tau_1 \rangle_1 = \frac{1}{24}$ by (6)). The proofs of the string and dilaton equations are the same as the usual proofs when no λ-class is present (see, for example, [**33**, p. 191]), so we suppress them. In particular, by induction, we obtain the following repeated form of the dilaton equation from the dilaton equation: if $a = a_0 + a_1 + \dots$, then

$$\langle \tau_0^{a_0} \tau_1^{a_1} \tau_2^{a_2} \dots \lambda_k \rangle_g = \frac{(a + 2g - 3)!}{(a + 2g - 3 - a_1)!} \langle \tau_0^{a_0} \tau_2^{a_2} \dots \lambda_k \rangle_g \tag{11}$$

(except when the equation does not make sense, that is, when $g = 0$ and $a - a_1 < 3$, or $g = 1$ and $a - a_1 = k = 0$), expressing the consequence of eliminating each τ_1. The string and dilaton equations can be easily translated into differential equations for G_g.

2.3. *The Hurwitz generating series H*

Fix a genus g, a degree d, and a partition $(\alpha_1, \dots, \alpha_m)$ of d with m parts. Let

$$r = d + m + 2(g-1); \tag{12}$$

so a branched cover of $\mathbb{P}^1$, with monodromy above ∞ given by α, and r other specified simple branch points (and no other branching) has genus g (by the Riemann–Hurwitz formula). Let H^g_α be the number of such branched covers that are connected. (We do not take the branched points over ∞ to be labelled.)

The remarkable formula of Ekedahl, Lando, Shapiro and Vainshtein [**8**, Theorem 1.1; **9**]

$$H^g_\alpha = \frac{r!}{\#\mathrm{Aut}(\alpha)} \prod_{i=1}^{m} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \int_{\overline{\mathcal{M}}_{g,m}} \frac{1 - \lambda_1 + \dots \pm \lambda_g}{\prod (1 - \alpha_i \psi_i)} \tag{13}$$

expresses Hurwitz numbers in terms of Hodge integrals.

A proof of (13) using virtual localization [**24**] in the moduli space of stable maps to $\mathbb{P}^1$ appears in [**25**]. It is explained there how (13) follows quickly from virtual localization on an appropriate 'relative' moduli space, not yet defined in the algebraic category (yielding relative Gromov–Witten invariants; see [**31**, § 7] and [**27**] for discussion in the symplectic category, and [**16**] for some discussion in the algebraic category in the case $g = 0$). In the case where there is no ramification above ∞ (that is, $\alpha = (1^d)$), the argument reduces to Fantechi and Pandharipande's independent proof of (13) [**15**, Theorem 2].

DEFINITION 2.2. The *Hurwitz generating series* is

$$H = \sum_{g \geqslant 0} H_g y^{g-1},$$

where H_g is the generating series

$$H_g = H_g(x, p) = \sum_{d \geqslant 1, \alpha \vdash d} \frac{H^g_\alpha}{r!} p_\alpha x^d$$

for the H^g_α, where $p_1, p_2, \dots$ and x are indeterminates, and where $2-2g = d-r+l(\alpha)$ and $p_\alpha = p_{\alpha_1} \dots p_{\alpha_m}$.

Note that e^H counts all covers, not just connected ones. (In [**20**] H_g is denoted by F_g.)

Goulden and Jackson have conjectured that H_g is of a particular form in terms of an implicitly defined set of variables $\{\phi_i(s, p) \colon i \geqslant 0\}$ defined as follows. Let

$$\phi_i(z, p) = \sum_{n \geqslant 1} \frac{n^{n+i}}{n!} p_n z^n, \tag{14}$$

where i is an integer, be a formal power series (denoted by $\psi_i(z, p)$ in [**20**]). Then, through the functional equation

$$s = x e^{\phi_0(s, p)}, \tag{15}$$

s is uniquely defined as a formal power series in x (and p).

In particular, H_0 and H_1 are given in (24) and (25), respectively. The remaining H_g are the subject of the following conjecture.

CONJECTURE 2.3 (Goulden and Jackson [**20**, Conjecture 1.2]). For $g \geqslant 2$,

$$H_g(x, p) = \sum_{e=2g-1}^{5g-5} \frac{1}{(1-\phi_1(s, p))^e} \cdot \sum_{n=e-1}^{e+g-1} \sum_{\substack{\theta \vDash n \\ l(\theta)=e-2(g-1)}} \frac{K_\theta^g}{\#\operatorname{Aut}(\theta)} \phi_{\theta_1}(s, p)\phi_{\theta_2}(s, p)\ldots \tag{16}$$

for some rational numbers K_θ^g.

We prove this conjecture (Theorem 3.2). Remarkably, each unknown constant K_θ^g turns out to be a single Hodge integral, up to sign.

REMARK 2.4. Goulden and Jackson proved Conjecture 2.3 for $g = 2$, and conjectured explicit values for certain K_θ^g (for $g = 3$ and all θ [**20**, Appendix A], and for $(e, l(\theta)) = (2g-1, 1)$ and all admissible g and n [**20**, p. 3]); we discuss these further in § 3.3.

