Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®
European Journal of Combinatorics 25 (2004) 437–455
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Distributions of regular branched surface coverings

I.P. Goulden[a], Jin Ho Kwak[b], Jaeun Lee[c]

[a] *Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*
[b] *Combinatorial and Computational Mathematics Center, Pohang University of Science and Technology, Pohang 790–784, Republic of Korea*
[c] *Mathematics, Yeungnam University, Kyongsan 712–749, Republic of Korea*

Received 22 March 2002; accepted 18 November 2002

Abstract

In a study of surface branched coverings, one can ask naturally: *In how many different ways can a given surface be a branched covering of another given surface?* This problem was studied by many authors in Quart. J. Math. Oxford Ser. 2 46 (1995) 485, Math. Scand. 84 (1999) 23, Discrete Math. 156 (1996) 141, Discrete Math. 183 (1998) 193, Discrete Math. (in press), European J. Combin. 22 (2001) 1125, Sibirsk. Mat. Zh. 25 (1984) 606 etc. In this paper, as a complete answer to the question for regular coverings, we determine the distribution of the regular branched coverings of any nonorientable surface $\mathbb{S}$ when the covering transformation group and a set of branch points are freely assigned.

MSC: 05C10; 05A15; 57M12; 20C15

Keywords: Surface branched coverings; Distribution; Character

1. Introduction

Throughout this paper, a surface $\mathbb{S}$ means a compact connected 2-manifold without boundary, which is homeomorphic to one of the following:

$$\mathbb{S}_k = \begin{cases} \text{the orientable surface with } k \text{ handles} & \text{if } k \geq 0, \\ \text{the nonorientable surface with } -k \text{ crosscaps} & \text{if } k < 0, \end{cases}$$

whose Euler characteristic $\chi(\mathbb{S}_k)$ is $2 - 2k$ if $k \geq 0$ and $2 + k$ if $k < 0$.

A continuous function $p : \tilde{\mathbb{S}} \to \mathbb{S}$ from a surface $\tilde{\mathbb{S}}$ onto another $\mathbb{S}$ is called a *branched covering* if there exists a finite subset B of $\mathbb{S}$ such that the restriction of p to $\tilde{\mathbb{S}} - p^{-1}(B)$,

E-mail address: julee@yu.ac.kr (J. Lee).

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2002.11.001

$p|_{\tilde{\mathbb{S}}-p^{-1}(B)} : \tilde{\mathbb{S}} - p^{-1}(B) \to \mathbb{S} - B$, is a covering projection in the usual sense. The smallest subset B of $\mathbb{S}$ which has this property is called the *branch set*.

Let $\mathcal{A}$ be a finite group. For any *action* of $\mathcal{A}$ on a topological space X, a *fixed point* of the action is a point $x \in X$ such that $gx = x$ for some nonidentity element g of $\mathcal{A}$. An action with no fixed points is called *free* and an action with a finite number of fixed points is called *pseudofree*. A branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is *regular* if there exists a (finite) group $\mathcal{A}$ which acts pseudofreely on $\tilde{\mathbb{S}}$ so that the surface $\mathbb{S}$ is homeomorphic to the quotient space $\tilde{\mathbb{S}}/\mathcal{A}$, say by h, and the quotient map $\tilde{\mathbb{S}} \to \tilde{\mathbb{S}}/\mathcal{A}$ is the composition $h \circ p$ of p and h. We call it simply a *branched $\mathcal{A}$-covering*. In this case, the group $\mathcal{A}$ becomes the covering transformation group of the branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$. Two branched coverings $p : \tilde{\mathbb{S}} \to \mathbb{S}$ and $q : \tilde{\mathbb{S}}' \to \mathbb{S}$ are *equivalent* if there exists a homeomorphism $h : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}}'$ such that $p = q \circ h$.

Mednykh [12] obtained a formula for the number of nonequivalent coverings over an arbitrary compact Riemann surface with prescribed ramification type. This gives a complete answer for the question when covering surfaces are orientable.

Kwak et al. [7] introduced a polynomial $R_{(\mathbb{S},B,\mathcal{A})}(x)$, called a *branched covering distribution polynomial*, for a surface $\mathbb{S}$ defined as follows: for a finite group $\mathcal{A}$,

$$R_{(\mathbb{S},B,\mathcal{A})}(x) = \sum_{i=-\infty}^{\infty} a_i(\mathbb{S}, B, \mathcal{A})x^i,$$

where $a_i(\mathbb{S}, B, \mathcal{A})$ denotes the number of equivalence classes of branched $\mathcal{A}$-coverings $p : \mathbb{S}_i \to \mathbb{S}$ with branch set B. This polynomial can have at most finitely many nonzero terms by the Riemann–Hurwitz equation: $\chi(\tilde{\mathbb{S}}) = |\mathcal{A}|\chi(\mathbb{S}) - \sum_{x\in B} \mathrm{def}_p(x)$ for a branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$, where $\mathrm{def}_p(x) = |\mathcal{A}| - |p^{-1}(x)|$ is called the *deficiency* of x.

The polynomial $R_{(\mathbb{S},B,\mathcal{A})}(x)$ has been computed in [7, 8, 11] when $\mathcal{A}$ is the cyclic group $\mathbb{Z}_p$, the dihedral group $\mathbb{D}_p$ of order $2p$, or the direct sum of m copies of the cyclic group $\mathbb{Z}_p$, here p is prime, respectively. Recently, Kwak et al. [9] computed the total number of equivalence classes of branched orientable $\mathcal{A}$-coverings of a nonorientable surface for any finite Abelian group $\mathcal{A}$.

For a finite group $\mathcal{A}$, Jones [6] enumerated the equivalence classes of the branched $\mathcal{A}$-coverings of any given surface according to the degrees of branch points. This enables us to determine the distribution of the branched $\mathcal{A}$-coverings of an orientable surface for any finite group $\mathcal{A}$. But, this does not work when the base is nonorientable because a covering surface of a nonorientable surface can be orientable. As a main result of this paper, we compute the branched covering distribution polynomial $R_{(\mathbb{S},B,\mathcal{A})}(x)$ for any nonorientable surface $\mathbb{S}$, any branch set B and any finite group $\mathcal{A}$.

2. A classification of regular branched coverings

Let $\mathfrak{B}_m$ denote the bouquet of m loops, say $\ell_1, \ldots, \ell_m$. Every loop of a graph $\mathfrak{B}_m$ gives rise to a pair of oppositely directed loops. A surface $\mathbb{S}_k$ can be represented by a $4k$-gon with identification data $\prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1}$ on its boundary if $k > 0$; $-2k$-gon with identification data $\prod_{s=1}^{-k} a_s a_s$ on its boundary if $k < 0$; and bigon with identification data aa^{-1} on its boundary if $k = 0$.

Let B be a finite subset of $\mathbb{S}_k$. For our purpose, we assume that $|B| > 0$ when $k = 0$. For a point $* \in \mathbb{S}_k - B$, the fundamental group $\pi_1(\mathbb{S}_k - B, *)$ of the punctured surface $\mathbb{S}_k - B$ with base point $*$ can be presented as follows:

$$\left\langle a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_{|B|};\ \prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1} \prod_{t=1}^{|B|} c_t = 1 \right\rangle \text{ if } k > 0;$$

$$\left\langle a_1, \ldots, a_{-k}, c_1, \ldots, c_{|B|};\ \prod_{s=1}^{-k} a_s a_s \prod_{t=1}^{|B|} c_t = 1 \right\rangle \qquad \text{if } k < 0;$$

$$\left\langle c_1, \ldots, c_{|B|};\ \prod_{t=1}^{|B|} c_t = 1 \right\rangle \qquad \text{if } k = 0.$$

We call it the *standard presentation* of the fundamental group $\pi_1(\mathbb{S}_k - B, *)$. For each $t = 1, 2, \ldots, |B|$, we take a simple closed curve based at $*$ lying in the face determined by the polygonal representation of the surface $\mathbb{S}_k$ so that it represents the homeotopy class of the generator c_t. Then, it induces a 2-cell embedding of a bouquet $\mathfrak{B}_m$ of m circles into the surface $\mathbb{S}_k$ such that the embedding has $|B|$1-sided regions and one $(|B|+4k)$-sided region if $k > 0$; $|B|$1-sided regions and one $(|B| - 2k)$-sided region if $k < 0$; and $|B|$1-sided regions and one $|B|$-sided region if $k = 0$, where m is the number of the generators of the corresponding fundamental group, that is, $m = 2k + |B|$ if $k \geq 0$ and $m = -k + |B|$ if $k < 0$. We call this embedding $\imath : \mathfrak{B}_m \to \mathbb{S}_k$ the *standard embedding,* and denote it by $\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B$.

Let us denote the set of directed loops of $\mathfrak{B}_m$ by $D(\mathfrak{B}_m)$ and identify the loops $\ell_1, \ldots, \ell_m$ considering as positively directed loops with the generators of the corresponding fundamental group $\pi_1(\mathbb{S}_k - B, *)$. Let $C^1(\mathfrak{B}_m; \mathcal{A})$ denote the set of all functions $\varphi : D(\mathfrak{B}_m) \to \mathcal{A}$ such that $\varphi(e^{-1}) = \varphi(e)^{-1}$ for each $e \in \mathfrak{B}_m$, where e^{-1} denotes the reverse loop to a directed loop e. We call an element of $C^1(\mathfrak{B}_m; \mathcal{A})$ an $\mathcal{A}$-voltage assignment of $\mathfrak{B}_m$. In fact, such an $\mathcal{A}$-voltage assignment of $\mathfrak{B}_m$ can be identified with an ordered m-tuple of elements in $\mathcal{A}$, so that one can assume $C^1(\mathfrak{B}_m; \mathcal{A}) = \mathcal{A} \times \cdots \times \mathcal{A}$, ($m$-times). Let $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$ denote the set of $\mathcal{A}$-voltage assignments φ of $\mathfrak{B}_m$ satisfying the following two conditions:

(C1) $\varphi(\ell_1), \ldots, \varphi(\ell_m)$ generate $\mathcal{A}$, and

(C2) if $k \geq 0$, then $\varphi(\ell_i) \neq id_{\mathcal{A}}$ for each $i = 2k+1, \ldots, 2k+|B| = m$ and

$$\prod_{i=1}^{k} \varphi(\ell_i)\varphi(\ell_{k+i})\varphi(\ell_i)^{-1}\varphi(\ell_{k+i})^{-1} \prod_{i=1}^{|B|} \varphi(\ell_{2k+i}) = 1,$$

if $k < 0$, then $\varphi(\ell_i) \neq id_{\mathcal{A}}$ for each $i = -k+1, \ldots, -k+|B| = m$ and

$$\prod_{i=1}^{-k} \varphi(\ell_i)\varphi(\ell_i) \prod_{i=1}^{|B|} \varphi(\ell_{-k+i}) = 1.$$

By using a method to construct surface branched coverings from a voltage assignment given in [1], Kwak et al. [7] obtained the following variant of the Hurwitz existence and classification of branched surface coverings.

