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Enumerating branched orientable surface coverings over a non-orientable surface

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Abstract

The isomorphism classes of several types of graph coverings of a graph have been enumerated by many authors [M. Hofmeister, Graph covering projections arising from finite vector space over finite fields, Discrete Math. 143 (1995) 87–97; S. Hong, J.H. Kwak, J. Lee, Regular graph coverings whose covering transformation groups have the isomorphism extention property, Discrete Math. 148 (1996) 85–105; J.H. Kwak, J.H. Chun, J.Lee, Enumeration of regular graph coverings having finite abelian covering transformation groups, SIAM J. Discrete Math. 11 (1998) 273–285; J.H. Kwak, J. Lee, Isomorphism classes of graph bundles, Canad. J. Math. XLII (1990) 747–761; J.H. Kwak, J. Lee, Enumeration of connected graph coverings, J. Graph Theory 23 (1996) 105–109]. Recently, Kwak et al [Balanced regular coverings of a signed graph and regular branched orientable surface coverings over a non-orientable surface, Discrete Math. 275 (2004) 177–193] enumerated the isomorphism classes of balanced regular coverings of a signed graph, as a continuation of an enumeration work done by Archdeacon et al [Bipartite covering graphs, Discrete Math. 214 (2000) 51–63] the isomorphism classes of branched orientable regular surface coverings of a non-orientable surface having a finite abelian covering transformation group. In this paper, we enumerate the isomorphism classes of connected balanced (regular or irregular) coverings of a signed graph and those of unbranched orientable coverings of a non-orientable surface, as an answer of the question raised by Liskovets [Reductive enumeration under mutually orthogonal group actions, Acta-Appl.

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Math. 52 (1998) 91–120]. As a consequence of these two results, we also enumerate the isomorphism classes of branched orientable surface coverings of a non-orientable surface.

Keywords: Graph covering; Signed graph; Surface branched covering; Enumeration

1. Introduction

Let G be a finite connected graph with vertex set $V(G)$ and edge set $E(G)$. Every edge of a graph G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of G by $D(G)$.

The *neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of directed edges emanating from v. We use $|X|$ for the cardinality of a set X. The number $\beta(G) = |E(G)| - |V(G)| + 1$ is equal to the number of independent cycles in G and it is referred to as the *betti number* of G.

A graph $\tilde{G}$ is called a *covering* of G with projection $p : \tilde{G} \to G$ if there is a surjection $p : V(\tilde{G}) \to V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We say that the projection $p : \tilde{G} \to G$ is an *n-fold covering* of G if p is n-to-one. Two coverings $p_i : \tilde{G}_i \to G$, $i = 1, 2$ are said to be *isomorphic* if there exists a graph isomorphism $\Phi : \tilde{G}_1 \to \tilde{G}_2$ such that $p_1 = p_2 \circ \Phi$. Such a Φ is called a *covering isomorphism*.

Let S_n denote a symmetric group on n elements $\{1, 2, \ldots, n\}$. A *permutation voltage assignment* (or, *voltage assignment*) of G is a function $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The *permutation-derived graph* G^ϕ is defined as follows: $V(G^\phi) = V(G) \times \{1, \ldots, n\}$ and $E(G^\phi) = E(G) \times \{1, 2, \ldots, n\}$, so that an edge (e, i) of G^ϕ joins a vertex (u, i) to $(v, \phi(e)i)$ for $e = uv \in D(G)$ and $i = 1, 2, \ldots, n$. The first coordinate projection $p^\phi : G^\phi \to G$ is an n-fold covering. Following Gross and Tucker [2], every n-fold covering $\widetilde{G}$ of a graph G can be derived from a voltage assignment which assigns the identity element on the directed edges of a fixed spanning tree T of G. We call such a ϕ reduced. That is, for a covering $p : \tilde{G} \to G$, there exists a reduced voltage assignment ϕ of G such that the derived covering $p^\phi : G^\phi \to G$ is isomorphic to $p : \tilde{G} \to G$. Moreover, for a reduced voltage assignment $\phi : D(G) \to S_n$, the derived graph G^ϕ is connected if and only if the subgroup of S_n generated by the image of the voltage assignment ϕ acts transitively on the set $\{1, 2, \ldots, n\}$, (see [2]). Such a voltage assignment is said to be *transitive*. From now on, we assume that every voltage assignment is reduced and transitive.

A *signed graph* is a pair $G_\theta = (G, \theta)$ of a graph G and a function $\theta : E(G) \to \mathbb{Z}_2$, $\mathbb{Z}_2 = \{1, -1\}$. We call G the *underlying graph* of G_θ and θ the *signing* of G. An edge e of G_θ is said to be positive (resp. negative) if $\theta(e) = 1$ (resp. $\theta(e) = -1$). A signed graph G_θ is *balanced* if its vertex set can be partitioned into two partite sets such that a positive edge has the ends in the same partite set, and a negative edge has the ends in different partite sets. Notice that a signed graph G_θ is balanced if and only if every closed walk in G_θ contains even number of negative edges, i.e., the value $\prod_{e \in W} \theta(e) = 1$ for every closed walk W in G_θ.

A graph map $h : G_\theta \to H_{\theta'}$ between two signed graphs is a graph map between their underlying graphs G and H which preserves the signing of each edge, i.e., $\theta'(h(e)) = \theta(e)$ for each $e \in E(G)$. A signed graph $\tilde{G}_{\tilde{\theta}}$ is called a *covering* of another G_θ with projection $p : \tilde{G}_{\tilde{\theta}} \to G_\theta$ if $p : \tilde{G} \to G$ is a covering and $\theta \circ p = \tilde{\theta}$. Two coverings $p : \tilde{G}_{\tilde{\theta}} \to G_\theta$ and $p' : \tilde{G}'_{\tilde{\theta}'} \to G_\theta$ of a signed graph G_θ are *isomorphic* if the underlying coverings $p : \tilde{G} \to G$ and $p' : \tilde{G}' \to G$ of the graph G are isomorphic.

By a method similar to the construction of a covering graph in [2], one can see that every covering of a signed graph can be derived from a reduced voltage assignment ϕ of G. That is, for a covering $p : \tilde{G}_{\tilde{\theta}} \to G_\theta$, there exists a reduced voltage assignment ϕ of G such that the derived covering $p^\phi : G^\phi \to G$ is isomorphic to $\tilde{p} : \tilde{G} \to G$. We define a signing $\theta^\phi : E(G^\phi) \to \{-1, 1\}$ by $\theta^\phi(e_i) = \theta(e)$ for each $e_i \in E(G^\phi)$. Then, the covering $p^\phi : (G^\phi)_{\theta^\phi} \to G_\theta$ is isomorphic to $p : \tilde{G}_{\tilde{\theta}} \to G_\theta$.