2.4. *The relationship between H_g and G_g*

The following is a useful result that connects H_g and G_g. Throughout this section and the next we will make use of the mapping

$$\Xi\colon t_k \mapsto \phi_k(x, p),$$

for $k \geqslant 0$, extended as a ring homomorphism to $\mathbb{Q}[[t]]$.

THEOREM 2.5. *If $g > 0$, then $H_g(x, p) = \Xi G_g(t)$.*

Proof. For $g > 0$, by (13),

$$\begin{aligned}
H_g &= \sum_{\alpha \vdash d} \frac{1}{\#\operatorname{Aut}(\alpha)} \frac{\prod \alpha_i^{\alpha_i}}{\prod \alpha_i!} p_\alpha x^d \int_{\overline{\mathcal{M}}_{g,m}} \frac{1 - \lambda_1 + \ldots \pm \lambda_g}{\prod (1 - \alpha_i \psi_i)} \\
&= \sum_{\alpha_1 + \ldots + \alpha_m = d} \frac{1}{m!} \frac{\prod \alpha_i^{\alpha_i}}{\prod \alpha_i!} p_\alpha x^d \int_{\overline{\mathcal{M}}_{g,m}} \frac{1 - \lambda_1 + \ldots \pm \lambda_g}{\prod (1 - \alpha_i \psi_i)} \\
&= \sum_m \frac{1}{m!} \sum_{\alpha_1, \ldots, \alpha_m \geqslant 1} \prod_i \left(\frac{\alpha_i^{\alpha_i} p_{\alpha_i} x^{\alpha_i}}{\alpha_i!} \right) \\
&\quad \cdot \sum_{\substack{b_1 + \ldots + b_m = 3g-3+m-k \\ 0 \leqslant k \leqslant g,\, b_i \geqslant 0}} \int_{\overline{\mathcal{M}}_{g,m}} (\alpha_1 \psi_1)^{b_1} \ldots (\alpha_m \psi_m)^{b_m} (-1)^k \lambda_k \\
&= \sum_m \frac{1}{m!} \sum_{\substack{b_1 + \ldots + b_m = 3g-3+m-k \\ 0 \leqslant k \leqslant g,\, b_i \geqslant 0}} (-1)^k \langle \tau_{b_1} \ldots \tau_{b_m} \lambda_k \rangle_g \\
&\quad \cdot \sum_{\alpha_1, \ldots, \alpha_m \geqslant 1} \prod_i \left(\frac{\alpha_i^{\alpha_i + b_i} p_{\alpha_i} x^{\alpha_i}}{\alpha_i!} \right).
\end{aligned}$$

Hence

$$H_g = \sum_{m \geqslant 0} \frac{1}{m!} \sum_{b_1,\dots,b_m \geqslant 0,\, 0 \leqslant k \leqslant g} (-1)^k \left(\prod_{i=1}^{m} \phi_{b_i}(x,p) \right) \langle \tau_{b_1} \dots \tau_{b_m} \lambda_k \rangle_g.$$

The result then follows from (7). □

If $g = 0$, the above statement must be modified. The formula (13) applies when $l(\alpha) \geqslant 3$; so if $H_g[m]$ is the summand of H_g corresponding to all α with $l(\alpha) = m$, then

$$H_0 = H_0[1] + H_0[2] + \sum_{m \geqslant 3} H_0[m] = H_0[1] + H_0[2] + \Xi G_0;$$

so

$$H_0 = H_0[1] + H_0[2] + \Xi F_0.$$

A. J. de Jong has pointed out that the change of variables Ξ is not invertible. In other words, ignoring the irrelevant variable x by setting it equal to 1, we find that Ξ is not invertible. To see this, let ρ: $p_n \mapsto np_n$ and σ: $t_n \mapsto t_{n+1}$. Then $\rho\Xi = \Xi\sigma$. But ρ is invertible and σ is not. Thus Ξ is not invertible.

3. *Structure theorems for G and H*

For $k \geqslant 0$, let

$$I_k = \sum_{i \geqslant 0} t_{k+i} \frac{I_0^i}{i!}. \tag{17}$$

When $k = 0$, this is a functional equation that, by Lagrange inversion, uniquely defines $I_0 \in \mathbb{Q}[[t]]$, and thence I_k is uniquely defined as a series in $\mathbb{Q}[[t]]$ for all $k \geqslant 0$. If $t_0 = 0$, the unique solution of (17) is $I_0 = 0$; so that with this specialization

$$I_k = t_k \quad \text{for } k \geqslant 1. \tag{18}$$

3.1. *Structure theorem for G*

The following is a generalization of the IZ genus expansion ansatz. This argument also gives a much more direct proof of the original IZ genus expansion ansatz, by 'setting $\lambda_k = 0$' for $k > 0$ (excising terms for all θ such that $\sum_j (1-j)\theta_j + 3g - 3 > 0$). (The only proof of the IZ ansatz in the literature known to the authors is in [**10**].) Denote $\partial/\partial t_i$ by ∂_i for the sake of brevity.