Theorem 1 ([7] Existence and Classification of Regular Branched Coverings). (a) *For a finite group $\mathcal{A}$, every connected branched $\mathcal{A}$-covering of $\mathbb{S}_k$ with branch set B can be derived from a voltage assignment φ in $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$. (Such a covering is denoted by $\tilde{p}_\varphi : \mathbb{S}_k^\varphi \to \mathbb{S}_k$.)*

(b) *For any two voltage assignments φ and ψ in $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$, the branched $\mathcal{A}$-coverings $\tilde{p}_\varphi : \mathbb{S}_k^\varphi \to \mathbb{S}_k$ and $\tilde{p}_\psi : \mathbb{S}_k^\psi \to \mathbb{S}_k$ are equivalent if and only if there exists a group automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that $\psi(\ell_i) = \sigma(\varphi(\ell_i))$ for all $\ell_i \in D(\mathfrak{B}_m)$, where $m = 2k + |B|$ if $k \geq 0$ and $m = -k + |B|$ if $k < 0$.*

Remark. (1) The condition (C1) guarantees that the covering surface $\mathbb{S}_k^\varphi$ is connected, and the condition (C2) does that the set B is nothing but the branch set of the covering $\tilde{p}_\varphi : \mathbb{S}_k^\varphi \to \mathbb{S}_k$.

(2) Let $\mathrm{Aut}(\mathcal{A})$ denote the automorphism group of the group $\mathcal{A}$. Then, from Theorem 1(b), one can define naturally an $\mathrm{Aut}(\mathcal{A})$-action on the set $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$. It is free (no fixed point) because of the condition (C1).

(3) Let $\mathrm{Epi}(\pi_1(\mathbb{S}_k - B, *), \mathcal{A})$ denote the set of all epimorphisms from $\pi_1(\mathbb{S}_k - B, *)$ to $\mathcal{A}$. Then the set $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$ can be identified with the subset of $\mathrm{Epi}(\pi_1(\mathbb{S}_k - B, *), \mathcal{A})$ consisting of epimorphisms φ satisfying the condition that $\varphi(\ell_i) \neq id_{\mathcal{A}}$ as in (C2).

Notice that every branched covering surface of an orientable surface is orientable. But a branched covering surface of a nonorientable surface can be orientable or nonorientable. So, to compute the branched covering distribution polynomial $R_{(\mathbb{S}_k, B, \mathcal{A})}(x)$ for a nonorientable surface $\mathbb{S}_k$, it is necessary to compute the number of equivalence classes of its branched orientable coverings.

For $k < 0$, let φ be a voltage assignment in $C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$. It is known that the derived branched covering surface $\mathbb{S}_k^\varphi$ is orientable if and only if there exists a subgroup $\mathcal{S}$ of index 2 in $\mathcal{A}$ such that $\varphi(\ell_i) \in \mathcal{A} - \mathcal{S}$ for $i = 1, \ldots, -k$ and $\varphi(\ell_i) \in \mathcal{S}$ for $i = -k+1, \ldots, -k+|B|$ (see Theorem 4.1.5 in [2]). From this fact and Theorem 1, one can see that if φ and ψ are two voltage assignments in $C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$ which induce orientable covering surfaces and their corresponding subgroups of index 2 are $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, then two coverings $\tilde{p}_\varphi : \mathbb{S}_k^\varphi \to \mathbb{S}_k$ and $\tilde{p}_\psi : \mathbb{S}_k^\psi \to \mathbb{S}_k$ are equivalent if and only if there exists an automorphism σ on $\mathcal{A}$ such that $\sigma(\mathcal{S}_1) = \mathcal{S}_2$ and $\psi(\ell_i) = \sigma(\varphi(\ell_i))$ for $i = 1, \ldots, -k+|B|$.

Let $\mathcal{A}$ be a finite group and let $\mathcal{S}$ be a subgroup of index 2 in $\mathcal{A}$. For a finite subset B of a nonorientable surface $\mathbb{S}_k$, let $C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S}))$ denote the subset of $C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$ consisting of voltage assignments φ satisfying the condition:

(C3) $\varphi(\ell_1), \ldots, \varphi(\ell_{-k})$ belongs to $\mathcal{A} - \mathcal{S}$, while $\varphi(\ell_{-k+1}), \ldots, \varphi(\ell_{-k+|B|})$ belongs to $\mathcal{S}$ but not $id_{\mathcal{A}}$.

Then, one can say

$$C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S})) = \{\varphi \in \mathrm{Epi}(\pi_1(\mathbb{S}_k - B, *), \mathcal{A}) : \varphi \text{ satisfies (C3)}\},$$

which contains all representatives of branched orientable $\mathcal{A}$-coverings of a nonorientable surface $\mathbb{S}_k$ whose corresponding subgroup of index 2 is $\mathcal{S}$. We summarize our discussions as follows.

Theorem 2. *Let $\mathcal{A}$ be a finite group and let $k < 0$ so that the surface $\mathbb{S}_k$ is nonorientable. Then we have the following.*

(a) *Every connected branched orientable $\mathcal{A}$-covering of $\mathbb{S}_k$ with branch set B can be derived from a voltage assignment in $C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S}))$ for a subgroup $\mathcal{S}$ of index 2 in $\mathcal{A}$.*

(b) *For any two voltage assignments $\varphi \in C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S}_1))$ and $\psi \in C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S}_2))$, the branched $\mathcal{A}$-coverings $\tilde{p}_\varphi : \mathbb{S}_k^\varphi \to \mathbb{S}_k$ and $\tilde{p}_\psi : \mathbb{S}_k^\psi \to \mathbb{S}_k$ are equivalent if and only if there exists an automorphism σ on $\mathcal{A}$ such that $\sigma(\mathcal{S}_1) = \mathcal{S}_2$ and $\psi(\ell_i) = \sigma(\varphi(\ell_i))$ for all $\ell_i \in D(\mathfrak{B}_{-k+|B|})$.*

Clearly, if the group $\mathcal{A}$ does not have a subgroup of index 2, then every $\mathcal{A}$-covering of a nonorientable surface $\mathbb{S}_k$ is nonorientable. We say that two subgroups $\mathcal{S}_1$ and $\mathcal{S}_2$ of a group $\mathcal{A}$ are *similar* if there exists an automorphism σ on $\mathcal{A}$ such that $\sigma(\mathcal{S}_1) = \mathcal{S}_2$. Now, from Theorem 2, one can see that $\cup_{\mathcal{S}} C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S}))$ contains all representatives of branched orientable $\mathcal{A}$-coverings of the nonorientable surface $\mathbb{S}_k$ whose branch sets are B, where $\mathcal{S}$ runs over all representatives of similarity classes of subgroups of index 2 in $\mathcal{A}$.

3. Polynomial $R_{(\mathbb{S}_k, B, \mathcal{A})}(x)$

From now on, we only consider nonorientable surface $\mathbb{S}_k$, that is, $k < 0$. Let $\mathcal{A}$ be a finite group and let $|B| = b$ for convenience. For a sequence of positive integers $\alpha_1, \alpha_2, \ldots, \alpha_b$ greater than 1, let $E_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b)$ denote the number of m-tuples $(x_1, \ldots, x_{-k}, y_1, \ldots, y_b)$ in $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$ such that the order $o(y_i)$ of y_i in $\mathcal{A}$ is α_i for $i = 1, \ldots, b$. That is,

$$\begin{aligned} &E_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b) \\ &\quad = |\{\varphi \in \mathrm{Epi}(\pi_1(\mathbb{S}_k - B, *), \mathcal{A}) : o(\varphi(c_i)) = \alpha_i \text{ for } i = 1, \ldots, b\}|. \end{aligned}$$

Let $E_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b)$ denote the number of $(-k + b)$-tuples $(x_1, \ldots, x_{-k}, y_1, \ldots, y_b)$ in $C^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S}))$ such that the order $o(y_i)$ of y_i is α_i for $i = 1, \ldots, b$. That is,

$$\begin{aligned} E_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b) = |\{\varphi \in \mathrm{Epi}(\pi_1(\mathbb{S}_k - B, *), \mathcal{A}) &: o(\varphi(c_i)) \\ &= \alpha_i \text{ for } i = 1, \ldots, b, \text{ and } \varphi \text{ satisfies (C3)}\}|. \end{aligned}$$

Clearly, $E_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b)$ or $E_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b)$ is zero if there exists an α_i which is not a divisor of the order of $\mathcal{A}$. Notice that the Euler characteristic of the surface $\mathbb{S}_k^\varphi$ is

$$\chi(\mathbb{S}_k^\varphi) = |\mathcal{A}| \left(\chi(\mathbb{S}_k) - b + \sum_{i=1}^{b} \frac{1}{o(\varphi(c_i))} \right)$$

for a voltage assignment φ in $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$, because the deficiency of a branch point is $|\mathcal{A}|(1 - \frac{1}{\alpha_i})$ for $i = 1, \ldots, b$. Let $\mathrm{Aut}(\mathcal{A}, \mathcal{S})$ be the set of all automorphisms on $\mathcal{A}$ which fixes the subgroup $\mathcal{S}$, i.e. $\sigma(\mathcal{S}) = \mathcal{S}$. Now, the following comes from Theorems 1 and 2.

