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Enumerative Properties of $NC^{(B)}(p, q)^*$

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Abstract. We determine the rank generating function, the zeta polynomial and the Möbius function for the poset $NC^{(B)}(p, q)$ of annular non-crossing partitions of type B, where p and q are two positive integers. We give an alternative treatment of some of these results in the case $q = 1$, for which this poset is a lattice. We also consider the general case of multiannular non-crossing partitions of type B, and prove that this reduces to the cases of non-crossing partitions of type B in the annulus and the disc.

Keywords: annular non-crossing partitions of type B, rank generating function, zeta polynomial, Möbius function

1. Introduction

The enumerative properties of the lattice $NC(n)$ of non-crossing partitions of $\{1, \ldots, n\}$ have been studied since the early 1970's, starting with the paper [8] of Kreweras. An important feature of this lattice is its connection to the symmetric group $\mathcal{S}_n$. More precisely, one has a natural poset isomorphism

$$NC(n) \simeq [\varepsilon, \alpha_n] := \{\tau \in \mathcal{S}_n \mid \varepsilon \leq \tau \leq \alpha_n\}, \tag{1.1}$$

where "$\leq$" is a natural partial order on $\mathcal{S}_n$, ε is the unit of $\mathcal{S}_n$, and α_n is the long cycle $(1, \ldots, n)$ (see [3,5]).

In 1997, Reiner [11] introduced the lattice $NC^{(B)}(n)$ of non-crossing partitions of type B. Soon after that (see [2,4,6]) it was noticed that one has a poset isomorphism analogous to the one from (1.1):

$$NC^{(B)}(n) \simeq [\varepsilon, \gamma_n] := \{\tau \in B_n \mid \varepsilon \leq \tau \leq \gamma_n\}, \tag{1.2}$$

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where "$\leq$" is a natural partial order on the hyperoctahedral group B_n, ε is the unit of B_n, and γ_n is the long cycle $(1,\ldots,n,-1,\ldots,-n)$. (Here B_n is viewed as the group of permutations τ of $\{1,\ldots,n\}\cup\{-1,\ldots,-n\}$ that satisfy the condition $\tau(-i)=-\tau(i)$, $1\leq i\leq n$.)

The recent paper [10] introduced a family of posets $NC^{(B)}(p,q)$, where p,q are two positive integers. One has a poset isomorphism

$$NC^{(B)}(p,q)\simeq[\varepsilon,\gamma_{p,q}]\subseteq B_{p+q}, \tag{1.3}$$

where the partial order on the hyperoctahedral group B_{p+q} is the same as in (1.2), and $\gamma_{p,q}$ is now the permutation with two cycles

$$\gamma_{p,q}:=(1,\ldots,p,-1,\ldots,-p)(p+1,\ldots,p+q,-(p+1),\ldots,-(p+q))\in B_{p+q}.$$

The elements of $NC^{(B)}(p,q)$ are certain partitions of the set $\{1,\ldots,p+q\}\cup\{-1,\ldots,-(p+q)\}$, and the partial order considered on $NC^{(B)}(p,q)$ is given by reverse refinement: $\pi\leq\rho$ if and only if every block of π is contained in a block of ρ. The distinctive feature of the partitions in $NC^{(B)}(p,q)$ is that one can draw them as non-crossing diagrams in an *annulus* with $2p$ points marked on its outside circle and $2q$ points marked on its inside circle. (This is unlike the diagrams drawn for partitions in $NC^{(B)}(n)$, which are drawn in a *disc* with $2n$ points marked on its boundary.) The poset $NC^{(B)}(p,q)$ is not generally a lattice, but we have a notable exception occurring in the case when $q=1$. In this case the meet operation coincides with the usual "intersection meet" for partitions — the blocks of the meet $\pi\wedge\rho\in NC^{(B)}(p,1)$ are precisely the non-empty intersections $A\cap B$ where A is a block of π and B is a block of ρ.

In the present paper, we determine the rank generating function, the zeta polynomial, and the Möbius function of the poset $NC^{(B)}(p,q)$. Here is how the paper is organized. In Section 2, we give a brief review of $NC^{(B)}(p,q)$ and its properties that are needed in the present paper. Then in Section 3, we discuss the special "lattice" case $q=1$, when the formulas for both the rank generating function and the Möbius function are nicer, and have simpler derivations. It is amusing to note that $NC^{(B)}(n-1,1)$ has the same rank generating function as $NC^{(B)}(n)$. Nevertheless, one has $NC^{(B)}(n-1,1)\not\simeq NC^{(B)}(n)$ for all $n\geq 3$, by looking at Möbius functions.

Section 4 is about the rank generating function of $NC^{(B)}(p,q)$ for general p,q. We observe that we still have nice formulas when we focus on partitions in $NC^{(B)}(p,q)$ that have a given connectivity (the *connectivity* of a partition $\pi\in NC^{(B)}(p,q)$ is the number of pairs of blocks $A,-A$ of π such that $A\neq -A$ and A intersects both sets $\{\pm1,\ldots,\pm p\}$ and $\{\pm(p+1),\ldots,\pm(p+q)\}$). But when we just enumerate the partitions in $NC^{(B)}(p,q)$ by their rank we get 1-parameter sums (which can be summed up to a "closed form" when $q=1$, but not for general q). A nice fact arising in our analysis, stated as Theorem 4.5.3, is that the total number of partitions in $NC^{(B)}(p,q)$ is given by

$$\left|NC^{(B)}(p,q)\right|=\frac{p+q+pq}{p+q}\cdot\binom{2p}{p}\binom{2q}{q}.$$

Section 5 is devoted to determining the Möbius function for $NC^{(B)}(p,q)$. The method used there is to count multichains via suitable "systems of parentheses", on the same lines that were used by Edelman [7] to count multichains in $NC(n)$ and then by Reiner [11] to count multichains in $NC^{(B)}(n)$. A benefit of this approach is that it also yields concrete formulas for the zeta polynomial for $NC^{(B)}(p,q)$, and for the number of maximal chains in $NC^{(B)}(p,q)$. The formulas obtained are again not in closed form, but (again) they can be summed up to closed form in the particular case when $q=1$.