In [11], Kwak et al. enumerate the isomorphism classes of regular balanced coverings of a signed graph and those of regular bipartite coverings of a graph [4–7,9]. As an application of the results, they also enumerate the isomorphism classes of regular branched orientable surface coverings of a non-orientable surface having a finite abelian covering transformation group. It gives a partial answer for the question raised by Liskovets in [13].

In Section 2, we discuss an enumeration method for the isomorphism classes of (regular or not) balanced coverings of an unbalanced graph. In Section 3, we enumerate the isomorphism classes of balanced coverings of a signed graph. As an application of our results, we enumerate the isomorphism classes of bipartite coverings over a non-bipartite graph G and in Section 4, we obtain an enumeration formula for the isomorphism classes of branched or unbranched orientable surface coverings of a non-orientable surface.

2. A characterization of balanced coverings

Let $\mathrm{Isoc}(G; n)$ denote the number of the isomorphism classes of connected n-fold coverings of a graph G. Similarly, $\mathrm{Isoc}(G_\theta; n)$ denotes the signed ones. Notice that $\mathrm{Isoc}(G; n) = \mathrm{Isoc}(G_\theta; n)$ for any signing θ on a graph G. Let $\mathrm{Isoc}^{\mathscr{B}}(G_\theta; n)$ denote the number of the isomorphism classes of connected balanced n-fold coverings of a signed graph G_θ.

Notice that if G_θ is balanced, then every covering of G_θ is also balanced, which gives $\mathrm{Isoc}^{\mathscr{B}}(G_\theta; n) = \mathrm{Isoc}(G_\theta; n) = \mathrm{Isoc}(G; n)$ for any natural number n. We also notice that every odd-fold covering of an unbalanced graph is also unbalanced. Therefore, we restrict our discussion to the even-fold coverings of an unbalanced graph G_θ. We observe that there are some analogous properties between the balancedness and bipartiteness. For example, a graph is bipartite if and only if every cycle has an even length, and analogously a signed graph is balanced if and only if every cycle contains an even number of negative edges. Some of such analogous properties are listed in Table 1. From the correspondence in Table 1, one can image that the enumeration method for the bipartite coverings of a graph used in [1] can be applied to derive some enumeration formulae for the balanced coverings of a signed graph.

Let G_θ be an unbalanced graph and let T be a fixed spanning tree of G. A cotree edge e in $E(G) - E(T)$ is said to be *unbalanced* (resp., *balanced*) if $T + e$ has an unbalanced

Table 1
A relationship

Graph G		Signed graph G_θ
Bipartiteness	$\longleftrightarrow$	balancedness
Even cycle	$\longleftrightarrow$	balanced cycle
Odd cycle	$\longleftrightarrow$	unbalanced cycle

(resp., balanced) cycle. Let $E_T^{\mathscr{B}}(G_\theta)$ (resp. $E_T^{\mathscr{U}}(G_\theta)$) denote the set of all balanced (resp. unbalanced) cotree edges in $E(G) - E(T)$. Let $\beta_{\mathscr{U}}(G,T) = |E_T^{\mathscr{U}}(G_\theta)|$ and $\beta_{\mathscr{B}}(G,T) = |E_T^{\mathscr{B}}(G_\theta)|$, so that $\beta(G) = \beta_{\mathscr{B}}(G,T) + \beta_{\mathscr{U}}(G,T)$.

For convenience, let $\mathscr{P}_{2n}$ be the set of all permutations in the symmetric group S_{2n} which preserve the parity in $\{1, 2, \ldots, 2n\}$, and let $\mathscr{R}_{2n}$ be the set of all permutations in S_{2n} which reverse the parity in $\{1, 2, \ldots, 2n\}$. Then $\mathscr{P}\mathscr{R}_{2n} := \mathscr{P}_{2n} \cup \mathscr{R}_{2n}$ is a subgroup of S_{2n} and $|\mathscr{P}\mathscr{R}_{2n}| = 2(n!)^2$.

Let $v : D(G) \to S_2 = \{1, \tau = (12)\}$ be a reduced voltage assignment defined by $v(e) = 1$ for each $e \in E_T^{\mathscr{B}}(G_\theta)$ and $v(e) = \tau$ for each $e \in E_T^{\mathscr{U}}(G_\theta)$. Then the derived double covering graph $(G^v)_{\theta^v}$ is balanced. By a method similar to the proof of Theorem 3.1 in [1], one can show that every balanced covering of the unbalanced graph G_θ is a covering of the double covering graph $(G^v)_{\theta^v}$ as a signed graph. More precisely, for any balanced covering $p : \tilde{G}_{\tilde{\theta}} \to G_\theta$ of G_θ there exists a covering map $q : \tilde{G}_{\tilde{\theta}} \to (G^v)_{\theta^v}$ such that $p = p^v \circ q$. We call $(G^v)_{\theta^v}$ the *canonical balanced double covering* of G_θ. Notice that $p^{-1}(v) = q^{-1}((p^v)^{-1}(v)) = q^{-1}(v_1) \cup q^{-1}(v_2)$. Now, by labelling $q^{-1}(v_1) = \{v_i : i = 1, 3, \ldots, 2n-1\}$ and $q^{-1}(v_2) = \{v_i : i = 2, 4, \ldots, 2n\}$, one can have the following theorem.

Theorem 1. *Let G_θ be an unbalanced graph and let T be a spanning tree of G. Let $p : \tilde{G}_{\tilde{\theta}} \to G_\theta$ be an even-fold (say, $2n$) connected balanced covering. Then there exists a reduced transitive voltage assignment $\phi : D(G) \to \mathscr{P}\mathscr{R}_{2n}$ such that $\phi(e) \in \mathscr{P}_{2n}$ if $e \in E_T^{\mathscr{B}}(G_\theta)$, $\phi(e) \in \mathscr{R}_{2n}$ if $e \in E_T^{\mathscr{U}}(G_\theta)$, and $p^\phi : (G^\phi)_{\theta^\phi} \to G_\theta$ is isomorphic to $p : \tilde{G}_{\tilde{\theta}} \to G_\theta$.*

Let $C_T^{\mathscr{B}}(G_\theta, 2n)$ denote the set of all reduced transitive voltage assignments $\phi : D(G) \to \mathscr{P}\mathscr{R}_{2n}$ such that $\phi(e) \in \mathscr{P}_{2n}$ if $e \in E_T^{\mathscr{B}}(G_\theta)$ and $\phi(e) \in \mathscr{R}_{2n}$ if $e \in E_T^{\mathscr{U}}(G_\theta)$.