Theorem 3. *Let $\mathcal{A}$ be a finite group and let B be a b-subset of a nonorientable surface $\mathbb{S}_k$. Then*

$$\begin{aligned} &a_i(\mathbb{S}_k, B, \mathcal{A}) \\ &= \begin{cases} \displaystyle\sum_{\mathcal{S}} \frac{1}{|\mathrm{Aut}(\mathcal{A},\mathcal{S})|} \sum_{f'(\alpha)=2(1-i)} E_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b) & \text{if } i \geq 0, \\ 0 & \text{if } k < i < 0, \\ \displaystyle\frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{f'(\alpha)=2+i} E_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b) & \\ \displaystyle\qquad - \sum_{\mathcal{S}} \frac{1}{|\mathrm{Aut}(\mathcal{A},\mathcal{S})|} \sum_{f'(\alpha)=2+i} E_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b) & \text{if } i \leq k, \end{cases} \end{aligned}$$

where $\mathcal{S}$ ranges over the representatives of similarity classes of subgroups of index 2 in $\mathcal{A}$ and $f'(\alpha) = |\mathcal{A}| \left(2 + k - b + \sum_{j=1}^{b} \frac{1}{\alpha_j}\right)$.

Now, we compute the numbers $E_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b)$ and $E_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b)$. Let $H_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b)$ denote the number of m-tuples $(x_1, \ldots, x_{-k}, y_1, \ldots, y_b)$ in $C^1(\mathfrak{B}_m; \mathcal{A})$ which satisfy the condition (C2) and the order $o(y_i)$ of y_i is α_i for $i = 1, \ldots, b$. Then $H_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b)$ is equal to the number of homeomorphisms from $\pi_1(\mathbb{S}_k - B)$ to $\mathcal{A}$ such that the image of the generator corresponding to the ith branch point is of order α_i for $i = 1, \ldots, b$. That is,

$$\begin{aligned} &H_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b) \\ &\quad = |\{\varphi \in \mathrm{Hom}(\pi_1(\mathbb{S}_k - B, *), \mathcal{A}) : o(\varphi(c_i)) = \alpha_i \text{ for } i = 1, \ldots, b\}|. \end{aligned}$$

It comes from Möbius inversion that

$$E_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b) = \sum_{K \leq \mathcal{A}} \mu(K)\, H_k(K; \alpha_1, \alpha_2, \ldots, \alpha_b),$$

where μ is the *Möbius function* for a group $\mathcal{A}$, which assigns an integer $\mu(K)$ to each subgroup K of $\mathcal{A}$ by the recursive formula

$$\sum_{H \geq K} \mu(H) = \delta_{K,\mathcal{A}} = \begin{cases} 1 \text{ if } K = \mathcal{A}, \\ 0 \text{ if } K < \mathcal{A}. \end{cases}$$

Similarly, let $H_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b)$ denote the number of homeomorphisms from $\pi_1(\mathbb{S}_k - B, *)$ to $\mathcal{A}$ such that the image of the generator corresponding to the ith branch point is of order α_i for $i = 1, \ldots, b$, and the image of a generator lies in $\mathcal{S}$ if and only if the generator corresponds to a branch point. That is,

$$\begin{aligned} H_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b) = |\{\varphi \in \mathrm{Hom}(\pi_1(\mathbb{S}_k - B, *), \mathcal{A}) : o(\varphi(c_i)) = \alpha_i \text{ for } i \\ = 1, \ldots, b, \text{ and } \varphi \text{ satisfies (C3)}\}|. \end{aligned}$$

Let $\mathcal{S}$ be a subgroup of index 2 in $\mathcal{A}$. A subgroup K of $\mathcal{A}$ is said to be a *subgroup of the pair* $(\mathcal{A}, \mathcal{S})$ if $K \cap \mathcal{S} \neq K$, denoted by $K \leq (\mathcal{A}, \mathcal{S})$. Then, an analogous argument gives that

$$E_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b) = \sum_{K \leq (\mathcal{A}, \mathcal{S})} \mu(K) H_k((K, K \cap \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b).$$

For computational convenience, we describe Theorem 3 in another way. Let

$$H_k(r, b, \mathcal{A}) = \sum_{g(\alpha)=r} H_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b);$$

$$H_k(r, b, (\mathcal{A}, \mathcal{S})) = \sum_{g(\alpha)=r} H_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b),$$

where $g(\alpha) = \sum_{j=1}^{b} \frac{|\mathcal{A}|}{\alpha_j}$. It implies that

$$\sum_{g(\alpha)=r} E_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b) = \sum_{K \leq \mathcal{A}, \frac{|\mathcal{A}|}{|K|} \mid r} \mu(K) H_k\left(\frac{|K|}{|\mathcal{A}|} r, b, K\right);$$

$$\sum_{g(\alpha)=r} E_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b) = \sum_{K \leq (\mathcal{A}, \mathcal{S}), \frac{|\mathcal{A}|}{|K|} \mid r} \mu(K) H_k\left(\frac{|K|}{|\mathcal{A}|} r, b, (K, K \cap \mathcal{S})\right),$$

which are the numbers of epimorphisms with the total deficiency $\sum_{b \in B} \mathrm{def}_p(b) = b\,|\mathcal{A}| - g(\alpha)$ in the respective cases. In the following result, we can rephrase Theorem 3.

Theorem 4. *Let $\mathcal{A}$ be a finite group and let B be a b-subset of a nonorientable surface $\mathbb{S}_k$. Then*

$$a_i(\mathbb{S}_k, B, \mathcal{A}) = \begin{cases} \sum_{\mathcal{S}} \frac{1}{|\mathrm{Aut}(\mathcal{A}, \mathcal{S})|} \sum_{K \leq (\mathcal{A}, \mathcal{S}), \frac{|\mathcal{A}|}{|K|} \mid \gamma'(i)} \mu(K)\, H_k\left(\frac{|K|}{|\mathcal{A}|}\gamma'(i), b, (K, K \cap \mathcal{S})\right) & \text{if } i \geq 0, \\ 0 & \text{if } k < i < 0, \\ \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{K \leq \mathcal{A}, \frac{|\mathcal{A}|}{|K|} \mid \gamma''(i)} \mu(K)\, H_k\left(\frac{|K|}{|\mathcal{A}|}\gamma''(i), b, K\right) & \\ - \sum_{\mathcal{S}} \frac{1}{|\mathrm{Aut}(\mathcal{A}, \mathcal{S})|} \sum_{K \leq (\mathcal{A}, \mathcal{S}), \frac{|\mathcal{A}|}{|K|} \mid \gamma''(i)} \mu(K)\, H_k\left(\frac{|K|}{|\mathcal{A}|}\gamma''(i), b, (K, K \cap \mathcal{S})\right) & \text{if } i \leq k, \end{cases}$$

where $\gamma'(i) = |\mathcal{A}|(b - k - 2) - 2(i - 1)$ and $\gamma''(i) = |\mathcal{A}|(b - k - 2) + i + 2$, and $\mathcal{S}$ ranges over the representatives of similarity classes of subgroups of index 2 in $\mathcal{A}$.

To complete the computation of $a_i(\mathbb{S}_k, B, \mathcal{A})$, one should determine the numbers $H_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b)$ and $H_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b)$ for any finite group $\mathcal{A}$ and its subgroup $\mathcal{S}$ of index 2. This will be carried out in the following section.

4. Enumeration of homeomorphisms

By using the computational method given in [5], Jones [6] expressed the number $H_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b)$ in terms of irreducible characters of $\mathcal{A}$ and computed it explicitly in some special cases. Throughout this section, let $\mathrm{Irr}(\mathcal{A})$ denote the set of all irreducible complex characters of the group $\mathcal{A}$, and for $\xi \in \mathrm{Irr}(\mathcal{A})$, let $\xi\langle\alpha_i\rangle = \sum_{g\in\mathcal{A},\, o(g)=\alpha_i} \xi(g)$.

Theorem 5 ([6]). *Let $\mathcal{A}$ be a finite group and let $k < 0$. Then*

$$H_k(\mathcal{A}; \alpha_1, \alpha_2, \ldots, \alpha_b) = |\mathcal{A}|^{-k-1} \sum_{\xi \in \mathrm{Irr}(\mathcal{A})} c_\xi^{-k} \xi(1)^{2+k-b} \xi\langle\alpha_1\rangle \cdots \xi\langle\alpha_b\rangle,$$

where c_ξ denotes the Frobenius–Schur indicator of ξ defined by $c_\xi = \frac{1}{|\mathcal{A}|}\sum_{g\in\mathcal{A}} \xi(g^2)$, and it is equal to

$$\begin{cases} 1 & \text{if } \rho \text{ is real}, \\ -1 & \text{if } \xi \text{ is real but } \rho \text{ is not real}, \\ 0 & \text{if } \xi \text{ is not real}, \end{cases}$$

and ρ is the representation corresponding to ξ.