THEOREM 3.1 (Genus expansion ansatz). *If* $g > 1$,

$$G_g(t) = \frac{1}{(1-I_1)^{2g-2}} G_g\left(0, 0, \frac{I_2}{1-I_1}, \frac{I_3}{1-I_1}, \dots\right) \tag{19}$$

$$= \sum_{\substack{\sum_{2 \leqslant j \leqslant 3g-2}(j-1)l_j \\ +k=3g-3}} (-1)^k \frac{\langle \tau_2^{l_2} \tau_3^{l_3} \dots \tau_{3g-2}^{l_{3g-2}} \lambda_k \rangle_g}{(1-I_1)^{2(g-1)+\sum l_j}} \frac{I_2^{l_2}}{l_2!} \cdots \frac{I_{3g-2}^{l_{3g-2}}}{l_{3g-2}!}. \tag{20}$$

(It is straightforward to show that the right-hand sides of equations (19) and (20) are the same.)

In [**14**, § 2.1], Faber and Pandharipande use the terminology 'primitive' to denote Hodge integrals without τ_0 or τ_1. Essentially, the formal derivation here (like the work of [**28**]) is to write an explicit formula for G_g in terms of primitive Hodge integrals. Viewed in this way, it is clear there are only finitely many degrees of freedom for each genus (as there are only finitely many primitive Hodge integrals for a fixed genus); the interesting part is the precise form.

Proof. Let $\Delta = \sum_{m \geqslant 0} t_{m+1}\partial_m - \partial_0$. Then, from the string equation (8),

$$\Delta G_g(t) = 0,$$

for $g > 0$, and $G_g(t)$ is the unique such series with the initial value $G_g(0, t_1, \dots)$ at $t_0 = 0$. We begin the proof by exploiting this uniqueness to establish that

$$G_g(t) = G_g(0, I_1, I_2, \dots) \quad \text{for } g > 0. \tag{21}$$

Let $\zeta_i = 0$ if $i < 0$ and $\zeta_i = 1$ if $i \geqslant 0$. Then, from (17), for $m, k \geqslant 0$,

$$\partial_m I_k = \zeta_{m-k}\frac{I_0^{m-k}}{(m-k)!} + \left(\sum_{i \geqslant 1} t_{k+i}\frac{I_0^{i-1}}{(i-1)!}\right)\partial_m I_0,$$

so

$$\partial_m I_k = \zeta_{m-k}\frac{I_0^{m-k}}{(m-k)!} + I_{k+1}\partial_m I_0.$$

Then, substituting $k = 0$ above, we obtain for $m \geqslant 0$,

$$\partial_m I_0 = \frac{1}{m!}\frac{I_0^m}{1 - I_1},$$

so, for $k, m \geqslant 0$,

$$\partial_m I_k = \zeta_{m-k}\frac{I_0^{m-k}}{(m-k)!} + \frac{I_0^m}{m!}\frac{I_{k+1}}{1 - I_1}. \tag{22}$$

Now, by the chain rule,

$$\Delta G_g(0, I_1, I_2, \dots) = \sum_{k \geqslant 1}\left(\sum_{m \geqslant 0} t_{m+1}\partial_m I_k - \partial_0 I_k\right)\frac{\partial}{\partial I_k} G_g(0, I_1, I_2, \dots).$$

But, from (22),

$$\sum_{m \geqslant 0} t_{m+1}\partial_m I_k - \partial_0 I_k = \sum_{m \geqslant k} t_{m+1}\frac{I_0^{m-k}}{(m-k)!} + \frac{I_{k+1}}{1 - I_1}\sum_{m \geqslant 0} t_{m+1}\frac{I_0^m}{m!} - \frac{I_{k+1}}{1 - I_1} = 0,$$

for $k \geqslant 1$. Thus $\Delta G_g(0, I_1, I_2, \dots) = 0$. But $G_g(0, I_1, I_2, \dots)|_{t_0 = 0} = G_g(0, t_1, t_2, \dots)$ from (18), and thus we have established (21) by the uniqueness argument.

To complete the proof, we use the repeated form (11) of the dilaton equation for $g > 1$:

$$\begin{aligned} G_g(0, I_1, I_2, \dots) &= \sum_{b_1, b_2, \dots \geqslant 0} (-1)^{\sum_{i \geqslant 1}(1-i)b_i + 3g - 3}\langle \tau_1^{b_1}\tau_2^{b_2}\dots\lambda_k\rangle_g \frac{I_1^{b_1}}{b_1!}\frac{I_2^{b_2}}{b_2!}\cdots \\ &= \sum_{b_2, b_3, \dots \geqslant 0} (-1)^{\sum_{i \geqslant 2}(1-i)b_i + 3g - 3}\langle \tau_2^{b_2}\tau_3^{b_3}\dots\lambda_k\rangle_g \frac{I_2^{b_2}}{b_2!}\frac{I_3^{b_3}}{b_3!}\cdots \\ &\quad \cdot \sum_{b_1 \geqslant 0}\binom{-(b_1 + b_2 + \dots) - 2g + 2}{b_1} I_1^{b_1} \end{aligned}$$

from (11). Thus

$$G_g(0, I_1, I_2, \ldots) = \frac{1}{(1-I_1)^{2g-2}} G_g\left(0, 0, \frac{I_2}{1-I_1}, \frac{I_3}{1-I_1}, \ldots\right) \quad \text{for } g > 1,$$

and the result now follows from (21). □

3.2. *Structure theorem for H*

We now give the main structure theorem for H.