Theorem 2. *Let G_θ be an unbalanced graph and let T be a fixed spanning tree of G. Let ϕ and ψ be two voltage assignments in $C_T^{\mathscr{B}}(G_\theta, 2n)$. Then $(G^\phi)_{\theta^\phi}$ and $(G^\psi)_{\theta^\psi}$ are isomorphic as coverings if and only if there exists a permutation $\omega \in \mathscr{P}\mathscr{R}_{2n}$ such that $\psi(e) = \omega \circ \phi(e) \circ \omega^{-1}$ for all $e \in D(G - T)$.*

Proof. It is clear that $(G^\phi)_{\theta^\phi}$ and $(G^\psi)_{\theta^\psi}$ are isomorphic if and only if G^ϕ and G^ψ are isomorphic as coverings. Since ϕ and ψ are reduced, by Theorem 2 in [8], G^ϕ and G^ψ are isomorphic as coverings if and only if there exists a permutation $\omega \in S_{2n}$ such that

$\psi(e)=\omega\circ\phi(e)\circ\omega^{-1}$ for all $e\in D(G-T)$. Since G^{ϕ} and G^{ψ} are connected and ϕ,ψ are elements of $C_T^{\mathscr{B}}(G_\theta,2n)$, such an ω should be an element of $\mathscr{PR}_{2n}$. This completes the proof. □

We observe that $C_T^{\mathscr{B}}(G_\theta,2n)$ can be identified with the set of all transitive $\beta(G)$-tuples $(\sigma_1,\ldots,\sigma_{\beta_{\mathscr{B}}(G,T)},\sigma_{\beta_{\mathscr{B}}(G,T)+1},\ldots,\sigma_{\beta(G)})$ in $(\mathscr{P}_{2n})^{\beta_{\mathscr{B}}(G,T)}\times(\mathscr{R}_{2n})^{\beta_{\mathscr{U}}(G,T)}$. Define an $\mathscr{PR}_{2n}$-action on $C_T^{\mathscr{B}}(G_\theta,2n)$ by

$$\begin{aligned}&\omega\cdot(\sigma_1,\ldots,\sigma_{\beta_{\mathscr{B}}(G,T)},\sigma_{\beta_{\mathscr{B}}(G,T)+1},\ldots,\sigma_{\beta(G)})\\&\quad=(\omega\sigma_1\omega^{-1},\ldots,\omega\sigma_{\beta_{\mathscr{B}}(G,T)}\omega^{-1},\omega\sigma_{\beta_{\mathscr{B}}(G,T)+1}\omega^{-1},\ldots,\omega\sigma_{\beta(G)}\omega^{-1})\end{aligned}$$

for any $\omega\in\mathscr{PR}_{2n}$. From Theorem 2 and the Burnside lemma, we have the following.

Corollary 3. *The number of the isomorphism classes of connected balanced* $2n$*-fold coverings of an unbalanced graph* G_θ *is equal to*

$$\mathrm{Isoc}^{\mathscr{B}}(G_\theta,2n)=\frac{1}{2(n!)^2}\sum_{\omega\in\mathscr{PR}_{2n}}|\mathrm{Fix}_\omega|,$$

where $\mathrm{Fix}_\omega=\{\phi\in C_T^{\mathscr{B}}(G_\theta,2n):\omega\cdot\phi=\phi\}$.

3. Enumeration of balanced coverings

In this section, we shall give an enumerating formula for $\mathrm{Isoc}^{\mathscr{B}}(G_\theta;2n)$. To do this, we start with the following.

With a fixed spanning tree *T*, let $C_T(G,n)$ denote the set of all reduced transitive voltage assignments $\phi:D(G)\to S_n$. For a convenience, we identify this set $C_T(G,n)$ with the set of all transitive $\beta(G)$-tuples $(\sigma_1,\ldots,\sigma_{\beta(G)})$ in $(S_n)^{\beta(G)}$, where the transitivity of $\beta(G)$-tuples means that the subgroup generated by $\{\sigma_1,\ldots,\sigma_{\beta(G)}\}$ acts transitively on the set $\{1,2,\ldots,n\}$. Let $\mathscr{S}_{\mathscr{A}}(n)$ denote the number of subgroups of index *n* in a group $\mathscr{A}$ and let $t_{m,\ell}$ denote the number of transitive ℓ-tuples in $(S_m)^{\ell}$. Then, the number of subgroups of index *n* in the fundamental group of the graph *G*, which is a free group $\mathscr{F}_{\beta(G)}$ on $\beta(G)$ elements, is equal to $|C_T(G,n)|/(n-1)!$, i.e., $\mathscr{S}_{\mathscr{F}_{\beta(G)}}(n)=t_{n,\beta(G)}/(n-1)!$, (see [12]).

Lemma 4. (1) *For any two natural numbers* $m,\ell\geqslant 1$, *we have*

$$t_{m,\ell}=(m!)^{\ell}-\sum_{j=1}^{m-1}\binom{m-1}{j-1}((m-j)!)^{\ell}\,t_{j,\ell},$$

where the summation over the empty index set is defined to be 0.

(2) *Let* G_θ *be an unbalanced graph. Then* $|C_T^{\mathscr{B}}(G_\theta,2n)|=n!\,t_{n,2\beta(G)-1}$.

Proof. (1) comes from the Hall's recursive formula in [3] for $\mathscr{S}_{\mathscr{F}_{\beta(G)}}(n)$ and the relation $\mathscr{S}_{\mathscr{F}_{\beta(G)}}(n)=t_{n,\beta(G)}/(n-1)!$.

