Journal of Integer Sequences, Vol. 8 (2005),
Article 05.4.7

# On a Sequence Arising in Algebraic Geometry

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**Abstract**

We derive recurrence relations for the sequence of Maclaurin coefficients of the function $\chi = \chi(t)$ satisfying $(1 + \chi)\ln(1 + \chi) = 2\chi - t$.

[1]Supported by a Discovery Grant from NSERC
[2]Partly supported by ISF Grant 533-03
[3]Supported by a grant from the Gil'adi Foundation of the Israeli Ministry of Absorption

# 1 Introduction

Consider the function $\chi = \chi(t)$ satisfying

$$(1+\chi)\ln(1+\chi) = 2\chi - t \tag{1}$$

The sequence of coefficients in the Maclaurin expansion of $\chi$ plays an important role in algebraic geometry. Namely, the $n$-th coefficient is equal to the dimension of the cohomology ring of the moduli space of $n$-pointed stable curves of genus 0. These coefficients are also related to WDVV equations of physics. Exact definitions can be found in [4, 6, 7, 8] and references therein.

It follows from (1) that

$$\chi' := \frac{d\chi}{dt} = \frac{1+\chi}{1+t-\chi}, \tag{2}$$

and $\chi$ has the critical point $t = e-2$. Using this, Manin [7, Chap.4, p.194] provides for the coefficients in the Maclaurin expansion of $\chi$,

$$\chi(t) = t + \sum_{n=2}^{\infty} m_n \frac{t^n}{n!}, \tag{3}$$

the following expression:

$$m_n \sim \frac{1}{\sqrt{n}} \left( \frac{n}{e^2 - 2e} \right)^{n-\frac{1}{2}}. \tag{4}$$

Exact computation of the defined numbers is a challenging problem. Indeed, taking into account that

$$2\chi - (1+\chi)\ln(1+\chi) = \chi + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n} \chi^n,$$

and differentiating $n$ times the identity $t = t(\chi(t))$, we deduce from the Bruno formula [9, p.36, (45a)] that

$$m_n = \sum \frac{n!(-1)^j (j-2)!}{j_1! \cdots j_{n-1}!} \left(\frac{m_1}{1!}\right)^{j_1} \left(\frac{m_2}{2!}\right)^{j_2} \cdots \left(\frac{m_n}{(n-1)!}\right)^{j_{n-1}}, \quad n \geq 2, \tag{5}$$

where $m_1 = 1$, $j = j_1 + j_2 + \cdots + j_{n-1}$, and the sum is over all non-negative integral solutions to $j_1 + 2j_2 + \cdots + (n-1)j_{n-1} = n$. This allows recurrent computation of the numbers $m_n$. Indeed, by (5),

$$\begin{aligned} m_2 &= 0! m_1^2 = 1 \\ m_3 &= -1! m_1^3 + 0!(3m_1 m_2) = 2 \\ m_4 &= 2! m_1^4 - 1!(6m_1^2 m_2) + 0!(3m_2^2 + 4m_1 m_3) = 7 \\ &\vdots \end{aligned}$$

However, when $n$ increases, (5) becomes intractable due to the fast growth of the number of partitions of $n$.

Koganov [5] used the Bürmann-Lagrange inversion formula and generalizations of the Stirling numbers of the second kind [2], to deduce an efficient 3-dimensional scheme for computation of $m_n$'s. Here the Stirling numbers of the second kind of first and second order ($S_1(n,k)$ and $S_2(n,k)$) are defined by the two-dimensional recurrences:

$$S_1(n+1,k) = kS_1(n,k) + S_1(n,k-1),$$

$$n \geq k \geq 1, S_1(n,0) = \delta_{0,n}, S_1(n,1) = 1,$$
$$S_2(n+1,k) = kS_2(n,k) + nS_2(n-1,k-1),$$
$$n \geq k \geq 1, S_2(n,0) = \delta_{1,n}, S_2(n,1) = 1.$$

Then, [5],

$$m_n = 1 + (n-1)! \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} n(n+1)\cdots(n+k-1) \cdot$$

$$\sum_{q=0}^{n-1} \sum_{\ell=0}^{\min(k,q)} \frac{S_1(q+1,\ell+1)}{q!}(-2)^{k-\ell} \frac{S_2(n-1-q-(k-\ell),k-\ell)}{(n-1-q-(k-\ell))!}.$$

This made possible [5] computing the first 10 numbers $m_n$.