Now, we aim to determine $H_k((\mathcal{A}, \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b)$ for $k < 0$ in terms of irreducible characters of $\mathcal{A}$ and $\mathcal{S}$. Let $\mathrm{C}_1 = \{1\}, \ldots, \mathrm{C}_k$ be the conjugacy classes of the group $\mathcal{A}$. Then, for each character ξ of $\mathcal{A}$, $\xi(g)$ is constant for all $g \in \mathrm{C}_i$, and we call this common value $\xi(i)$ for each $i = 1, 2, \ldots, k$. In the group algebra $\mathbb{C}[\mathcal{A}]$ over the complex field $\mathbb{C}$, let $\mathcal{C}_i = \sum_{x\in\mathrm{C}_i} x$ for each $i = 1, 2, \ldots, k$, and for each irreducible character $\xi \in \mathrm{Irr}(\mathcal{A})$, we define $e^{(\xi)} = \frac{\xi(1)}{|\mathcal{A}|}\sum_{i=1}^{k} \overline{\xi}(i)\mathcal{C}_i$. Then, one can show that

$$\mathcal{C}_i = |\mathrm{C}_i| \sum_{\xi \in \mathrm{Irr}(\mathcal{A})} \frac{\xi(i)}{\xi(1)} e^{(\xi)},$$

and $e^{(\xi)}e^{(\eta)} = e^{(\xi)}\delta_{\xi,\eta}$, forming mutually orthogonal central idempotents in the group algebra $\mathbb{C}[\mathcal{A}]$ (see e.g. [10], p. 147). These results allow us to describe Theorem 5 in a compact manner. Also, from these results, we have

$$\begin{aligned} \mathcal{C}_{i_1} \cdots \mathcal{C}_{i_m} &= |\mathrm{C}_{i_1}| \cdots |\mathrm{C}_{i_m}| \sum_{\xi \in \mathrm{Irr}(\mathcal{A})} \frac{\xi(i_1) \cdots \xi(i_m)}{\xi(1)^m} e^{(\xi)} \\ &= \frac{|\mathrm{C}_{i_1}| \cdots |\mathrm{C}_{i_m}|}{|\mathcal{A}|} \sum_{i=1}^{k} \left(\sum_{\xi \in \mathrm{Irr}(\mathcal{A})} \frac{\xi(i_1) \cdots \xi(i_m)\overline{\xi}(i)}{\xi(1)^{m-1}} \right) \mathcal{C}_i. \end{aligned}$$

Now, the following lemma comes from the fact that the set $\{g \in \mathcal{A} : o(g) = \alpha_i\}$ is a union of conjugacy classes of $\mathcal{A}$.

Lemma 1. *Let $\mathcal{A}$ be a finite group and let α_i be natural numbers, $i = 1, \ldots, b$. Then, we have the following equation in the group algebra $\mathbb{C}[\mathcal{A}]$:*

$$\left(\sum_{g\in\mathcal{A},O(g)=\alpha_1} g\right)\cdots\left(\sum_{g\in\mathcal{A},\,O(g)=\alpha_b} g\right) = \sum_{\xi\,\in\,\mathrm{Irr}(\mathcal{A})} \frac{\xi\langle\alpha_1\rangle\cdots\xi\langle\alpha_b\rangle}{\xi(1)^b}\, e^{(\xi)}$$
$$= \frac{1}{|\mathcal{A}|}\sum_{i=1}^{k}\left(\frac{\xi\langle\alpha_1\rangle\cdots\xi\langle\alpha_b\rangle\overline{\xi}(i)}{\xi(1)^{b-1}}\right)\mathcal{C}_i.$$

Let $\mathcal{S}$ be a subgroup of index 2 in $\mathcal{A}$ and fix an element $t \in \mathcal{A}-\mathcal{S}$. Then for each $\xi \in \mathrm{Irr}(\mathcal{S})$ the character ξ_t defined by $\xi_t(s) = \xi(tst^{-1})(s \in \mathcal{S})$ is also an irreducible character of $\mathcal{S}$. An irreducible character ξ of $\mathcal{S}$ is said to be of type 1 (or $\xi \in \Im_1(\mathcal{S})$) if ξ and ξ_t are distinct, and of type 2 (or $\xi \in \Im_2(\mathcal{S})$) otherwise. It is known [4, 10] that $\xi \in \Im_1(\mathcal{S})$ if and only if there exists an irreducible character $\hat{\xi}$ of $\mathcal{A}$ such that $\hat{\xi}_{\mathcal{S}} = \xi+\xi_t$, and that $\xi \in \Im_2(\mathcal{S})$ if and only if there exists an irreducible character $\hat{\xi}$ of $\mathcal{A}$ such that $\hat{\xi}_{\mathcal{S}} = \xi$, where $\hat{\xi}_{\mathcal{S}}$ is the restriction of $\hat{\xi}$ to $\mathcal{S}$.

Let $\mathrm{C}_1 = \{1\},\ldots,\mathrm{C}_h$ be the conjugacy classes in the subgroup $\mathcal{S}$. Notice that the union of two distinct conjugacy classes in $\mathcal{S}$ can be a conjugacy class in $\mathcal{A}$. Hence $\sum_{g\in\mathcal{A}} g^2 = \sum_{i=1}^{h} N_i\,\mathcal{C}_i$ is an element of the group algebra of $\mathcal{S}$, because $g^2 \in \mathcal{S}$ for each $g\in\mathcal{A}$. If $\xi \in \Im_1(\mathcal{S})$, then

$$c_{\hat{\xi}}\,|\mathcal{A}| = \sum_{g\in\mathcal{A}}\hat{\xi}(g^2) = \sum_{g\in\mathcal{A}}\xi(g^2) + \sum_{g\in\mathcal{A}}\xi_t(g^2) = 2\sum_{g\in\mathcal{A}}\xi(g^2),$$

and hence

$$|\mathcal{S}|\,c_{\hat{\xi}} = \sum_{g\in\mathcal{A}}\xi(g^2) = \sum_{i=1}^{h} N_i\,|\mathrm{C}_i|\,\xi(i).$$

Multiplying this equation by $\frac{1}{|\mathcal{S}|}\overline{\xi}(a)$ and summing over $\xi \in \Im_1(\mathcal{S})$, one can obtain

$$\sum_{i=1}^{h} N_i\frac{|\mathrm{C}_i|}{|\mathcal{S}|}\sum_{\xi\in\Im_1(\mathcal{S})}\overline{\xi}(a)\,\xi(i) = \sum_{\xi\in\Im_1(\mathcal{S})}\overline{\xi}(a)\,c_{\hat{\xi}}.$$

If $\xi \in \Im_2(\mathcal{S})$, then, by a similar computation, we have

$$2|\mathcal{S}|\,c_{\hat{\xi}} = |\mathcal{A}|\,c_{\hat{\xi}} = \sum_{g\in\mathcal{A}}\xi(g^2) = \sum_{i=1}^{h} N_i\,|\mathrm{C}_i|\,\xi(i),$$

and

$$\sum_{i=1}^{h} N_i\frac{|\mathrm{C}_i|}{|\mathcal{S}|}\sum_{\xi\in\Im_2(\mathcal{S})}\overline{\xi}(a)\,\xi(i) = 2\sum_{\xi\in\Im_2(\mathcal{S})}\overline{\xi}(a)\,c_{\hat{\xi}}.$$

Hence,

$$\sum_{i=1}^{h} N_i\frac{|\mathrm{C}_i|}{|\mathcal{S}|}\sum_{\xi\in\mathrm{Irr}(\mathcal{S})}\overline{\xi}(a)\,\xi(i) = \sum_{\xi\in\Im_1(\mathcal{S})}\overline{\xi}(a)\,c_{\hat{\xi}} + 2\sum_{\xi\in\Im_2(\mathcal{S})}\overline{\xi}(a)\,c_{\hat{\xi}}.$$

Since $\frac{|\mathrm{C}_i|}{|\mathcal{S}|}\sum_{\xi\in\mathrm{Irr}(\mathcal{S})}\overline{\xi}(a)\,\xi(i)=\delta_{a,i}$, we have

$$N_a=\sum_{\xi\in\Im_1(\mathcal{S})}\overline{\xi}(a)\,c_{\hat{\xi}}+2\sum_{\xi\in\Im_2(\mathcal{S})}\overline{\xi}(a)\,c_{\hat{\xi}}$$

for each $a=1,\ldots,h$, and hence

$$\sum_{g\in\mathcal{A}}g^2=\sum_{i=1}^{h}\left(\sum_{\xi\in\Im_1(\mathcal{S})}\overline{\xi}(i)\,c_{\hat{\xi}}+2\sum_{\xi\in\Im_2(\mathcal{S})}\overline{\xi}(i)\,c_{\hat{\xi}}\right)\mathcal{C}_i.$$

Since $\mathcal{C}_i=|\mathrm{C}_i|\sum_{\xi\in\mathrm{Irr}(\mathcal{A})}\frac{\xi(i)}{\xi(1)}e^{(\xi)}$ and $\sum_{i=1}^{h}|\mathrm{C}_i|\,\xi(i)\,\overline{\eta}(i)=|\mathcal{S}|\,\delta_{\xi,\eta}$, we have

$$\sum_{g\in\mathcal{A}}g^2=|\mathcal{S}|\left(\sum_{\xi\in\Im_1(\mathcal{S})}\frac{c_{\hat{\xi}}}{\xi(1)}e^{(\xi)}+\sum_{\xi\in\Im_2(\mathcal{S})}2\frac{c_{\hat{\xi}}}{\xi(1)}e^{(\xi)}\right).$$

By a similar computation, we can see that

$$\sum_{g\in\mathcal{S}}g^2=|\mathcal{S}|\sum_{\xi\in\mathrm{Irr}(\mathcal{S})}\frac{c_{\xi}}{\xi(1)}e^{(\xi)}$$

and

$$\sum_{g\in\mathcal{A}-\mathcal{S}}g^2$$
$$=\sum_{g\in\mathcal{A}}g^2-\sum_{g\in\mathcal{S}}g^2=|\mathcal{S}|\left(\sum_{\xi\in\Im_1(\mathcal{S})}\frac{c_{\hat{\xi}}-c_{\xi}}{\xi(1)}e^{(\xi)}+\sum_{\xi\in\Im_2(\mathcal{S})}\frac{2c_{\hat{\xi}}-c_{\xi}}{\xi(1)}e^{(\xi)}\right).$$

We summarize our discussions as follows.