(2) The number $\mathscr{S}_{\mathscr{F}_{2\beta(G)-1}}(n)$ can be considered in two ways: first, the equality $\mathscr{S}_{\mathscr{F}_{2\beta(G)-1}}(n) = t_{n,2\beta(G)-1}/(n-1)!$ is known already as in (1). On the other hand, this number is equal to the number of subgroups on index n in the fundamental group of the canonical double covering $(G^v)_{\theta^v}$, because $\beta(G^v) = 2\beta(G) - 1$. Furthermore, those subgroups of the fundamental group of $(G^v)_{\theta^v}$ are in one-to-one correspondence to the subgroups of index $2n$ in the fundamental group of G which correspond to the connected balanced $2n$-fold coverings of G by Theorem 1. Hence, we have the equality $\mathscr{S}_{\mathscr{F}_{2\beta(G)-1}}(n) = |C_T^{\mathscr{B}}(G_\theta, 2n)|/(n-1)!n!$ by noting that the number of elements in $\mathscr{P}\mathscr{R}_{2n}$ which fix the symbol 1 is $(n-1)!n!$. Now, a comparison of the two equalities gives the proof. □

Now, we aim to compute $|\mathrm{Fix}_\omega|$ for an $\omega \in \mathscr{P}\mathscr{R}_{2n}$. First, note that an ω has a fixed point if and only if ω is a regular permutation, which means by definition that ω is a product of disjoint $2n/\ell$ cycles of length ℓ. Moreover, a voltage assignment $\phi \in C_T^{\mathscr{B}}(G_\theta, 2n)$ is fixed by ω if and only if for each cotree arc e, $\phi(e)$ belongs to the centralizer $Z_{S_{2n}}(\omega)$ of ω in S_{2n}. We recall that ω is of type ℓ if, in the cycle decomposition of ω, every cycle has length ℓ. For such an ω, it is well-known that the centralizer $Z(\omega)$ of such an ω in S_{2n} can be represented as the wreath product $\mathbb{Z}_\ell \wr S_{2n/\ell}$, where $\mathbb{Z}_\ell$ is the cyclic group of order ℓ. Notice that each element $a \in \mathbb{Z}_\ell \wr S_{2n/\ell}$ can be written in the form $a = (c_1, c_2, \ldots, c_{2n/\ell}; \tilde{a})$, where $c_1, \ldots, c_{2n/\ell} \in \mathbb{Z}_\ell$ and $\tilde{a} \in S_{2n/\ell}$. For a $\phi \in \mathrm{Fix}_\omega$, so that $\phi(e) \in Z_{S_{2n}}(\omega)$ for each $e \in D(G) - D(T)$. For such a $\phi \in \mathrm{Fix}_\omega$, the function $\tilde{\phi} : D(G) \to S_{2n/\ell}$ define by $\tilde{\phi}(e) = \widetilde{\phi(e)}$ is also a voltage assignment.

First, let $\omega \in \mathscr{P}_{2n}$ and let ω be a product of disjoint $2n/\ell$ cycles of length ℓ for some $\ell|2n$. Since each of these cycles of ω consists of only either odd numbers or even numbers, $2n/\ell$ must be even and hence ℓ is a divisor of n. Moreover, $Z(\omega) \cap \mathscr{P}_{2n} = \mathbb{Z}_\ell \wr (S_{n/\ell} \wr \{1\})$, $Z(\omega) \cap \mathscr{R}_{2n} = \mathbb{Z}_\ell \wr (S_{n/\ell} \wr S_2) - \mathbb{Z}_\ell \wr (S_{n/\ell} \wr \{1\})$, and $Z(\omega) \cap \mathscr{P}\mathscr{R}_{2n} = (\mathbb{Z}_\ell \wr S_{n/\ell}) \wr S_2 = \mathbb{Z}_\ell \wr (S_{n/\ell} \wr S_2)$.

Let $\mathrm{Fix}^{\mathscr{P}}(\ell^m)$ denote the set of all reduced voltage assignments $\phi : D(G) \to \mathbb{Z}_\ell \wr (S_m \wr S_2)$ such that $\phi(e) \in \mathbb{Z}_\ell \wr (S_m \wr \{1\})$ if $e \in E_T^{\mathscr{B}}(G_\theta)$, $\phi(e) \in \mathbb{Z}_\ell \wr (S_m \wr S_2) - \mathbb{Z}_\ell \wr (S_m \wr \{1\})$ if $e \in E_T^{\mathscr{U}}(G_\theta)$, and $\tilde{\phi} \in C_T^{\mathscr{B}}(G_\theta, 2m)$. Recall that $C_T^{\mathscr{B}}(G_\theta, 2m)$ is the set of all reduced transitive voltage assignments ψ such that the $2m$-fold covering $G^{\psi}_{\theta^{\tilde{\phi}}}$ of G_θ is balanced. Let $\mathrm{Fix}^{\mathscr{P}}_{\ell^{n/\ell}}$ be the set of all voltage assignments $\phi \in C_T^{\mathscr{B}}(G_\theta, 2n)$ which is fixed by ω. Then, by a method similar to Liskovets [12], we have the following.

Lemma 5. *Let $\omega \in \mathscr{P}_{2n}$. Then $|\mathrm{Fix}_\omega| \neq 0$ if and only if ω is a regular permutation which is a product of disjoint $2n/\ell$ cycles of length ℓ for some $\ell|n$. For such an ω, we have*

$$|\mathrm{Fix}_\omega| = |\mathrm{Fix}^{\mathscr{P}}_{\ell^{n/\ell}}| = \left(\sum_{d|\ell} \mu(d)\, d^{n/\ell-1} \left(\frac{\ell}{d}\right)^{(n/\ell)(2\beta(G)-1)} t_{n/\ell, 2\beta(G)-1} \right) \left(\frac{n}{\ell}\right)!\ \ell^{n/\ell},$$

where μ is the Möbius function.

Next, let $\omega \in \mathscr{R}_{2n}$. Then ω is a product of disjoint $2n/\ell'$ cycles of length ℓ'. Since each cycle of ω reverses the parity, $\ell' = 2\ell$ for some divisor ℓ of n, i.e., ω is a product of disjoint n/ℓ cycles of length 2ℓ for some divisor ℓ of n. Moreover, $Z(\omega) \cap \mathscr{P}_{2n} = \langle 2 \rangle \wr S_{n/\ell}$, $Z(\omega) \cap \mathscr{R}_{2n} = (\mathbb{Z}_\ell - \langle 2 \rangle) \wr S_{n/\ell}$.

Let $\mathrm{Fix}^{\mathscr{R}}(\ell^m)$ denote the set of all reduced voltage assignments $\phi : D(G) \to \mathbb{Z}_{2\ell} \wr S_m$ such that $\phi(e) \in \langle 2\rangle \wr S_m$ if $e \in E_T^{\mathscr{B}}(G_\theta)$, $\phi(e) \in (\mathbb{Z}_\ell - \langle 2\rangle) \wr S_m$ if $e \in E_T^{\mathscr{U}}(G_\theta)$, and $\tilde{\phi} \in C_T(G; m)$ such that $G_\theta^{\tilde{\phi}}$ is unbalanced. Notice that the number $|\{\psi \in C_T(G; m) : G_\theta^{\psi}$ is balanced$\}|$ is 0 if m is odd and $(m-1)!t_{m/2,2\beta(G)-1}/(m/2-1)!$ if m is even. Let $\mathrm{Fix}^{\mathscr{R}}_{\ell^{n/\ell}}$ be the set of all voltage assignments $\phi \in C_T^{\mathscr{B}}(G_\theta, 2n)$ which is fixed by ω. Then, by a slight modification of the method in Liskovets [12], we have the following.