THEOREM 3.2 [**20**, Conjecture 1.2]. *Conjecture* 2.3 *is true, with*

$$K_\theta^g = (-1)^k \langle \tau_{\theta_1}\tau_{\theta_2}\ldots\lambda_k\rangle_g, \tag{23}$$

where $k = \sum_j (1-j)\theta_j + 3g - 3$.

Proof. From Theorem 2.5 with $g > 0$, $H_g(x, p) = \Xi G_g(t)$ where, from Theorem 3.1 (20), for $g > 1$,

$$G_g = \sum (-1)^k \frac{\langle \tau_2^{l_2}\tau_3^{l_3}\ldots\tau_{3g-2}^{l_{3g-2}}\lambda_k\rangle_g}{(1-I_1)^{2(g-1)+\sum l_j}} \frac{I_2^{l_2}}{l_2!}\cdots\frac{I_{3g-2}^{l_{3g-2}}}{l_{3g-2}!},$$

where the sum is over those l_j and k such that

$$\sum_{2\leqslant j\leqslant 3g-2} (j-1)l_j + k = 3g-3,$$

as in (20). We want to prove (16), for $g \geqslant 2$; that is,

$$H_g(x, p) = \sum_{e=2g-1}^{5g-5} \frac{1}{(1-\phi_1(s,p))^e} \cdot \sum_{n=e-1}^{e+g-1} \sum_{\substack{\theta\vdash n\\ l(\theta)=e-2(g-1)}} \frac{K_\theta^g}{\#\operatorname{Aut}(\theta)} \phi_{\theta_1}(s,p)\phi_{\theta_2}(s,p)\ldots,$$

where K_θ^g satisfies (23). Since this can be rewritten in the form

$$H_g(x,p) = \sum \frac{K^g_{(2^{l_2}3^{l_3}\ldots)}}{(1-\phi_1(s,p))^{2(g-1)+\sum l_j}} \frac{\phi_2(s,p)^{l_2}}{l_2!}\cdots\frac{\phi_{3g-2}(s,p)^{l_{3g-2}}}{l_{3g-2}!},$$

where the sum (as in (20)) is over those l_j and k such that

$$\sum_{2\leqslant j\leqslant 3g-2} (j-1)l_j + k = 3g-3,$$

the proof is therefore complete if we can establish that $\Xi I_k(t) = \phi_k(s, p)$ for $k \geqslant 1$, thereby making the identification $K_\theta^g = (-1)^k\langle\tau_{\theta_1}\tau_{\theta_2}\ldots\lambda_k\rangle_g$.

From (14) and (15), for $k \geqslant 0$,

$$\begin{aligned}\phi_k(s,p) &= \sum_{n\geqslant 0} \frac{n^{n+k}}{n!} p_n x^n e^{n\phi_0(s,p)}\\ &= \sum_{m,n\geqslant 0} \frac{n^{n+k+m}}{n!} p_n x^n \frac{\phi_0(s,p)^m}{m!},\end{aligned}$$

so

$$\phi_k(s, p) = \sum_{m \geqslant 0} \phi_{k+m}(x, p) \frac{\phi_0(s, p)^m}{m!}.$$

By comparing this with the definition (17) of I_k, it follows that $\Xi I_k(t) = \phi_k(s, p)$ for $k \geqslant 0$, which completes the proof. □

We record the observation on the action of Ξ that

$$\Xi I_k = \phi_k(s, p) \quad \text{for } k \geqslant 0.$$

Thus we have established the connexion between the indeterminates x and p_i on the Hurwitz side and the indeterminates t_r and I_r on the Gromov–Witten side (see § 4.3).

3.3. *Analogous statements in genus* 0 *and* 1

We note that [**18**, Proposition 3.1(1)]

$$\left(x \frac{\partial}{\partial x}\right)^2 H_0(x, p) = \phi_0(s, p). \tag{24}$$

In the light of Theorem 2.5, stating that $\Xi G_g(t) = H_g(x, p)$ for $g > 0$, earlier statements in geometry and in combinatorics can now be seen to be equivalent. In genus 1,

$$H_1(x, p) = \Xi G_1(t) = \tfrac{1}{24}(\log(1 - \phi_1(s, p))^{-1} - \phi_0(s, p)) \tag{25}$$

[**39**; **19**, Theorem 4.2], and

$$\Xi F_1(t) = \tfrac{1}{24} \log(1 - \phi_1(s, p))^{-1}$$

[**28**, (5.30); **10**, (3.7); **4**]. The difference $-\frac{1}{24}\phi_0(s, p)$ can be seen to be the contribution to $\Xi G_1(t)$ from λ_1.

Surprisingly, the picture is least clear in genus 0. Here $F_0(t) = G_0(t)$, and the difference $H_0(x, p) - \Xi G_0(t)$ arises from where (13) breaks down: it is a generating series for covers of $\mathbb{P}^1$ with at most two pre-images of ∞, $H_0[1](x, p) + H_0[2](x, p)$. By [**18**] or [**3**],

$$H_0[1](x, p) = \phi_{-2}(x, p).$$

By [**1**] or [**18**],

$$H_0[2](x, p) = \sum_{i, j \geqslant 1} \frac{(i + j - 1)!}{(i-1)!(j-1)!} i^{i-1} j^{j-1} p_i p_j x^{i+j}.$$

From (17), $\Xi F_0(t) + H_0[1](x, p) + H_0[2](x, p) = H_0(x, p)$ so, using formula (9) for F_0 and [**18**, Theorem 1.1] for H_0, we see that this gives an explicit relation. However, it does not seem enlightening.