Lemma 2. *Let $\mathcal{A}$ be a finite group and let $\mathcal{S}$ be a subgroup of index 2 in $\mathcal{A}$. Then*

(a) $\sum_{g\in\mathcal{S}}g^2=|\mathcal{S}|\sum_{\xi\in\mathrm{Irr}(\mathcal{S})}\frac{c_\xi}{\xi(1)}e^{(\xi)}=\sum_{i=1}^{h}\left(\sum_{\xi\in\mathrm{Irr}(\mathcal{S})}c_\xi\,\overline{\xi}(i)\right)\mathcal{C}_i$.

(b) $\sum_{g\in\mathcal{A}-\mathcal{S}}g^2=|\mathcal{S}|\sum_{\xi\in\mathrm{Irr}(\mathcal{S})}\frac{d_\xi}{\xi(1)}e^{(\xi)}=\sum_{i=1}^{h}\left(\sum_{\xi\in\mathrm{Irr}(\mathcal{S})}d_\xi\,\overline{\xi}(i)\right)\mathcal{C}_i$,

where $d_\xi=c_{\hat{\xi}}-c_\xi$ if $\xi\in\Im_1(\mathcal{S})$ and $d_\xi=2c_{\hat{\xi}}-c_\xi$ if $\xi\in\Im_2(\mathcal{S})$.

Now, we are ready to compute the number $H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b)$. First, observe that $H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b)$ is equal to the coefficient of $\mathcal{C}_1=1$ in the element

$$\left(\sum_{g\in\mathcal{A}-\mathcal{S}}g^2\right)^{-k}\left(\sum_{g\in\mathcal{S},o(g)=\alpha_1}g\right)\cdots\left(\sum_{g\in\mathcal{S},\,o(g)=\alpha_b}g\right)$$

of the group algebra $\mathbb{C}[\mathcal{A}]$. By using Lemmas 1 and 2, and the fact that $e^{(\xi)}e^{(\eta)}=e^{(\xi)}\delta_{\xi,\eta}$, we have

$$\left(\sum_{g\in\mathcal{A}-\mathcal{S}} g^2\right)^{-k}\left(\sum_{g\in\mathcal{S},o(g)=\alpha_1} g\right)\cdots\left(\sum_{g\in\mathcal{S},\,o(g)=\alpha_b} g\right)$$
$$= |\mathcal{S}|^{-k}\sum_{\xi\,\in\,\mathrm{Irr}(\mathcal{S})}\left(\frac{d_\xi}{\xi(1)}\right)^{-k}\frac{\xi\langle\alpha_1\rangle\cdots\xi\langle\alpha_b\rangle}{\xi(1)^b}\,e^{(\xi)}$$
$$= |\mathcal{S}|^{-k-1}\sum_{i=1}^{h}\left(\sum_{\xi\,\in\,\mathrm{Irr}(\mathcal{S})}\frac{d_\xi^{-k}}{\xi(1)^{-k+b-1}}\,\xi\langle\alpha_1\rangle\cdots\xi\langle\alpha_b\rangle\overline{\xi}(i)\right)\mathcal{C}_i,$$

where the second equality comes from the definition of $e^{(\xi)}$. Now, by taking the coefficient of $\mathcal{C}_1 = 1$, we have the following theorem.

Theorem 6. *Let $\mathcal{A}$ be a finite group with a subgroup $\mathcal{S}$ of index 2 and let $k < 0$. Then*

$$H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b) = |\mathcal{S}|^{-k-1}\sum_{\xi\,\in\,\mathrm{Irr}(\mathcal{S})}\frac{d_\xi^{-k}}{\xi(1)^{-k+b-2}}\,\xi\langle\alpha_1\rangle\cdots\xi\langle\alpha_b\rangle,$$

where $d_\xi = c_{\hat{\xi}} - c_\xi$ if $\xi \in \Im_1(\mathcal{S})$ and $d_\xi = 2c_{\hat{\xi}} - c_\xi$ if $\xi \in \Im_2(\mathcal{S})$.

Notice that if $\mathcal{A} = \mathcal{S}\times\mathbb{Z}_2$, then every irreducible character of $\mathcal{S}$ is of type 2 and $c_{\hat{\xi}} = c_\xi$. Hence, $d_\xi = 2c_{\hat{\xi}} - c_\xi = c_\xi$ for each irreducible character ξ of $\mathcal{S}$.

Corollary 1. *Let $\mathcal{S}$ be any finite group and let $k < 0$. Then*

$$H_k((\mathcal{S}\times\mathbb{Z}_2,\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b) = H_k(\mathcal{S};\alpha_1,\alpha_2,\ldots,\alpha_b).$$

Notice that

$$\sum_{\alpha_i\,||\mathcal{A}|,\alpha_i\neq 1}\xi\langle\alpha_1\rangle\cdots\xi\langle\alpha_b\rangle z^{\frac{|\mathcal{A}|}{\alpha_1}+\cdots+\frac{|\mathcal{A}|}{\alpha_b}} = \left(\sum_{\alpha\,\|\mathcal{A}|,\alpha\neq 1}\xi\langle\alpha\rangle z^{\frac{|\mathcal{A}|}{\alpha}}\right)^b$$

for any fixed irreducible character ξ of a group $\mathcal{A}$.

For convenience, let $[z^r]\,f(z)$ be the coefficient of z^r in the polynomial $f(z)$. Now, the following comes from Theorems 5 and 6.

Corollary 2. *Let $\mathcal{A}$ be a finite group and let $k < 0$. Then we have*

$$H_k(r,b,\mathcal{A}) = [z^r]|\mathcal{A}|^{-k-1}\sum_{\xi\,\in\,\mathrm{Irr}(\mathcal{A})} c_\xi^{-k}\xi(1)^{2+k-b}\left(\sum_{\alpha\,\|\mathcal{A}|,\alpha\neq 1}\xi\langle\alpha\rangle z^{\frac{|\mathcal{A}|}{\alpha}}\right)^b$$

and for any subgroup $\mathcal{S}$ of index 2 in $\mathcal{A}$,

$$H_k(r,b,(\mathcal{A},\mathcal{S})) = [z^r]\,|\mathcal{S}|^{-k-1}\sum_{\xi\,\in\,\mathrm{Irr}(\mathcal{S})} d_\xi^{-k}\xi(1)^{2+k-b}\left(\sum_{\alpha\,\|\mathcal{S}|,\alpha\neq 1}\xi\langle\alpha\rangle z^{\frac{|\mathcal{A}|}{\alpha}}\right)^b,$$

where $d_\xi = c_{\hat{\xi}} - c_\xi$ if $\xi \in \Im_1(\mathcal{S})$ and $d_\xi = 2c_{\hat{\xi}} - c_\xi$ if $\xi \in \Im_2(\mathcal{S})$.

As the last part of this section, we compute the number $H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b)$ for an finite Abelian group $\mathcal{A}$. In this case, it is possible to find a formula for the number in terms of only $\xi \in \mathrm{Irr}(\mathcal{S})$ and c_ξ, but not involving $c_{\hat{\xi}}$. For a fixed element g in $\mathcal{A}-\mathcal{S}$, let

$$C_{\mathcal{S}}(g,\alpha_1,\alpha_2,\ldots,\alpha_b)=\{(x,y_1,\ldots,y_b)\in\mathcal{S}^{b+1}: x^2y_1\ldots y_b=g^2, o(y_i)=\alpha_i, \alpha_i\neq 1, i=1,\ldots,b\}.$$

For any irreducible character ξ of $\mathcal{S}$, it comes from the definition of c_ξ that $c_\xi \neq 0$ if and only if ξ takes real values. Then we can determine the cardinality of the set $C_{\mathcal{S}}(g,\alpha_1,\alpha_2,\ldots,\alpha_b)$ as follows:

Lemma 3. *Let $\mathcal{A}$ be a finite group and let $\mathcal{S}$ be a subgroup of index 2 in $\mathcal{A}$. Then, for a fixed element g in $\mathcal{A}-\mathcal{S}$, we have*

$$|C_{\mathcal{S}}(g,\alpha_1,\alpha_2,\ldots,\alpha_b)|=\sum_{\xi\in\mathrm{Irr}(\mathcal{S})} c_\xi\,\xi(1)^{1-b}\xi\langle\alpha_1\rangle\ldots\xi\langle\alpha_b\rangle\xi(g^2).$$

Let $\mathcal{A}$ be a finite Abelian group and let $\mathcal{S}$ be a subgroup of index 2 in $\mathcal{A}$. Then the number $H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b)$ is equal to the cardinality of the set

$$X=\{(x_1,\ldots,x_{-k},y_1,\ldots,y_b)\in(\mathcal{A}-\mathcal{S})^{-k}\times\mathcal{S}^b : x_1^2\ldots x_{-k}^2y_1\ldots y_b=1, o(y_i)=\alpha_i\}.$$

From this, we can see that if k is even, then the number $H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b)$ is equal to the cardinality of the set

$$Y=\{(x_1,\ldots,x_{-k-1},y,y_1,\ldots,y_b)\in(\mathcal{A}-\mathcal{S})^{-k-1}\times\mathcal{S}^{b+1} : y^2y_1\ldots y_b=1, o(y_i)=\alpha_i\},$$

and that if k is odd, then $H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b)$ is equal to the cardinality of the set

$$Z=\{(x_1,\ldots,x_{-k-1},y,y_1,\ldots,y_b)\in(\mathcal{A}-\mathcal{S})^{-k-1}\times\mathcal{S}^{b+1} : y^2y_1\ldots y_b=g^2, o(y_i)=\alpha_i\}$$

for a fixed element g in $\mathcal{A}-\mathcal{S}$. The correspondence between X and Y which sends $(x_1,\ldots,x_{-k},y_1,\ldots,y_b)$ to $(x_1,\ldots,x_{-k-1},x_1\ldots x_{-k},y_1,\ldots,y_b)$ is a bijection and the correspondence between X and Z which sends $(x_1,\ldots,x_{-k},y_1,\ldots,y_b)$ to $(x_1,\ldots,x_{-k-1},gx_1\ldots x_{-k},y_1,\ldots,y_b)$, $g\in\mathcal{A}-\mathcal{S}$, is a bijection. Hence,

$$H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b)=\begin{cases}|\mathcal{S}|^{-k-1}|C_{\mathcal{S}}(g,\alpha_1,\alpha_2,\ldots,\alpha_b)| & \text{if } k \text{ is odd,}\\ |\mathcal{S}|^{-k-1}H_1(\mathcal{S};\alpha_1,\alpha_2,\ldots,\alpha_b) & \text{if } k \text{ is even,}\end{cases}$$

where g is a fixed element in $\mathcal{A}-\mathcal{S}$. Now, the following comes from Theorem 5 and Lemma 3.