Lemma 6. *Let $\omega \in \mathscr{R}_{2n}$. Then $|\mathrm{Fix}_\omega| \neq 0$ if and only if ω is a regular permutation which is a product of disjoint n/ℓ cycles of length 2ℓ for some $\ell|n$. For such an ω, we have*

$$|\mathrm{Fix}_\omega| = |\mathrm{Fix}^{\mathscr{R}}_{\ell^{n/\ell}}| = \sum_{d|\ell, d:\mathrm{odd}} \mu(d)\, d^{(n/\ell)-1} \left(\frac{\ell}{d}\right)^{(n/\ell)\beta(G)} t^{-}_{n/\ell,\beta(G)},$$

where μ is the Möbius function and

$$t^{-}_{m,\beta(G)} = \begin{cases} t_{m,\beta(G)} & \text{if } m \text{ is odd}, \\ t_{m,\beta(G)} - (m-1)!\dfrac{t_{m/2,2\beta(G)-1}}{((m/2)-1)!} & \text{if } m \text{ is even}. \end{cases}$$

Now, we are ready to estimate the number $\mathrm{Isoc}^{\mathscr{B}}(G_\theta; 2n)$.

Theorem 7. *Let G_θ be an unbalanced graph. Then*

$$\mathrm{Isoc}^{\mathscr{B}}(G_\theta; 2n) = \frac{1}{2n} \sum_{\ell|n} \left(\sum_{d|(n/\ell)} \mu\left(\frac{n}{d\ell}\right) d^{2\ell(\beta(G)-1)+1} \frac{t_{\ell,2\beta(G)-1}}{(\ell-1)!} + \sum_{d|(n/\ell), n/(d\ell):\mathrm{odd}} \mu\left(\frac{n}{d\ell}\right) d^{\ell(\beta(G)-1)+1} \frac{t^{-}_{\ell,\beta(G)}}{(\ell-1)!} \right),$$

where μ is the Möbius function.

Proof. Let ω be a permutation which can be expressed as a product of disjoint $2n/\ell$ cycles of length ℓ, and ℓ is a divisor of n. Then, the number of such elements in $\mathscr{P}_{2n}$ is $(n!/(n/\ell)!\ell^{n/\ell})^2$. Let ω be a permutation which can be expressed as a product of disjoint n/ℓ cycles of length 2ℓ for a divisor ℓ of n. Then, the number of such elements in $\mathscr{R}_{2n}$ is $(n!)^2/(n/\ell)!\,\ell^{n/\ell}$. By Corollary 3, we have

$$\mathrm{Isoc}^{\mathscr{B}}(G_\theta; 2n) = \frac{1}{2(n!)^2} \left(\sum_{\ell|n} \left(\frac{n!}{(n/\ell)!\,\ell^{n/\ell}} \right)^2 |\mathrm{Fix}^{\mathscr{P}}_{\ell^{n/\ell}}| + \sum_{\ell|n} \frac{(n!)^2}{(n/\ell)!\,\ell^{n/\ell}} |\mathrm{Fix}^{\mathscr{R}}_{\ell^{n/\ell}}| \right).$$

Now, by Lemmas 5 and 6, we have the theorem. □

In Table 2, we list the numbers $\mathrm{Isoc}(G_\theta; 2n)$ and $\mathrm{Isoc}^{\mathscr{B}}(G_\theta; 2n)$ for small n and small $\beta = \beta(G)$.

Table 2
Two numbers $\mathrm{Isoc}(G_\theta; 2n)$ and $\mathrm{Isoc}^{\mathscr{B}}(G_\theta; 2n)$

β	2			4		
n	1	2	3	1	2	3
Isoc	3	26	624	15	14120	371515454
$\mathrm{Isoc}^{\mathscr{B}}$	1	5	24	1	71	23778

As a direct application of our results, one can enumerate the isomorphism classes of bipartite $2n$-fold connected coverings over a non-bipartite graph G.

Let G be a graph and let $\varpi : E(G) \to \{1, -1\}$ be a signing defined by $\varpi(e) = -1$ for all $e \in E(G)$. Then, G is bipartite if and only if G_ϖ is balanced. Hence, we have the following.

Corollary 8. *Let G be a connected non-bipartite graph. Then, for any n,* $\mathrm{Isoc}^{\mathscr{B}}(G_\varpi; n)$ *is equal to the number of the isomorphism classes of bipartite n-fold connected coverings of G.*

4. Applications to orientable branched coverings

In this section, with an aid of the enumeration formula for the connected balanced coverings of a signed graph, we enumerate the isomorphism classes of branched orientable surface coverings of a non-orientable surface. It gives an answer for the following question raised by Liskovets in [13]: count unramified orientable coverings of a non-orientable surface.

A *surface* means a compact connected 2-manifold $\mathbb{S}$ without boundary. A continuous function $p : \tilde{\mathbb{S}} \to \mathbb{S}$ from a surface $\tilde{\mathbb{S}}$ onto another $\mathbb{S}$ is called a *branched covering* if there exists a finite set B in $\mathbb{S}$ such that the restriction of p to $\tilde{\mathbb{S}} - p^{-1}(B)$, $p|_{\tilde{\mathbb{S}} - p^{-1}(B)} : \tilde{\mathbb{S}} - p^{-1}(B) \to \mathbb{S} - B$, is a covering projection in a usual sense. The smallest subset B of $\mathbb{S}$ which has this property is called the *branch set*. Two branched coverings $p : \tilde{\mathbb{S}} \to \mathbb{S}$ and $q : \tilde{\mathbb{S}}' \to \mathbb{S}$ are *isomorphic* if there exists a homeomorphism $h : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}}'$ such that $p = q \circ h$.