In what follows we present a simple computational method for $m_n$ based on a quadratic recurrence.

**Theorem 1.1** *The numbers $m_n$ satisfy*

$$m_n = \sum_{i=1}^{n-1} \binom{n-1}{i} m_i m_{n-i} - (n-2)m_{n-1}, \quad n \geq 2, \tag{6}$$

*with the initial condition $m_1 = 1$.*

**Proof** Multiplying both sides of (2) by $1 + t - \chi$ and rearranging, we obtain

$$\chi' = \chi\chi' + \chi - t\chi' + 1.$$

Applying (3) to this equation, we get

$$\sum_{n=1}^{\infty} m_n \frac{t^{n-1}}{(n-1)!} = \sum_{i=1}^{\infty} m_i \frac{t^i}{i!} \sum_{j=1}^{\infty} m_j \frac{t^{j-1}}{(j-1)!}$$

$$+ \sum_{n=1}^{\infty} m_n \frac{t^n}{n!} - \sum_{n=1}^{\infty} m_n \frac{t^n}{(n-1)!} + 1.$$

Equating the coefficients of $t^{n-1}/(n-1)!$ in this equation we accomplish the proof. □

## 2 A generalization

A natural generalization of the numbers $m_n$ is related to configuration spaces [3] and was introduced in [7, §4.3]. For an integer $k$, $k \geq 1$, consider the function $\chi_k = \chi_k(t)$ defined by

$$k(1+\chi_k)\ln(1+\chi_k) = (k+1)\chi_k - t, \tag{7}$$

for some fixed $k$. The previously considered $\chi$ thus coincides with $\chi_1$. Evidently,

$$\frac{d}{dt}\chi_k = \frac{1+\chi_k(t)}{1+t-k\chi_k(t)}, \tag{8}$$

and expanding at $t=0$ we get ,

$$\chi_k(t) = t + \sum_{n=2}^{\infty} m_n(k)\frac{t!}{n!}. \tag{9}$$

In particular, $m_1(k) = 1$. Using (8) analogously to the previous section we have the following generalization of Theorem 1.1.

**Theorem 2.1** *The numbers $m_n(k)$ are polynomials of degree $(n-1)$ in $k$, with integer coefficients defined by the recursion*

$$m_n(k) = k\sum_{i=1}^{n-1}\binom{n-1}{i} m_i(k)m_{n-i}(k) - (n-2)m_{n-1}(k), \quad n \geq 2, \tag{10}$$

*with initial condition $m_1(k) = 1$.* □

### 2.1 Coefficients of $m_n(k)$

Set

$$m_n(k) = \mu_1(n)k^{n-1} + \mu_2(n)k^{n-2} + \cdots + \mu_{n-1}(n)k + \mu_n(n). \tag{11}$$

Computation of the coefficients $\mu_n(n), \mu_{n-1}(n), \mu_{n-2}(n), \ldots$ is enabled by the following theorem.

**Theorem 2.2** *For $n \geq 2$ and $\ell = 1, \ldots, n$, the following recurrence holds:*

$$\mu_\ell(n) = \sum_{j=1}^{\ell}\sum_{i=1}^{n-1}\binom{n-1}{i}\mu_j(i)\mu_{\ell+1-j}(n-i) - (n-2)\mu_{\ell-1}(n-1). \tag{12}$$

**Proof** The relation (12) is obtained by equating coefficients of $k^{n-\ell}$ in equation (10). □

Using (12) for $\ell = n, n-1, \dots$ we calculate recursively

$$\begin{array}{rcr}
\mu_n(n) = & \delta_{n,1} & n \geq 1, \\
\mu_{n-1}(n) = & (-1)^n (n-2)! & n \geq 2, \\
\mu_{n-2}(n) = & (-1)^{n-1}(n-2)! \left(n-2+2\sum_{i=1}^{n-2} \frac{1}{i}\right) & n \geq 3, \\
\mu_{n-3}(n) = & (-1)^n (n-2)! \cdot & \\
& \cdot \left(\frac{1}{2}(n-3)(n+4) + (2n-7)\sum_{i=2}^{n-2} \frac{1}{i} + 6\sum_{i=2}^{n-2} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j}\right) & n \geq 4, \\
& \vdots &
\end{array}$$