Theorem 7. *Let $k<0$ and let $\mathcal{A}$ be a finite Abelian group with a subgroup $\mathcal{S}$ of index 2. Then*

$$H_k((\mathcal{A},\mathcal{S});\alpha_1,\alpha_2,\ldots,\alpha_b) = \begin{cases} |\mathcal{S}|^{-k-1} \sum_{\xi\in \mathrm{Irr}(\mathcal{S})} c_\xi\, \xi\langle\alpha_1\rangle\ldots\xi\langle\alpha_b\rangle\xi(g^2) & \text{if } k \text{ is odd},\\ |\mathcal{S}|^{-k-1} \sum_{\xi\in \mathrm{Irr}(\mathcal{S})} c_\xi\, \xi\langle\alpha_1\rangle\ldots\xi\langle\alpha_b\rangle & \text{if } k \text{ is even}, \end{cases}$$

where g is a fixed element in $\mathcal{A}-\mathcal{S}$.

Corollary 3. *Let $\mathcal{A}$ be a finite Abelian group. Then we have*

$$H_k(r,b,\mathcal{A}) = [z^r]\begin{cases} |\mathcal{A}|^{2k-1} \sum_{\xi\in \mathrm{Irr}(\mathcal{A})} \left(\sum_{\alpha\,\|\mathcal{A}|,\alpha\neq 1} \xi\langle\alpha\rangle z^{\frac{|\mathcal{A}|}{\alpha}}\right)^b & \text{if } k\geq 0,\\ |\mathcal{A}|^{-k-1} \sum_{\xi\in \mathrm{Irr}(\mathcal{A})} c_\xi^{-k} \left(\sum_{\alpha\,\|\mathcal{A}|,\alpha\neq 1} \xi\langle\alpha\rangle z^{\frac{|\mathcal{A}|}{\alpha}}\right)^b & \text{if } k<0, \end{cases}$$

and for any subgroup $\mathcal{S}$ of index 2 in $\mathcal{A}$,

$$H_k(r,b,(\mathcal{A},\mathcal{S})) = [z^r]\begin{cases} |\mathcal{S}|^{-k-1} \sum_{\xi\in \mathrm{Irr}(\mathcal{S})} c_\xi\, \xi(g^2) \left(\sum_{\alpha\,\|\mathcal{S}|,\alpha\neq 1} \xi\langle\alpha\rangle z^{\frac{|\mathcal{A}|}{\alpha}}\right)^b & \text{if } k \text{ is odd},\\ |\mathcal{S}|^{-k-1} \sum_{\xi\in \mathrm{Irr}(\mathcal{S})} c_\xi \left(\sum_{\alpha\,\|\mathcal{S}|,\alpha\neq 1} \xi\langle\alpha\rangle z^{\frac{|\mathcal{A}|}{\alpha}}\right)^b & \text{if } k \text{ is even}, \end{cases}$$

where g is any fixed element in $\mathcal{A}-\mathcal{S}$.

5. Applications

In this section, as a demonstration of our computational formulas, we compute explicit formulas for the distribution polynomial when the covering transformation group $\mathcal{A}$ is the cyclic group $\mathbb{Z}_n$ of order n or the dihedral group $\mathbb{D}_n$ of order $2n$.

Notice that there are n irreducible characters ξ of $\mathbb{Z}_n$ which are obtained by mapping a generator of $\mathbb{Z}_n$ to an nth root of unity, i.e. each irreducible character of $\mathbb{Z}_n$ is a group homeomorphism from $\mathbb{Z}_n$ to the unit circle in the complex plane. If ξ maps a generator α of $\mathbb{Z}_n$ to the ath power of the primitive nth root of unity, then $\xi\langle\alpha\rangle$ is the sum of the ath power of the primitive αth roots of unity. This is a Ramanujan sum, so it comes from Theorem 272 in [3] that

$$\xi\langle\alpha\rangle = \mu\left(\frac{\alpha}{\gcd(a,\alpha)}\right)\frac{\phi(\alpha)}{\phi\left(\frac{\alpha}{\gcd(a,\alpha)}\right)},$$

where μ is the Möbius function and ϕ is Euler's function. From this, one can obtain

$$\sum_{\xi\in \operatorname{Irr}(\mathbb{Z}_n)}\left(\sum_{\alpha|n,\alpha\neq 1}\xi\langle\alpha\rangle z^{\frac{n}{\alpha}}\right)^b=\sum_{a=1}^{n}\left(\sum_{\alpha|n,\alpha\neq 1}\mu\left(\frac{\alpha}{\gcd(a,\alpha)}\right)\frac{\phi(\alpha)}{\phi\left(\frac{\alpha}{\gcd(a,\alpha)}\right)}z^{\frac{n}{\alpha}}\right)^b$$

$$=\sum_{a|n}\phi\left(\frac{n}{a}\right)\left(\sum_{\alpha|n,\alpha\neq 1}\mu\left(\frac{\alpha}{\gcd(a,\alpha)}\right)\frac{\phi(\alpha)}{\phi\left(\frac{\alpha}{\gcd(a,\alpha)}\right)}z^{\frac{n}{\alpha}}\right)^b.$$

Let ξ be an irreducible character of $\mathbb{Z}_n$. Then $c_\xi \neq 0$ if and only if ξ sends a generator of $\mathbb{Z}_n$ to 1 or to -1. So, if n is odd, then ξ is the principal character, and if n is even, ξ is either the principal character or the alternating character. In this case, $c_\xi = 1$. Moreover, $\mathbb{Z}_n$ has a normal subgroup of index 2 if and only if n is even. By Corollary 3 and this fact, for any $k < 0$, we have

$$H_k(r,b,\mathbb{Z}_n)$$
$$=[z^r]n^{-k-1}\begin{cases}\left(\sum_{\alpha|n,\alpha\neq 1}\phi(\alpha)z^{\frac{n}{\alpha}}\right)^b & \text{if } n \text{ is odd},\\ \left(\sum_{\alpha|n,\alpha\neq 1}\phi(\alpha)z^{\frac{n}{\alpha}}\right)^b+\left(\sum_{\alpha|\frac{n}{2},\alpha\neq 1}\phi(\alpha)z^{\frac{n}{\alpha}}-\sum_{\alpha\nmid\frac{n}{2},\alpha|n}\phi(\alpha)z^{\frac{n}{\alpha}}\right)^b & \text{if } n \text{ is even},\end{cases}$$

and

$$H_k(r,b,(\mathbb{Z}_n,\mathbb{Z}_{\frac{n}{2}}))=[z^r]\left(\frac{n}{2}\right)^{-k-1}$$
$$\times\begin{cases}0 & \text{if } n \text{ is odd},\\ \left(\sum_{\alpha|\frac{n}{2},\alpha\neq 1}\phi(\alpha)z^{\frac{n}{\alpha}}\right)^b & \text{if } n\equiv 2 \pmod 4,\\ \left(\sum_{\alpha|\frac{n}{2},\alpha\neq 1}\phi(\alpha)z^{\frac{n}{\alpha}}\right)^b+(-1)^{-k}\left(\sum_{\alpha|\frac{n}{4},\alpha\neq 1}\phi(\alpha)z^{\frac{n}{\alpha}}-\sum_{\alpha\nmid\frac{n}{4},\alpha|\frac{n}{2}}\phi(\alpha)z^{\frac{n}{\alpha}}\right)^b & \text{if } n\equiv 0 \pmod 4.\end{cases}$$

Notice that a subgroup of a cyclic group $\mathbb{Z}_n$ is also cyclic, say $\mathbb{Z}_m$ with $m \mid n$ and that $\mu(\mathbb{Z}_m) = \mu\left(\frac{n}{m}\right)$, and for a subgroup K of $\mathbb{Z}_n$, $K \leq (\mathbb{Z}_n, \mathbb{Z}_{\frac{n}{2}})$ if and only if K is $\mathbb{Z}_m$ with $\frac{n}{m}$ is odd. Moreover, $|\operatorname{Aut}(\mathbb{Z}_n)| = \phi(n) = |\operatorname{Aut}(\mathbb{Z}_n, \mathbb{Z}_{\frac{n}{2}})|$ for any n. Now, by applying Theorem 4, we can find an explicit form of the polynomial $R_{(\mathbb{S}_k,B,\mathbb{Z}_n)}(x)$ for any $\mathbb{S}_k$, B and n (see Table 1). (All computations were carried out using Maple.) When p is prime the polynomial $R_{(\mathbb{S}_k,B,\mathbb{Z}_p)}(x)$ can be found in [7].