An *embedding* of a graph G into a surface $\mathbb{S}$ is a homeomorphism $i : G \to \mathbb{S}$ of G into $\mathbb{S}$. If every component of $\mathbb{S} - i(G)$, called a *region*, is homeomorphic to an open disk, then $i : G \to \mathbb{S}$ is called a 2-*cell* embedding. A *rotation system* ρ for a graph G is an assignment of a cycle permutation ρ_v on the neighborhood $N(v) = \{e \in D(G) : i_e = v\}$ to each $v \in V(G)$. An *embedding scheme* (ρ, θ) for a graph G consists of a rotation scheme ρ and a signing θ, where $\theta : E(G) \to \{1, -1\}$.

It is well-known that every embedding scheme for a graph G determines a 2-cell embedding of G into an orientable or non-orientable surface, and every 2-cell embedding of G into a surface is determined by such a scheme (see [15,16]). The sign $\theta(e)$ for an edge e depends on the twistedness or untwistedness of the collar-neighborhood of the edge e in a surface $\mathbb{S}$.

If an embedding scheme (ρ, θ) for a graph G determines a 2-cell embedding of G into a surface $\mathbb{S}$, then the orientability of $\mathbb{S}$ can be detected by looking at the balancedness of cycles of G_θ. In fact, $\mathbb{S}$ is orientable if and only if every cycle of G_θ is balanced.

Let $i : G \to \mathbb{S}$ be a 2-cell embedding with the embedding scheme (ρ, θ), and let $\phi : D(G) \to S_n$ be a voltage assignment. The derived graph G^ϕ has the *derived embedding scheme* (ρ^ϕ, θ^ϕ), which is defined by $(\rho^\phi)_{v_i}(e_i) = (\rho_v(e))_i$ and $\theta^\phi(e_i) = \theta(e)$ for each $e_i \in D(G^\phi)$. Then, the embedding scheme (ρ^ϕ, θ^ϕ) determines a 2-cell embedding $\tilde{i} : G^\phi \to \mathbb{S}^\phi$ of the derived graph G^ϕ into a surface, say $\mathbb{S}^\phi$. Moreover, if G^ϕ is connected, then $\mathbb{S}^\phi$ is connected. It is a fact [2] that the surface $\mathbb{S}^\phi$ is a branched n-fold covering of the surface $\mathbb{S}$, which is said to be *induced* by an embedding $i : G \to \mathbb{S}$ and an voltage assignment $\phi : D(G) \to S_n$. It is known [2] that every branched n-fold covering of a surface is isomorphic to a surface branched covering induced by a suitable 2-cell embedding of a graph with suitable voltage assignment on it.

Let N_k be a non-orientable surface with k crosscaps, and let $\mathfrak{B}_m$ denote the graph consisting of one vertex and m self-loops, say $\ell_1, \ldots, \ell_m$. We call it the *bouquet of m circles* or simply, a *bouquet*.

As in [7], we construct the *standard embedding* $\mathfrak{B}_{b+k} \hookrightarrow N_k - B$ of the bouquet $\mathfrak{B}_{b+k}$ into the non-orientable surface N_k, where $b = |B|$, as follows: an embedding scheme (ρ, θ) for the standard embedding $\mathfrak{B}_{k+b} \hookrightarrow N_k - B$ is defined by

$$\rho_v = (\ell_1 \ell_1^{-1} \ell_2 \ell_2^{-1} \ldots \ell_{k+b} \ell_{k+b}^{-1}),$$

where v is the vertex of the bouquet $\mathfrak{B}_{k+b}$, with the *standard signing*

$$\theta(\ell_s) = \begin{cases} -1 & \text{if } s = 1, 2, \ldots, k, \\ 1 & \text{if } s = k+1, k+2, \ldots, k+b. \end{cases}$$

For example, we consider the standard embedding $(\mathfrak{B}_3, \theta) \hookrightarrow N_2 - B$ with one branch point set B; its embedding scheme is given by $\rho_v = (\ell_1\ \ell_1^{-1}\ \ell_2\ \ell_2^{-1}\ \ell_3\ \ell_3^{-1})$, $\theta(\ell_1) = \theta(\ell_2) = -1$ and $\theta(\ell_3) = 1$. Note that the loops ℓ_1 and ℓ_2 are embedded as the polygonal boundary in the polygonal representation of the Klein bottle N_2, which are corresponded to the center lines of two Möbius bands attached to the sphere with 2 holes to make 2 crosscaps in the Klein bottle, as in Fig. 1.

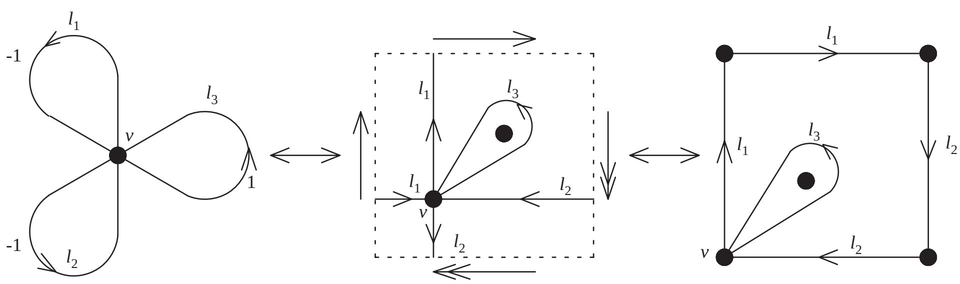

Fig. 1. An embedding scheme for $(\mathfrak{B}_3, \theta) \hookrightarrow N_2 - B$.

For a b-subset B of the surface N_k, let $\mathbf{B}^O((\mathfrak{B}_{b+k}, \theta) \hookrightarrow N_k - B; 2n)$ denote the set of voltage assignments $\phi : D(\mathfrak{B}_{b+k}) \to S_{2n}$ which satisfy the following two conditions:

(a) ϕ is transitive and $(\mathfrak{B}^{\phi}_{b+k}, \theta^{\phi})$ is balanced,
(b) $\phi(\ell_i) \neq 1$ for each $i = k+1, \ldots, k+b$ and $\prod_{i=1}^{k} \phi(\ell_i)\phi(\ell_i)\prod_{i=1}^{|B|} \phi(\ell_{k+i}) = 1$.