Remark 3.3. Using Theorem 3.2, one can see that the conjectures of Goulden and Jackson described in Remark 2.4 are true. The conjectured values of K_θ^3 can be checked using Faber's program [**12**]. The conjectured values of K_θ^g for $e = 2g - 1$, $l(\theta) = 1$ (involving coefficients of $(\frac{1}{2}z / \sin \frac{1}{2}z)^{k+1}$) turn out to be equivalent to [**13**, Theorem 2] and [**8**, Theorem 1.2].

4. *Consequences and applications*

4.1. *Combinatorial comments on Hodge integrals*

The terms that appear in Conjecture 2.3 can be given, in principle, a combinatorial interpretation. The left-hand side already has a combinatorial interpretation, through Hurwitz's encoding, in terms of transitive ordered factorizations of permutations into transpositions.

For the right-hand side, n^{n+i} is the number of rooted (vertex-)labelled trees with $i+1$ marked vertices (vertices may be multiply marked). The generating series for this number is $\phi_i(z, p)$, where p_n records the number of vertices in a tree. Therefore $\phi_0(z, p)$ is the number of rooted labelled trees with exactly one marked vertex. Similar interpretations can therefore be given to s and $1/(1-\phi_1(s, p))^e$. The right-hand side therefore has an interpretation in terms of structures obtained by gluing together and ordering collections of rooted labelled trees with marked vertices. This suggests that K_θ^g, which has been identified up to sign as a Hodge integral through Theorem 3.2, can be defined purely combinatorially, provided the mapping between the structures corresponding to the left-hand and right-hand sides of (16) is made explicit. In particular, this would involve determining how markers attached to the vertices of the trees from the right-hand side encode transitive ordered factorizations of permutations into transpositions, that occur on the left-hand side of (16). This is, of course, where the difficulty lies since the theorem itself provides no information about the elementwise action of such a mapping.

4.2. *Consequences of Theorem* 2.5

Theorem 2.5 gives a new combinatorial structure on G (and hence F), and one could hope to prove results about F using H, that is, the combinatorics of branched covers. For example, there is a simple differential operator T (the 'cut-and-join' operator) annihilating e^H, corresponding to the interpretation of H as counting factorizations of permutations ([**18**, Lemma 2.2], and independently [**37**, p. 8]), defined as follows.

Define $H^\# = H^\#(x, y, u, p)$ by substituting xu^2 for x, yu^2 for y, and $p_i u^{1-i}$ for p_i in H. Then

$$H_g^\# = \sum_{d \geqslant 0,\, \alpha \vdash d} \frac{H_\alpha^g}{r!} p_\alpha x^d u^r$$

where $r = l(\alpha) + d + 2g - 2$ is the number of simple branch points (now marked by u). Let

$$T = \tfrac{1}{2} \sum_{a, b \geqslant 1} \left[(a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} + \frac{1}{y} ab p_{a+b} \frac{\partial}{\partial p_a} \frac{\partial}{\partial p_b} \right] - \frac{\partial}{\partial u}.$$

Then $Te^{H^\#} = 0$, and $H^\#$ is uniquely determined by this equation and the condition $H^\#(x, y, 0, p) = p_1 x$ (that is, there is only one cover of $\mathbb{P}^1$ unbranched away from ∞).

Note that even the string equation becomes mysterious when translated to a statement about H:

$$\frac{\partial}{\partial t_0} H = \tfrac{1}{2} t_0^2 + x \frac{\partial}{\partial x} H.$$

It is not combinatorially clear why this should be true.

4.3. *Comments on the connexion between H and G (and F)*

It is worth noting how the variables that are used by physicists to study F (and that are equally useful for G) have exactly paralleled the variables used by combinatorialists to study H. Specifically, physicists (and geometers) use the following conventions.

P1. They write F in terms of the variables t_i; the power series F_g, $G_g \in \mathbb{Q}[[t]]$ are naturally generating series for all Hodge integrals.

P2. For $g > 1$, F_g and G_g lie in a much smaller ring. Via the genus reduction ansatz, Theorem 3.1, F_g and G_g can be rewritten as elements of $\mathbb{Q}[1/(1-I_1), I_2, I_3, \dots]$, and this representation is particularly simple (as only a finite number of monomials appear, and their coefficients are each single Hodge integrals).

P3. It is often physically enlightening [**28**, **10**] to rewrite the above in terms of other variables. Let $u_0 = \partial_0^2 F_0$. Then for $g > 1$,

$$F_g, G_g \in \mathbb{Q}[1/\partial_0 u_0, \partial_0 u_0, \partial_0^2 u_0, \dots]$$

(and in fact F_g has a particular bigrading in terms of these variables, where $\deg \partial_0^r u_0 = (1, r-1)$). In [**10**], these variables are used in the proof of the IZ genus reduction ansatz. It is not hard to translate between the $\partial_0^r u_0$ and the I_k; in particular, $u_0 = I_0$; see [**10**, p. 284].

Combinatorialists' conventions are as follows.