Table 1
The polynomial $R_{(\mathbb{S}_{-1},B,\mathbb{Z}_n)}(x)$ for small n and small b

n	$b=2$	$b=3$	$b=4$
4	$4x^{-4}$	x^2+12x^{-6}	$24x^{-8}+16x^{-10}$
6	$2x^2+4x^{-4}+4x^{-6}$	$4x^4+6x^{-6}+24x^{-8}+24x^{-10}$	$8x^6+32x^{-10}+120x^{-12}+128x^{-14}+16x^{-16}$
7	$6x^{-7}$	$36x^{-13}$	$216x^{-19}$
8	$2x^2+8x^{-8}$	$3x^4+4x^6+24x^{-12}+48x^{-14}$	$4x^6+16x^8+48x^{-16}+192x^{-18}+192x^{-20}+128x^{-22}$
9	$4x^{-7}+6x^{-9}$	$12x^{-13}+36x^{-15}+36x^{-17}$	$32x^{-19}+144x^{-21}+288x^{-23}+216x^{-25}$
10	$4x^4+4x^{-6}+8x^{-10}$	$16x^8+6x^{-10}+48x^{-14}+96x^{-18}$	$64x^{12}+8x^{-16}+48x^{-18}+48x^{-20}+384x^{-22}+128x^{-24}+768x^{-26}+128x^{-28}$
12	$2x^2+4x^4+8x^{-10}+8x^{-12}$	$9x^6+18x^8+4x^{10}+36x^{-16}+84x^{-18}+96x^{-20}+48x^{-22}$	$4x^8+40x^{10}+80x^{12}+32x^{14}+96x^{-22}+336x^{-24}+672x^{-26}+880x^{-28}+768x^{-30}+448x^{-32}+128x^{-34}$

To compute the covering distribution polynomial $R_{(\mathbb{S}_k,B,\mathbb{D}_n)}(x)$ for a dihedral group $\mathbb{D}_n$, let $\mathbb{D}_n = \langle \rho, \tau : \rho^n = \tau^2 = 1,\ \tau\rho\tau^{-1} = \rho^{-1}\rangle$ and let $\zeta = \exp(\frac{2\pi i}{n})$, a primitive nth root of unity. If n is even, then there are $\frac{n}{2}+3$ irreducible characters of $\mathbb{D}_n$ (see [10, pp. 65–66]):

$$\begin{cases} \xi^{(r,s)}(\rho^u\tau^v) = (-1)^{ur+vs}\ (r,s=0,1): & \text{four linear characters,}\\ \xi^{(a)}(\rho^u) = \zeta^{au} + \overline{\zeta^{au}}\ (1 \le a \le \frac{n}{2}-1): & \frac{n}{2}-1 \text{ characters of dimension 2,}\\ \xi^{(t)}(\rho^u\tau) = 0. \end{cases}$$

If n is odd, then there are $\frac{n-1}{2}+2$ irreducible characters of $\mathbb{D}_n$:

$$\begin{cases} \xi^{(r)}(\rho^u\tau^v) = (-1)^{vr}\ (r=0,1): & \text{two linear characters,}\\ \xi^{(a)}(\rho^u) = \zeta^{au} + \overline{\zeta^{au}}\ (1 \le a \le \frac{n-1}{2}): & \frac{n-1}{2} \text{ characters of dimension 2.}\\ \xi^{(t)}(\rho^u\tau) = 0. \end{cases}$$

Notice that $c_\xi = 1$ for each irreducible character ξ of $\mathbb{D}_n$. From this and Corollary 2, we can see that for any $k < 0$ the number $H_k(r, b, \mathbb{D}_n)$ is equal to $(2n)^{-k-1}$ times the coefficient of z^r in the polynomial

$$\left(\sum_{\alpha|n,\alpha\neq 1}\phi(\alpha)z^{\frac{2n}{\alpha}}+nz^n\right)^b+\left(\sum_{\alpha|n,\alpha\neq 1}\phi(\alpha)z^{\frac{2n}{\alpha}}-nz^n\right)^b$$
$$+(1+(-1)^n)\left(\sum_{\alpha|\frac{n}{2},\alpha\neq 1}\phi(\alpha)z^{\frac{2n}{\alpha}}-\sum_{\alpha|n,\alpha\nmid\frac{n}{2}}\phi(\alpha)z^{\frac{2n}{\alpha}}\right)^b$$
$$+2^{2+k}\sum_{a=1}^{\lfloor\frac{n-1}{2}\rfloor}\left(\sum_{\alpha|n,\alpha\neq 1}\mu\left(\frac{\alpha}{\gcd(a,\alpha)}\right)\frac{\phi(\alpha)}{\phi\left(\frac{\alpha}{\gcd(a,\alpha)}\right)}z^{\frac{2n}{\alpha}}\right)^b.$$

Notice that if n is odd, then the cyclic subgroup $\mathbb{Z}_n$ of $\mathbb{D}_n$ is the unique subgroup of index 2. If n is even, then there are two equivalence classes of subgroups of index 2; the subgroup $\mathbb{Z}_n$ itself and the other consisting of two subgroups that are isomorphic to $\mathbb{D}_{\frac{n}{2}}$, say $\mathcal{S}_1$ and $\mathcal{S}_2$. We observe that an irreducible character ξ of $\mathbb{Z}_n$ is of type 2 if and only if ξ is the principal character or the alternating character. Let ξ be an irreducible character of $\mathbb{D}_{\frac{n}{2}}$ for even n. If $n \equiv 2 \pmod 4$, then ξ is of type 2. If $n \equiv 0 \pmod 4$, ξ is of type 2 if and only if $\xi \neq \xi^{(1,0)}$ and $\xi \neq \xi^{(1,1)}$. Now, the following comes from the definition of d_ξ. For an irreducible character ξ of $\mathbb{Z}_n$ or $\mathbb{D}_{\frac{n}{2}}$,

$$d_\xi=\begin{cases}0 & \text{if } (\xi=\xi^{(1,0)} \text{ or } \xi=\xi^{(1,1)}) \text{ and } n\equiv 0 \pmod 4.\\ 1 & \text{otherwise.}\end{cases}$$

By Corollary 2, for any $k < 0$ and any n, we have

$$H_k(r,b,(\mathbb{D}_n,\mathbb{Z}_n))$$
$$=[z^r]n^{-k-1}\sum_{a|n}\phi\left(\frac{n}{a}\right)\left(\sum_{\alpha|n,\alpha\neq 1}\mu\left(\frac{\alpha}{\gcd(a,\alpha)}\right)\frac{\phi(\alpha)}{\phi\left(\frac{\alpha}{\gcd(a,\alpha)}\right)}z^{\frac{2n}{\alpha}}\right)^b,$$

and

$$H_k(r,b,(\mathbb{D}_n,\mathbb{D}_{\frac{n}{2}}))=[z^r]n^{-k-1}$$
$$\times\left(\begin{array}{l}\left(\sum_{\alpha|\frac{n}{2},\alpha\neq 1}\phi(\alpha)z^{\frac{2n}{\alpha}}+\frac{n}{2}z^n\right)^b+\left(\sum_{\alpha|\frac{n}{2},\alpha\neq 1}\phi(\alpha)z^{\frac{2n}{\alpha}}-\frac{n}{2}z^n\right)^b\\ +2^{2+k}\sum_{a=1}^{\lfloor\frac{n-2}{4}\rfloor}\left(\sum_{\alpha|\frac{n}{2},\alpha\neq 1}\mu\left(\frac{\alpha}{\gcd(a,\alpha)}\right)\frac{\phi(\alpha)}{\phi\left(\frac{\alpha}{\gcd(a,\alpha)}\right)}z^{\frac{2n}{\alpha}}\right)^b\end{array}\right).$$

Notice that a subgroup of $\mathbb{D}_n$ is isomorphic to $\mathbb{Z}_m$ or $\mathbb{D}_m$ for some $m \mid n$. If n is odd, then there is only one subgroup of index 2 of $\mathbb{D}_n$ which is $\mathbb{Z}_n$, and if n is even, there are two equivalence classes of subgroups of index 2; the subgroup $\mathbb{Z}_n$ itself and the other consisting of two subgroups that are isomorphic to $\mathbb{D}_{\frac{n}{2}}$, say $\mathcal{S}_1$ and $\mathcal{S}_2$. For any n, a subgroup $\mathcal{K}$ of $\mathbb{D}_n$

is a subgroup of $(\mathbb{D}_n, \mathbb{Z}_n)$ if and only if $\mathcal{K}$ is a subgroup of type $\mathbb{D}_m$ with $m|n$. Moreover, the number of such subgroups is $\frac{n}{m}$, $|\mathrm{Aut}(\mathbb{D}_n, \mathbb{Z}_n)| = n\phi(n)$, and $\mu(\mathbb{D}_m) = \mu(\frac{n}{m})$ in the lattice of such subgroups. For even n, a subgroup $\mathcal{K}$ of $\mathbb{D}_n$ is a subgroup of $(\mathbb{D}_n, \mathcal{S}_1) = (\mathbb{D}_n, \mathbb{D}_{\frac{n}{2}})$ if and only if $\mathcal{K}$ is a subgroup of type $\mathbb{Z}_m$, a subgroup of type $\mathbb{D}_m$ for some $m|n$ with $\frac{n}{m}$ is odd, or a subgroup of type $\mathbb{D}_m$ of $\mathcal{S}_2$ with $\frac{n}{m}$ is even. Notice that if $\frac{n}{m}$ is odd there is only one subgroup of type $\mathbb{Z}_m$ and there are $\frac{n}{m}$ subgroups of type $\mathbb{D}_m$, and if $\frac{n}{m}$ is even there are $\frac{n}{2m}$ subgroups of type $\mathbb{D}_m$. Moreover, by considering the lattice structure of such subgroups, one can see that $\mu(\mathbb{Z}_m) = -\frac{n}{m}\mu(\frac{n}{m})$ and $\mu(\mathbb{D}_m) = \mu(\frac{n}{m})$. Notice that $|\mathrm{Aut}(\mathbb{D}_n, \mathbb{D}_{\frac{n}{2}})| = \frac{n}{2}\phi(n)$. Now, by applying Theorem 4, we can find an explicit form of the polynomial $R_{(\mathbb{S}_k, B, \mathbb{D}_n)}(x)$ for any $\mathbb{S}_k$, B and n. When p is prime the polynomial $R_{(\mathbb{S}_k, B, \mathbb{D}_p)}(x)$ can be found in [8].