Note that condition (a) guarantees that the surface N_k^{ϕ} is connected and orientable, and condition (b) does that the set B is the same as the branch set of the branched covering $\tilde{p}^{\phi} : N_k^{\phi} \to N_k$. Furthermore, we have the following as in [10]: every voltage assignment in $\mathbf{B}^O((\mathfrak{B}_{b+k}, \theta) \hookrightarrow N_k - B; 2n)$ induces a connected branched orientable $2n$-fold covering of the surface N_k with branch set B. Conversely, every connected branched orientable $2n$-fold covering of N_k with branch set B can be derived from a voltage assignment in $\mathbf{B}^O((\mathfrak{B}_{b+k}, \theta) \hookrightarrow N_k - B; 2n)$. Moreover, for any two voltage assignments $\phi, \psi \in \mathbf{B}^O((\mathfrak{B}_{b+k}, \theta) \hookrightarrow N_k - B; 2n)$, two branched $2n$-fold surface coverings $\tilde{p}^{\phi} : N_k^{\phi} \to N_k$ and $\tilde{p}^{\psi} : N_k^{\psi} \to N_k$ are isomorphic if and only if two graph coverings $p^{\phi} : \mathfrak{B}^{\phi}_{b+k} \to \mathfrak{B}_{b+k}$ and $p^{\psi} : \mathfrak{B}^{\psi}_{b+k} \to \mathfrak{B}_{b+k}$ are isomorphic. Equivalently, there exists a permutation $\sigma \in S_{2n}$ such that $\psi(\ell_i) = \sigma\phi(\ell_i)\sigma^{-1}$ for all $\ell_i \in D(\mathfrak{B}_{b+k})$. It means that $\mathrm{Isoc}^O(N_k, B; 2n) = |\mathbf{B}^O((\mathfrak{B}_{b+k}, \theta) \hookrightarrow N_k - B; 2n)/S_{2n}|$.

Now, by using a method similar to the proof of Theorem 2 in [10], we have the following.

Theorem 9. *Let N_k be a non-orientable surface of genus k and let B be a b-subset of N_k. Then, every connected branched orientable covering of N_k must be even-fold, and for any n, the number of the isomorphism classes of connected branched orientable $2n$-fold coverings of N_k with branch set B is*

$$\mathrm{Isoc}^O(N_k, B; 2n) = (-1)^b \mathrm{Isoc}^O(N_k, \emptyset; 2n) + \sum_{t=0}^{b-1} (-1)^t \mathrm{Isoc}^{\mathscr{B}}((\mathfrak{B}_{b+k-t-1}, \theta); 2n),$$

where $\mathrm{Isoc}^O(N_k, \emptyset; 2n)$ *is the number of the isomorphism classes of connected unbranched $2n$-fold orientable coverings of N_k.*

Since the term $\mathrm{Isoc}^{\mathscr{B}}((\mathfrak{B}_{b+k-t-1}, \theta); 2n)$ is already computed in Theorem 7, we need to compute only the term $\mathrm{Isoc}^O(N_k, \emptyset; 2n)$.

To simplify this computation, we introduce a new voltage set. Let $\mathbf{B}^O_{2n}(N_k)$ be the subset of $\mathbf{B}^O((\mathfrak{B}_{b+k}, \theta) \hookrightarrow N_k - \emptyset; 2n)$ consisting of all transitive voltage assignments $\phi : D(\mathfrak{B}_k) \to \mathscr{PR}_{2n}$ such that $\phi(\ell_i) \in \mathscr{R}_{2n}$ for each $i = 1, 2, \ldots, k$ and $\prod_{i=1}^{k} \phi(\ell_i)^2 = 1$. Then, by Theorem 2, $\mathbf{B}^O_{2n}(N_k)$ contains all representatives of connected orientable $2n$-fold coverings of N_k and

$$\mathrm{Isoc}^O(N_k, \emptyset; 2n) = |\mathbf{B}^O((\mathfrak{B}_{b+k}, \theta) \hookrightarrow N_k - \emptyset; 2n)/S_{2n}| = |\mathbf{B}^O_{2n}(N_k)/\mathscr{PR}_{2n}|.$$

Hence, the number $\mathrm{Isoc}^O(N_k, \emptyset; 2n)$ of the isomorphism classes of connected unbranched $2n$-fold orientable coverings of N_k is equal to the number of orbits of the coordinatewise conjugacy action of the group $\mathscr{PR}_{2n}$ on $\mathbf{B}^O_{2n}(N_k)$. In order to apply the Burnside lemma in this situation, we compute the number of fixed points of ω for each element $\omega \in \mathscr{P}\mathscr{R}_{2n}$. To do this, we need the following lemma.

Let $\mathscr{S}_k(m)$ denote the number of subgroups of index m in the fundamental group $\pi_1(N_k, *)$ of the non-orientable surface N_k of genus k and let $\mathscr{S}_k(m)^+$ denote the number of subgroups of index m in the fundamental group of N_k whose corresponding covering surfaces are orientable.

Lemma 10 (*Mednykh and Pozdnyakova [14]*). *Let k and m be any two natural numbers. Then*

$$\mathscr{S}_k(m) = m \sum_{s=1}^{m} \frac{(-1)^{s+1}}{s} \sum_{\substack{i_1+i_2+\cdots+i_s=m \\ i_1,i_2,\ldots,i_s \geqslant 1}} \beta_{i_1}\beta_{i_2}\ldots\beta_{i_s},$$

where

$$\beta_h = \sum_{\chi\in D_h} \left(\frac{h!}{f^{(\chi)}}\right)^{k-2},$$

D_h is the set of all irreducible representations of the symmetric group S_h, and $f^{(\chi)}$ is the degree of the representation χ, and

$$\mathscr{S}_k^+(m) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \mathscr{S}_{2(k-1)}\left(\frac{m}{2}\right) & \text{if } m \text{ is even.} \end{cases}$$

Let $\omega \in \mathscr{P}\mathscr{R}_{2n}$ such that $|\mathrm{Fix}_\omega| \neq 0$. If $\omega \in \mathscr{P}_{2n}$, then $Z(\omega) \cap \mathscr{P}_{2n} = \mathbb{Z}_\ell \wr (S_{n/\ell} \wr \{1\})$ and $Z(\omega) \cap \mathscr{R}_{2n} = \mathbb{Z}_\ell \wr (S_{n/\ell} \wr S_2) - \mathbb{Z}_\ell \wr (S_{n/\ell} \wr \{1\})$ for some $\ell|n$.