C1. They write H in terms of the variables x and p_i; the power series $H_g \in \mathbb{Q}[[x, p]]$ is a generating series for all Hurwitz numbers.

C2. In fact, for $g > 1$, H_g lies in a much smaller ring:

$$H_g \in \mathbb{Q}[1/(1-\phi_1(s,p)), \phi_2(s,p), \phi_3(s,p), \dots],$$

which, via Ξ, is the same as P2 above.

C3. Also, H_g lies in $\mathbb{Q}[[\phi_0(x,p), \phi_1(x,p), \dots]]$; via Ξ this is the same as P1 above.

4.4. *Applications of Theorem* 3.2

Along with techniques from [**20**], Theorem 3.2 gives a machine for developing and proving recurrences and explicit formulas for Hurwitz numbers, given that the necessary Hodge integrals can be calculated by Faber's program [**12**]. As an example, in [**20**], a conjectured recursion of Graber and Pandharipande was proved using Theorem 3.2 in genus 2 (proved there). We now give further examples.

The examples are for the case in which there is no ramification over ∞. We will refer to the corresponding numbers as *simple Hurwitz numbers*. They are obtained by setting $p_1 = 1$ and $p_i = 0$ for $i \neq 1$. Under this specialization, $\phi_i(x,p) = x$ for all i, and, from (15), $s = w$ where w is the unique solution of

$$w = xe^w,$$

and is given explicitly by

$$w = \sum_{n \geqslant 1} n^{n-1} \frac{x^n}{n!}.$$

Then H_g becomes

$$\widetilde{H_g} = \sum_{d \geqslant 1} \frac{H^g_{(1^d)}}{(2d+2g-2)!} x^d,$$

the generating series for simple Hurwitz numbers.

EXAMPLE 4.1 (a recurrence equation for genus 3). From a geometric perspective, 'it is not likely such simple recursive formulas (similar to the Graber–Pandharipande formula in genus 2, and simpler recursions in genus 0 and 1 [**39**, Theorem 2.7] (our intercalation)) occur in $g \geqslant 3$' [**15**, p. 18]. However, using Theorem 3.2, one can obtain recurrences as follows. Let $D = x\,d/dx$. Then

$$\begin{aligned}
D^2\widetilde{H_0}(x) &= w,\\
\widetilde{H_1}(x) &= \tfrac{1}{24}(\log(1-w)^{-1} - w),\\
\widetilde{H_2}(x) &= \frac{1}{5760}\left(\frac{4w^2}{(1-w)^4} + \frac{28w^3}{(1-w)^5}\right),\\
\widetilde{H_3}(x) &= \frac{1}{80640}\frac{w^2}{(1-w)^6} + \frac{73}{90720}\frac{w^3}{(1-w)^7} + \frac{37}{5184}\frac{w^4}{(1-w)^8}\\
&\quad + \frac{89}{5184}\frac{w^5}{(1-w)^9} + \frac{245}{20736}\frac{w^6}{(1-w)^{10}}.
\end{aligned}$$

These are from [**20**], although the final two can now be obtained from Theorem 3.2, with the help of Faber's program [**12**] to compute the necessary Hodge integrals.

It is convenient to set $w = 1 - W^{-1}$, so $D = W^2(W-1)d/dW$. Then $D^n\widetilde{H_g}(x)$ is a polynomial in W provided $2g-2+n>0$. (The resemblance to the stability condition for $\overline{\mathcal{M}}_{g,n}$ is probably not coincidental; D can be interpreted as marking a point above a fixed general point of $\mathbb{P}^1$.) For $(g,n) = (0,1)$, $(0,2)$, $D^n\widetilde{H_g}(x)$ is a rational series in W. A number of these series are given below:

$$\begin{aligned}
D\widetilde{H_0}(x) &= (1 - W^{-2})/2,\\
\widetilde{H_1}(x) &= \frac{\log(W)W - W + 1}{24W},\\
D\widetilde{H_1}(x) &= (W-1)^2/24,\\
\widetilde{H_2}(x) &= (W-1)^2W^2(-6+7W)/1440,\\
\widetilde{H_3}(x) &= (W-1)^2W^4\\
&\quad \cdot (720 - 6696W + 19250W^2 - 21840W^3 + 8575W^4)/725760.
\end{aligned}$$

Various relations can be found between the $D^n\widetilde{H_g}(x)$ for $(g,n) \neq (0,0)$, $(1,0)$ by positing a general form for them and equating coefficients of powers of W to obtain a set of linear equations for the parameters appearing in this form.