Let us now describe a recurrence for computation of the initial coefficients $\mu_1(n), \mu_2(n), \dots$. Set

$$M_\ell(x) = \sum_{n=1}^{\infty} \mu_\ell(n) \frac{x^n}{n!}. \tag{13}$$

**Theorem 2.3** *Let* $M_n \equiv M_n(\frac{1}{2}(1-t^2)), \qquad t \geq 0$*. Then for* $n \geq 2$ *the following recursion holds:*

$$\frac{d}{dt}(tM_n) = \sum_{i=2}^{n-1} \left(\frac{d}{dt} M_i\right) M_{n+1-i} - tM_{n-1} - \frac{1}{2}(1-t^2)\frac{d}{dt} M_{n-1}, \tag{14}$$

*with initial conditions*

$$M_1 = 1-t, \qquad M_n|_{t=1} = 0. \tag{15}$$

**Proof** Multiply on both sides of (12) by $x^{n-1}/(n-1)!$, and sum over $n \geq 1$, to obtain the following system of equations for $M_\ell(x)$:

$$M_1'(x) = M_1(x)M_1'(x) + 1, \quad M_1(0) = 0, \tag{16}$$

$$M_\ell'(x) = \sum_{i=2}^{\ell-1} M_i'(x) M_{\ell-i+1}(x) + M_{\ell-1}(x) - xM_{\ell-1}'(x), M_\ell(0) = 0, \ell \geq 2. \tag{17}$$

From (16) we find

$$M_1(x) = \frac{1}{2} M_1^2(x) + x,$$

and

$$M_1(x) = 1 - \sqrt{1-2x} = 1-t \tag{18}$$

with $t = (1-2x)^{\frac{1}{2}}$. Finally, changing variables in (17) from $x$ to $t$, and using

$$M_1'(x) = t^{-1}, \qquad x = \frac{1}{2}(1-t^2), \qquad \frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} = -t^{-1}\frac{d}{dt},$$

we obtain (after multiplication by $-1$), for $n \geq 2$, the formula (14). $\square$

Notice that from Theorem 2.3 it follows by induction that for $n \geq 2$, $M_n$ is a polynomial in $t$ and $t^{-1}$ of the form:

$$M_n = \sum_{i=-(2n-3)}^{n} a_i(n) t^i. \tag{19}$$

Thus Theorem 2.3 recursively yields

$$\begin{aligned}
M_1 &= -t+1,\\
M_2 &= \frac{1}{6}t^2 - \frac{1}{2}t + \frac{1}{2} - \frac{1}{6}t^{-1},\\
M_3 &= \frac{1}{72}t^3 - \frac{1}{8}t + \frac{2}{9} - \frac{1}{8}t^{-1} + \frac{1}{72}t^{-3},\\
M_4 &= \frac{1}{270}t^4 - \frac{1}{144}t^3 - \frac{1}{72}t + \frac{1}{18} - \frac{1}{20}t^{-1} + \frac{1}{72}t^{-3} - \frac{1}{432}t^{-5},\\
M_5 &= \frac{23}{17280}t^5 - \frac{1}{270}t^4 + \frac{1}{576}t^3 + \frac{1}{405}t^2 - \frac{5}{1152}t + \frac{1}{90} - \frac{59}{4320}t^{-1}\\
&\quad + \frac{43}{5760}t^{-3} - \frac{5}{1728}t^{-5} + \frac{5}{10368}t^{-7},\\
&\vdots
\end{aligned}$$

Setting $(-1)!! = 1$, this easily implies

$$\begin{aligned}
\mu_1(n) &= (2n-3)!!, \quad n \geq 1,\\
\mu_2(n) &= -\frac{n-2}{3}(2n-3)!!, \quad n \geq 2,\\
\mu_3(n) &= \frac{(n-1)(n-2)(n-3)}{3^2}(2n-5)!!, \quad n \geq 2,\\
\mu_4(n) &= -\frac{(n-3)(n-4)(5(n-1)^2+1)}{3^4 \cdot 5}(2n-5)!!, \quad n \geq 3,\\
\mu_5(n) &= \frac{(n-3)(n-4)(n-5)(5(n-1)^3+4n-1)}{2 \cdot 3^5 \cdot 5}(2n-7)!!, \quad n \geq 3.\\
&\vdots
\end{aligned}$$

Finally, we will state a conjecture we have not been able to verify.