Let $\bar{F}(\ell^m)$ be the set of all transitive reduced voltage assignments $\phi : D(\mathfrak{B}_k) \to Z(\omega) \cap \mathscr{R}_{2n}$ such that $\tilde{\phi} \in B^O_{2n/\ell}(N_k)$. Then, by a method similar to that used in Section 3, one can see that

$$|\mathrm{Fix}_\omega| = \sum_{d|\ell} \mu(d)\ d^{2n/\ell-1}\, |\bar{F}((\ell/d)^{n/\ell})|.$$

Now, by Theorem 3.1 in [14] and the fact that $|\mathbf{B}^O_{2n}(N_k)|/(n-1)!\,n! = \mathscr{S}^+_k(2n) = \mathscr{S}_{2(k-1)}(n)$, one can have

$$|\bar{F}((\ell/d)^{n/\ell})| = \left(\frac{\ell}{d}\right)^{(2n/\ell)(k-1)+1} \left(\frac{n}{\ell} - 1\right)!\, \frac{n}{\ell}!\, \mathscr{S}_{2(k-1)}\left(\frac{n}{\ell}\right).$$

Hence, we have the following.

Lemma 11. *Let $\omega \in \mathscr{P}_{2n}$. Then $|\mathrm{Fix}_\omega| \neq 0$ if and only if ω is a regular permutation which is a product of disjoint $2n/\ell$ cycles of length ℓ for some $\ell|n$. For such an ω*

$$|\mathrm{Fix}_\omega| = \left(\sum_{d|\ell} \mu(d)\, d^{n/\ell-1} \left(\frac{\ell}{d}\right)^{(n/\ell)(2k-3)+1} \left(\frac{n}{\ell} - 1\right)!\, \mathscr{S}_{2(k-1)}\left(\frac{n}{\ell}\right)\right) \left(\frac{n}{\ell}\right)!\, \ell^{n/\ell},$$

where μ is the Möbius function.

Let $\omega \in \mathscr{R}_{2n}$. Then $Z(\omega) \cap \mathscr{R}_{2n} = (\mathbb{Z}_{2\ell} - \langle 2 \rangle) \wr S_{n/\ell}$. Let $\tilde{F}(\ell^m)$ be the set of all transitive reduced voltage assignments $\phi : D(G) \to \mathbb{Z}_{2\ell} \wr S_m$ such that $\phi(e) \in (\mathbb{Z}_{2\ell} - \langle 2 \rangle) \wr S_m$, and $\tilde{\phi} \in C_N^1(k; m)$, where $C_N^1(k; m)$ is the set of all transitive voltage assignments ϕ which induce non-orientable coverings. Then, by a method similar to that used in Section 3, one can see that

$$|\mathrm{Fix}_\omega| = \sum_{d|\ell, d:\mathrm{odd}} \mu(d) d^{n/\ell - 1} |\tilde{F}((\ell/d)^{n/\ell})|.$$

Now, by a slight modification of the proof of Theorem 3.1 in [14] and the fact that $|C_N^1(k; m)| = (m-1)! \left(\mathscr{S}_k(n/\ell) - \mathscr{S}_k^+(n/\ell)\right)$, one can have

$$|\tilde{F}((\ell/d)^{n/\ell})| = \left(\frac{\ell}{d}\right)^{(n/\ell)(k-1)} \left(\frac{n}{\ell} - 1\right)! \times \frac{1 + (-1)^{(kn/\ell)(\ell/d - 1)}}{2} \gcd\left(2, \frac{\ell}{d}\right) \left(\mathscr{S}_k\left(\frac{n}{\ell}\right) - \mathscr{S}_k^+\left(\frac{n}{\ell}\right)\right).$$

Hence, we have the following.

Lemma 12. *Let $\omega \in \mathscr{R}_{2n}$. Then, $|\mathrm{Fix}_\omega| \neq 0$ if and only if ω is a regular permutation which is a product of disjoint n/ℓ cycles of length 2ℓ for some $\ell | n$. For such an ω,*

$$|\mathrm{Fix}_\omega| = \sum_{d|\ell, d:\mathrm{odd}} \mu(d)\, d^{n/\ell - 1} \left(\frac{\ell}{d}\right)^{(n/\ell)(k-1)} \left(\frac{n}{\ell} - 1\right)! \times \frac{1 + (-1)^{(kn/\ell)((\ell/d) - 1)}}{2} \gcd\left(2, \frac{\ell}{d}\right) \left(\mathscr{S}_k\left(\frac{n}{\ell}\right) - \mathscr{S}_k^+\left(\frac{n}{\ell}\right)\right),$$

where μ is the Möbius function.

Now, by the Burnside lemma, we have the following.

Theorem 13. *Let N_k be a non-orientable surface of genus k. Then, every connected unbranched orientable covering of N_k must be even-fold, and for any n, the number of the isomorphism classes of connected unbranched $2n$-fold orientable coverings of N_k is*

$$\mathrm{Isoc}^O(N_k, \emptyset; 2n) = \frac{1}{2n} \sum_{\ell | n} \left(\sum_{d|(n/\ell)} \mu\left(\frac{n}{d\ell}\right) d^{2\ell(k-2)+2} \mathscr{S}_{2(k-1)}(\ell) + \sum_{d|(n/\ell), n/(d\ell):\mathrm{odd}} \mu\left(\frac{n}{d\ell}\right) d^{\ell(k-2)+1} \times \frac{1 + (-1)^{k\ell(d-1)}}{2} \gcd(2, d) \left(\mathscr{S}_k(\ell) - \mathscr{S}_k^+(\ell)\right) \right),$$

where μ is the Möbius function, and $\mathscr{S}_k(\ell)$ and $\mathscr{S}_k^+(\ell)$ are the numbers mentioned in Lemma 10.

Table 3
The number $\mathrm{Isoc}^O(N_k, \emptyset; 2n)$

n	N_1	N_2	N_3	N_4
1	1	1	1	1
2	0	3	9	39
3	0	3	57	1483
4	0	6	847	354009
5	0	4	15303	208211284

For example, if *n* is a prime *p*, then the number of the isomorphism classes of connected unbranched $2p$-fold orientable coverings of N_k is

$$\mathrm{Isoc}^O(N_k, \emptyset; 2p) = \frac{1}{2p}\begin{cases} \mathscr{S}_{2(k-1)}(2) + \mathscr{S}_k(2) + 2^{2k-2} \\ \quad +2^{k-1}(1+(-1)^k) - 2 & \text{if } p = 2, \\ \mathscr{S}_{2(k-1)}(p) + \mathscr{S}_k(p) + p^{2k-2} \\ \quad +p^{k-1} - 2 & \text{otherwise}. \end{cases}$$

In particular, the number of the isomorphism classes of connected unbranched $2p$-fold orientable coverings of the Klein bottle N_2 is $\mathrm{Isoc}^O(N_2, \emptyset; 2p) = \lceil (p+3)/2 \rceil$. In Table 3, we compute the number $\mathrm{Isoc}^O(N_k, \emptyset; 2n)$ for small *n* and *k*.

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