With the form containing the twenty-six terms $(D^p\widetilde{H_i})(D^q\widetilde{H_j})$ for $p+q=4$, $i+j=3$, and $D^p\widetilde{H_i}$, for $i=3$, $1 \leqslant p \leqslant 4$, for $i=2$, $1 \leqslant p \leqslant 5$, and for $i=1$, $1 \leqslant p \leqslant 7$, the null space has dimension 11. (We choose this form for potential recursions because this is the form of the recursions previously produced via Gromov–Witten theory.) Thus further conditions on the parameters may be applied, although it is not at all clear whether there is a geometrically natural choice to make. One such expression, obtained by imposing linearity, is

$$2880\widetilde{H_3} = -\left(\frac{2}{49} - \frac{227}{294}D + \frac{99845}{588}D^2\right)\widetilde{H_2} - \left(\frac{1}{490}D^2 - \frac{11}{294}D^3 + \frac{38845}{14112}D^4 - \frac{1225}{576}D^5\right)\widetilde{H_1}.$$

This gives the following explicit formula for $H^3_{(1^d)}$ linearly in terms of $H^2_{(1^d)}$ and $H^1_{(1^d)}$:

$$2880H^3_{(1^d)} = -(24 - 454d + 99845d^2)\binom{2d+4}{2}\frac{H^2_{(1^d)}}{294} + d^2(-288 + 5280d - 388450d^2 + 300125d^3)\binom{2d+4}{4}\frac{H^1_{(1^d)}}{5880}.$$

Similar recursions exist for all genera, and these may be obtained in the same way.

Example 4.2 (another recurrence equation for genus 3, of 'geometric form'). As another example to show how common recursions are, we give a genus 3 recursion that is of a potentially geometrically meaningful form:

$$H^3_{(1^d)} = f(d)\binom{d}{2}H^2_{(1^d)} + \sum_{i+j=d}\left(g(i,j)\binom{2d+2}{2i-2}ijH^0_{(1^i)}H^3_{(1^j)} + h(i,j)\binom{2d+2}{2i}ijH^1_{(1^i)}H^2_{(1^j)}\right),$$

where $f(d)$, $g(i,j)$, and $h(i,j)$ are polynomials of low degree.

Any formula coming from a divisorial relation on the space of maps would have such a form. Even though such a divisorial relation should not exist, a geometrically-motivated recursion of this form might still exist; the recursion for genus 1 plane curves of [**6**] has this property, for example. One might hope for some geometrical understanding from such a recursion.

The terms on the right-hand side of the equation correspond to divisors on the space of maps. The first term corresponds to degree d genus 2 covers where two of the d points mapping to the same point of $\mathbb{P}^1$ are attached; hence the multiplicity of $\binom{d}{2}$. The second term corresponds to maps where the cover is a genus 0 degree i cover (a general such cover has $2i-2$ branch points) and a genus 3 degree j cover (a general such cover has $2j+4$ branch points) such that two points mapping to the same point of $\mathbb{P}^1$ (one on each component) are glued together; the multiplicity ij comes from the choice of the two points, and the multiplicity $\binom{2d+2}{2i-2}$ comes from partitioning the branch points between the two components. The third corresponds to maps where the cover is a genus 1 degree i cover and a genus 2 degree j cover with a point of one glued to a point of the other; the multiplicity calculation is similar to the second term. These divisors might appear with various multiplicities, given by the polynomials f, g and h.

Unfortunately, many such recursions can be found (by the same method as in Example 4.1), even if the degrees of f, g, and h are required to be small. One such is

$$
\begin{aligned}
f(d) &= \frac{1}{1702263010}(1532127678d - 2213123851),\\
g(i,j) &= -\frac{2}{121590215}(760192125ij - 12054428314i\\
&\qquad - 2006745110j + 1033797958),\\
h(i,j) &= -\frac{4}{2553394515}(798201731250ij - 217500288725i\\
&\qquad - 473678414332j - 42109762821).
\end{aligned}
$$

There seems to be no reason why this recursion should admit a geometrical explanation.

Example 4.3 (a recurrence equation for genus 2). The method of Example 4.1 can be applied to the genus 2 case; we suppress the details. The linear differential equation that is satisfied is

$$4320\widetilde{H_2}(x) = -300D^2\widetilde{H_1} + 7(D^5 - D^4)\widetilde{H_0}.$$

The corresponding linear recurrence equation is

$$180H^2_{(1^d)} = -25d^2\binom{2d+2}{2}H^1_{(1^d)} + 7d^4(d-1)\binom{2d+2}{4}H^0_{(1^d)}.$$

For genus 2 and 3, $H^g_{(1^d)}$ has been expressed in terms of $H^{g-1}_{(1^d)}$ and $H^{g-2}_{(1^d)}$. A reason this is not entirely unexpected is that D preserves the parity of the degree of polynomials in W. But the degree in W of $D^nH^g(x)$ is $2n+5g-5$, and the parity of this modulo 2 is the parity of $g-1$ modulo 2. Polynomials of both parities are required on the right-hand side in the posited form of the differential equation to match terms on the left-hand side. This is to be expected to persist for $g \geqslant 2$.

Example 4.4 (recurrence equations for genus 1 and 0). The parity argument in the previous example suggests that, if there is a recurrence equation, it must be of degree (at least) 2 for the genus 1 case, and indeed a degree 2 example is known (due to Graber and Pandharipande [**38**, § 5.11] or [**15**, p. 18]). This recurrence can be rewritten as the differential equation

$$D\widetilde{H_1} = \tfrac{1}{24}D^3\widetilde{H_0} - \tfrac{1}{24}D^2\widetilde{H_0} + (D^2\widetilde{H_0})(D\widetilde{H_1})$$

which is an immediate consequence of the observations that $D\widetilde{H_1}(x) = \frac{1}{24}(W-1)^2$, $D^2\widetilde{H_0}(x) = 1 - W^{-1}$ and $D^3\widetilde{H_0}(x) = W - 1$.