**Conjecture 1** *The expressions for $M_n$ do not contain monomials corresponding to the integral negative degrees of $(1-2x)$.*

This conjecture is confirmed by our calculations for $n \leq 5$.

## 2.2 Yet another property of $m_n(k)$

In this section we consider another combinatorial property of the polynomials $m_n(k)$.

**Theorem 2.4**

$$m_n(-1) = (1-n)^{n-1} \tag{20}$$

**Proof** Substitute $k = -1$ into (8), to obtain

$$\chi'_{-1} = -\chi_{-1}\chi'_{-1} + \chi_{-1} - t\chi'_{-1} + 1. \tag{21}$$

Now let

$$f(t) = \sum_{n=1}^{\infty} (1-n)^{n-1} \frac{t^n}{n!},$$

and note that, from Lagrange's Theorem as stated in [1, §1.2] we obtain

$$f(t) = -\frac{t}{T} - 1,$$

where $T = -te^T$. Differentiating the functional equation for $T$ with respect to $t$, we obtain

$$\frac{dT}{dt} = \frac{-e^T}{1+te^T} = \frac{T}{t(1-T)},$$

so that

$$\frac{df}{dt} = -\frac{T - tT'}{T^2} = \frac{1}{1-T},$$

and it is now routine to check that $f$ is a solution to (21). We conclude from the initial condition $f(0) = 0$ that $\chi_{-1}(t)$ coincides with $f(t)$. □

# 3 Numerical Calculation

The derived result allows extending sequence A074059 of Sloane's on-line Encyclopedia of Integer Sequences which previously contained only 5 terms. We give here the first 19 terms of the sequence:

$$\begin{aligned} m = \quad & \{1, 1, 2, 7, 34, 213, 1630, 14747, 153946, 1821473, \\ & 24087590, 352080111, 5636451794, 98081813581, \\ & 1843315388078, 37209072076483, 802906142007946, \\ & 18443166021077145, 449326835001457846, \ldots\} \end{aligned}$$

The first 10 polynomials $m_n(k)$ for $n = 1, \ldots, 10$, are given in the following table:

| $n$ | $m_n(k)$ |
|---|---|
| 1 | $1$ |
| 2 | $k$ |
| 3 | $3k^2 - k$ |
| 4 | $15k^3 - 10k^2 + 2k$ |
| 5 | $105k^4 - 105k^3 + 40k^2 - 6k$ |
| 6 | $945k^5 - 1260k^4 + 700k^3 - 196k^2 + 24k$ |
| 7 | $10395k^6 - 17325k^5 + 12600k^4 - 5068k^3 + 1148k^2 - 120k$ |
| 8 | $135135k^7 - 270270k^6 + 242550k^5 - 126280k^4 + 40740k^3 - 7848k^2 + 720k$ |
| 9 | $2027025k^8 - 4729725k^7 + 5045040k^6 - 3213210k^5 + 1332100k^4 - 363660k^3 + 61416k^2 - 5040k$ |
| 10 | $34459425k^9 - 91891800k^8 + 113513400k^7 - 85345260k^6 + 43022980k^5 - 15020720k^4 + 3584856k^3 - 541728k^2 + 40320k$ |


# Acknowledgement

The third author is grateful to L. M. Koganov for introducing the problem.


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2000 *Mathematics Subject Classification*: Primary 11Y55.
*Keywords:* cohomology rings of the moduli space, exponential generating functions, recurrences.

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(Concerned with sequence A074059.)

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Received July 15 2005; revised version received October 12 2005. Published in *Journal of Integer Sequences*, October 12 2005.

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