6. Further remarks

For any nonorientable surface $\mathbb{S}_k$ and any finite group $\mathcal{A}$, the number $\mathbf{Isoc}^O(\mathbb{S}_k, B; \mathcal{A})$ of equivalence classes of connected branched orientable $\mathcal{A}$-coverings of $\mathbb{S}_k$ with branch set B is

$$\mathbf{Isoc}^O(\mathbb{S}_k, B; \mathcal{A}) = \sum_{i=0}^{\infty} a_i(\mathbb{S}_k, B, \mathcal{A}) = \sum_{\mathcal{S}} \frac{|C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S})|}{|\mathrm{Aut}(\mathcal{A}, \mathcal{S})|},$$

where $\mathcal{S}$ ranges over the representatives of similarity classes of subgroups of index 2 in $\mathcal{A}$. Notice that

$$|C^1(\mathfrak{B}_{-k+|B|} \hookrightarrow \mathbb{S}_k - B; (\mathcal{A}, \mathcal{S})| = \sum_{K \le (\mathcal{A}, \mathcal{S})} \mu(K) \left(\sum_{\alpha_i | \frac{|K|}{2}, \alpha_i \ne 1} H_k((K, K \cap \mathcal{S}); \alpha_1, \alpha_2, \ldots, \alpha_b) \right).$$

Now, the following corollary comes from Theorems 5 and 6.

Corollary 4. *Let $\mathcal{A}$ be a finite group and let B be a b-subset of a nonorientable surface $\mathbb{S}_k$. Then we have*

$$\mathbf{Isoc}^O(\mathbb{S}_k, B; \mathcal{A}) = \sum_{\mathcal{S}} \sum_{K \le (\mathcal{A}, \mathcal{S})} \frac{\mu(K)\,|K|^{-k-1}}{2^{-k-1}|\mathrm{Aut}(\mathcal{A}, \mathcal{S})|} \left(\left(\frac{|K|}{2} - 1 \right)^b + (-1)^b \sum_{\xi} d_{\xi}^{-k} \xi(1)^{2+k} \right),$$

where $\mathcal{S}$ ranges over the representatives of similarity classes of subgroups of index 2 in $\mathcal{A}$ and ξ does all irreducible characters of $K \cap \mathcal{S}$ except the principal character.

Corollary 5. *Let $\mathcal{A}$ be a finite Abelian group and let B be a b-subset of a nonorientable surface $\mathbb{S}_k$. Then we have*

$$\mathbf{Isoc}^O(\mathbb{S}_k, B; \mathcal{A}) = \begin{cases} \sum_{\mathcal{S}} \sum_{K \le (\mathcal{A},\mathcal{S})} \frac{\mu(K)\,|K|^{-k-1}}{2^{-k-1}|\mathrm{Aut}(\mathcal{A},\mathcal{S})|} \left(\left(\frac{|K|}{2} - 1\right)^b + (-1)^b \sum_{\xi} c_\xi \xi(g^2) \right) & \textit{if } k < 0 \textit{ and } k \textit{ is odd}, \\ \sum_{\mathcal{S}} \sum_{K \le (\mathcal{A},\mathcal{S})} \frac{\mu(K)\,|K|^{-k-1}}{2^{-k-1}|\mathrm{Aut}(\mathcal{A},\mathcal{S})|} \left(\left(\frac{|K|}{2} - 1\right)^b + (-1)^b \sum_{\xi} c_\xi \right) & \textit{if } k < 0 \textit{ and } k \textit{ is even}, \end{cases}$$

where $\mathcal{S}$ ranges over the representatives of similarity classes of subgroups of index 2 in $\mathcal{A}$, and ξ does all irreducible characters of $K \cap \mathcal{S}$ except the principal character, and g is a fixed element in $K - \mathcal{S}$.

We observe that the number $\mathbf{Isoc}^{OR}(\mathbb{S}_k, B; n)$ of equivalence classes of regular branched connected orientable n-fold coverings of a nonorientable surface $\mathbb{S}_k$ with branch set B is equal to

$$\mathbf{Isoc}^{OR}(\mathbb{S}_k, B; n) = \sum_{\mathcal{A}} \mathbf{Isoc}^O(\mathbb{S}_k, B; \mathcal{A}),$$

where $\mathcal{A}$ ranges over the representatives of isomorphism classes of groups of order n. Hence, we can express the numbers $\mathbf{Isoc}^{OR}(\mathbb{S}_k, B; n)$ in terms of irreducible characters of groups of order n and those of their subgroups of index 2. Moreover, if we know all groups of order n and all of their irreducible characters, then we can have an explicit formula for $\mathbf{Isoc}^{OR}(\mathbb{S}_k, B; n)$. For example, if $n = 2p$ (p is an odd prime), then there are two groups of order $2p$ up to isomorphisms; the cyclic group $\mathbb{Z}_{2p}$ and the dihedral group $\mathbb{D}_p$. By Corollaries 4 and 5 and this discussion, we have for any b-subset B of a nonorientable surface $\mathbb{S}_k$

$$\mathbf{Isoc}^{OR}(\mathbb{S}_k, B; 2p) = \begin{cases} \frac{2}{p-1}(p^{-k-1} - 1) & \text{if } b = 0, \\ p^{-k-2}((p-1)^{b-1}(p+1) + (-1)^b) & \text{if } b \neq 0. \end{cases}$$

Observe that for any two surfaces $\mathbb{S}_i$ and $\mathbb{S}_k$, the number $\mathbf{Isoc}^R(\mathbb{S}_k, B; \mathbb{S}_i; n)$ of equivalence classes of regular branched connected n-fold coverings $\rho : \mathbb{S}_i \to \mathbb{S}_k$ with branch set B is equal to

$$\mathbf{Isoc}^R(\mathbb{S}_k, B; \mathbb{S}_i; n) = \sum_{\mathcal{A}} a_i(\mathbb{S}_k, B, \mathcal{A}),$$

where $\mathcal{A}$ ranges over the representatives of isomorphism classes of groups of order n. This is an answer to the original question when the covering is regular. Now, by combining the results in Jones [6] and the results in this paper, we can obtain a complete answer. Notice that $\mathbf{Isoc}^R(\mathbb{S}_k, B; \mathbb{S}_i; n)$ is finite and constant for any finite subset B of $\mathbb{S}_k$ with the same cardinality.

For example, we consider the case $i = 3$ and $k = -2$. Assume that there is a branched $\mathcal{A}$-covering $\rho : \mathbb{S}_3 \to \mathbb{S}_{-2}$ with branch set B. Then, by the Riemann–Hurwitz equation, we can see that $1 \le |B| \le 2$. If $|B| = 1$, then $|\mathcal{A}| = 6$ or 8, and if $|B| = 2$, then $|\mathcal{A}| = 4$. Now, by the classification theorem for finite groups and our results, we have Table 2. In Table 2, the column entitled “the others” includes the dihedral group $\mathbb{D}_3$ of order 6, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Table 2
The number $a_3(\mathbb{S}_{-2}, B, \mathcal{A})$

$\mathcal{A}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{D}_4$	The others	Total
$\|B\| = 0$	0	0	0	0	0	0	0	0
$\|B\| = 1$	0	0	3	2	2	2	0	9
$\|B\| = 2$	2	2	0	0	0	0	0	4
$\|B\| \geq 3$	0	0	0	0	0	0	0	0
Total	2	2	3	2	2	2	0	13

and $\mathcal{Q}_8$ (the quaternion group) which are of order 8, as well as all groups whose order is not 4, 6 and 8. We conclude from Table 2, in the notation above, that

$$\mathbf{Isoc}^R(\mathbb{S}_{-2}, B; \mathbb{S}_3; n) = \begin{cases} 3 & \text{if } n = 6 \text{ and } |B| = 1, \\ 6 & \text{if } n = 8 \text{ and } |B| = 1, \\ 4 & \text{if } n = 4 \text{ and } |B| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Acknowledgements

Goulden was supported by NSERC, Jin Ho Kwak by Com^{2}MaC-KOSEF and Jaeun Lee by KRF(2000-015-DP0037). The last author gratefully thanks the University of Waterloo, which hosted him on his sabbatical leave during the time the research was completed.

References

[1] J.L. Gross, T.W. Tucker, Generating all graph coverings by permutation voltage assignments, Discrete Math. 18 (1977) 273–283.
[2] J.L. Gross, T.W. Tucker, Topological Graph Theory, John Wiley and Sons, New York, 1987.
[3] G.H. Hardy, E.M. Wright, Introduction to the Theory of Numbers, fifth ed., Oxford University Press, Oxford, 1979.
[4] G. James, M. Liebeck, Representations and Characters of Groups, Cambridge University Press, Cambridge, 1993.
[5] G.A. Jones, Enumeration of homeomorphisms and surface-coverings, Quart. J. Math. Oxford Ser. 2 46 (1995) 485–507.
[6] G.A. Jones, Counting subgroups of non-Euclidean crystallographic groups, Math. Scand. 84 (1999) 23–39.
[7] J.H. Kwak, S. Kim, J. Lee, Distributions of regular branched prime-fold coverings of surfaces, Discrete Math. 156 (1996) 141–170.
[8] J.H. Kwak, J. Lee, Distribution of branched $\mathbb{D}_p$-coverings of surfaces, Discrete Math. 183 (1998) 193–212.
[9] J.H. Kwak, J. Lee, S. Shin, Balanced coverings of a signed graph and some regular branched orientable surface coverings over a nonorientable surface, Discrete Math. (in press).
[10] W. Ledermann, Introduction to Characters, second ed., Cambridge University Press, Cambridge, 1987.
[11] J. Lee, J. Kim, Enumeration of the branched $m\mathbb{Z}_p$-coverings of surfaces, European J. Combin. 22 (2001) 1125–1138.
[12] A.D. Mednykh, Nonequivalent coverings of Riemann surfaces with a prescribed ramification type, Sibirsk. Mat. Zh. 25 (1984) 606–625.