An even simpler recursion exists originating from the differential equation

$$D\widetilde{H_1} = \tfrac{1}{24}(D^3\widetilde{H_0})^2.$$

This gives

$$H^1_{(1^d)} = \frac{1}{d}\binom{2d}{4}\sum_{i=1}^{d-1} i^3(d-i)^3\binom{2d-4}{2i-2}H^0_{(1^i)}H^0_{(1^{d-i})}. \tag{26}$$

The differential equation is an immediate consequence of the above expressions for $D\widetilde{H_1}$ and $D^3\widetilde{H_0}$. Although it might not be difficult to prove (26) geometrically, there was no geometrical reason to suspect its existence.

The sphere is included for completeness from this point of view. Again, by the parity argument, a recurrence of degree 2 is expected. The simplest such differential equation is

$$D^2\widetilde{H_0} = \tfrac{1}{2}(D^2\widetilde{H_0})^2 + D\widetilde{H_0},$$

which is an immediate consequence of the observations that $D^2\widetilde{H_0}(x) = 1 - W^{-1}$ and $D\widetilde{H_0}(x) = \frac{1}{2}(1 - W^{-2})$. The resulting recurrence equation is

$$H^0_{(1^d)} = \frac{1}{d(d-1)}\binom{2d-2}{2}\sum_{i=1}^{d-1} i^2(d-i)^2\binom{2d-4}{2i-2}H^0_{(1^i)}H^0_{(1^{d-i})}, \tag{27}$$

which is a well-known recurrence found by Pandharipande (see [**38**, §5.11] or [**15**, p. 17]). Other (more complicated) genus 0 recurrences can also be found in this manner.

Example 4.5 (closed form expressions for simple Hurwitz numbers). Closed form expressions for simple Hurwitz numbers can be found for all genera (using the method of [**20**, Corollary 4.1]). The expression for the genus g case can be obtained from Theorem 3.2, with the specializations of p, s and ϕ_i given above, and is the following:

$$\frac{H^g_{(1^d)}}{(2d+2g-2)!} = [x^d]\widetilde{H_g}(x) = \sum_{r=2g-1}^{5g-5}\sum_{n=r-1}^{r+g-1} K_{n,g,r}\left([x^d]\frac{w^n}{(1-w)^r}\right)$$

where

$$K_{n,g,r} = \sum_{\substack{\theta\vDash n \\ l(\theta)=r-2(g-1)}} (-1)^k\langle\tau_{\theta_1}\tau_{\theta_2}\dots\lambda_k\rangle_g$$

and $k = \sum_i (1-i)\theta_i + 3g - 3$. Thus $K_{n,g,r}$ can be computed by Faber's program [**12**]. The remaining term is obtained by Lagrange inversion as

$$\begin{aligned}[x^d]\frac{w^n}{(1-w)^r} &= \frac{1}{d}[\mu^{d-1}]\left(\frac{n\mu^{n-1}}{(1-\mu)^r} + \frac{r\mu^n}{(1-\mu)^{r+1}}\right)e^{d\mu} \\ &= \sum_{i=0}^{d-n}\binom{r+i-1}{r-1}\frac{nd^{d-n-i-1}}{(d-n-i)!} + \sum_{i=0}^{d-n-1}\binom{r+i}{r}\frac{rd^{d-n-i-2}}{(d-n-i-1)!}.\end{aligned}$$

For example, for $\widetilde{H_3}(x)$, by Lagrange inversion,

$$\begin{aligned}\frac{H^3_{(1^d)}}{(2d+4)!} &= \frac{1}{1008}A_4(d) - \frac{113}{10080}A_5(d) + \frac{2383}{51840}A_6(d) - \frac{16759}{181440}A_7(d) \\ &\quad + \frac{227}{2304}A_8(d) - \frac{557}{10368}A_9(d) + \frac{245}{20736}A_{10}(d),\end{aligned} \tag{28}$$

where

$$A_k(d) = \frac{k}{d}\sum_{r=0}^{d-1}\binom{k+r}{k}\frac{d^{d-r-1}}{(d-r-1)!}.$$

This can be rewritten as

$$H^3_{(1^d)} = \frac{(2d+4)!}{2^5 3^3 9!} \sum_{r=0}^{d-1} \frac{d^{d-r-2}}{(d-r-1)!} \binom{r+4}{5} (r+1)$$
$$\cdot (1225r^4 + 3770r^3 + 35r^2 - 2822r + 1680).$$

It is clear that in general the simple Hurwitz numbers have the form

$$H^g_{(1^d)} = (2d+2g-2)! \sum_{r=0}^{d-1} \frac{d^{d-r-2}}{(d-r-1)!} P_g(d-r-1)$$

where $P_g(r)$ is a polynomial in r of degree $5g-5$.

Acknowledgements. We are grateful for conversations with Gilberto Bini, Robbert Dijkgraaf, Carel Faber, Tom Graber, John Harer, A. Johan de Jong, Rahul Pandharipande, Malcolm Perry and Wati Taylor. We would also like to thank Carel Faber for making available to us his MAPLE program for evaluating Hodge integrals. DMJ would like to thank the Mathematics Departments at Cambridge University, Duke University and MIT for their hospitality during his sabbatical leave (1998/99) when much of this work was carried out.

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