In Section 6, we give a brief description of the general case of multiannular non-crossing partitions of type B. The main point of the section is to establish that, due to a topological restriction called the genus inequality, the general multiannular case reduces in fact to the cases of non-crossing partitions of type B in a disc or an annulus.

2. Review of $NC^{(B)}(p,q)$

In this section we review, following [10], a few basic facts about the poset $NC^{(B)}(p,q)$. We will start with a set $S_{nc}^{(B)}(p,q)$ of "annular non-crossing *permutations* of type B", and we will then define $NC^{(B)}(p,q)$ in terms of $S_{nc}^{(B)}(p,q)$.

Definition 2.1. (Partial order on B_{p+q} and the definition of $S_{nc}^{(B)}(p,q)$)
We will introduce $S_{nc}^{(B)}(p,q)$ via a natural partial order on the hyperoctahedral group B_{p+q}. Let us denote for convenience $n := p+q$. Recall that B_n is the group of permutations τ of $\{\pm 1,\dots,\pm n\}$ that satisfy the condition $\tau(-i) = -\tau(i)$, $\forall\, 1\leq i\leq n$.

1^o *We consider the following (non-minimal) set of n^2 generators of B_n:*

$$\{(i,j)(-i,-j) \mid 1\leq i,j\leq n,\ i\neq j\}\cup\{(i,-j)(-i,j) \mid 1\leq i,j\leq n,\ i\neq j\} \\ \cup\{(i,-i)\mid 1\leq i\leq n\}. \tag{2.1}$$

The generators from (2.1) *define a* length function ℓ_B *on B_n, as follows: For every $\tau\in B_n$ the length $\ell_B(\tau)$ is the smallest possible $k\geq 0$ such that τ can be factored as a product of k generators (with the convention that the product of 0 generators is equal to the unit ε of B_n).*

2^o *The length function ℓ_B satisfies the triangle inequality*

$$\ell_B(\sigma)\leq \ell_B(\tau)+\ell_B\left(\tau^{-1}\sigma\right), \tag{2.2}$$

and using the case of equality, we define a partial order *on B_n, where for $\tau,\sigma\in B_n$ we put*

$$\tau\leq\sigma \overset{\text{def}}{\Longleftrightarrow} \ell_B(\sigma)=\ell_B(\tau)+\ell_B\left(\tau^{-1}\sigma\right). \tag{2.3}$$

In other words, the order relation $\tau\leq\sigma$ means that one can find minimal factorizations for τ and for $\tau^{-1}\sigma$ into products of generators, such that the concatenation of these two factorizations gives a minimal factorization for σ.

3^o *We define*

$$S_{nc}^{(B)}(p,q) := \{\tau\in B_n \mid \tau\leq\gamma_{p,q}\}, \tag{2.4}$$

where the partial order considered in B_n is the one defined above, and $\gamma_{p,q} \in B_n$ is the permutation with two cycles

$$\gamma_{p,q} := (1, \ldots, p, -1, \ldots, -p)(p+1, \ldots, p+q, -(p+1), \ldots, -(p+q)) \in B_n. \tag{2.5}$$

Since the inequality $\varepsilon \leq \tau$ holds for every $\tau \in B_n$, we thus have that $\mathcal{S}_{nc}^{(B)}(p, q)$ is the interval $[\varepsilon, \gamma_{p,q}]$ in the group B_n.

We mention here that one could give several other equivalent descriptions for $\mathcal{S}_{nc}^{(B)}(p, q)$. Two such descriptions were discussed in [10] — one of them is in terms of "genus inequality", and the other is in terms of "annular crossing patterns" (see, [10, Section 2.5]). But these alternative descriptions will not be used in the present paper (although the genus inequality will be used in Section 6 for other reasons).

Definition 2.2. (Orbit partitions and the definition of $NC^{(B)}(p, q)$)
Let $p, q,$ and $n := p+q$ be as above.

1^o *For every $\tau \in B_n$, we will use the notation $\Omega(\tau)$ for the partition of $\{\pm 1, \ldots, \pm n\}$ into orbits of τ. $\big($Thus two numbers a, b from $\{\pm 1, \ldots, \pm n\}$ belong to the same block of $\Omega(\tau)$ if and only if there exists an $m \in \mathbb{Z}$ such that $\tau^m(a) = b.\big)$ It is obvious that if A is a block of $\Omega(\tau)$ then $-A$ is a block of $\Omega(\tau)$ as well. In the case when $A = -A$ we say that A is* inversion-invariant, *or that it is a* zero-block *of $\Omega(\tau)$. Clearly, the blocks of $\Omega(\tau)$ that are not inversion-invariant come in pairs (A and $-A$, with $A \neq -A$).*

2^o *For every $\tau \in B_n$ we will use the notation $\widetilde{\Omega}(\tau)$ for the partition of $\{\pm 1, \ldots, \pm n\}$ which is obtained from $\Omega(\tau)$ by grouping together all the inversion-invariant blocks of $\Omega(\tau)$ (if such blocks exist) into one block of $\widetilde{\Omega}(\tau)$. That is, if*

$$\Omega(\tau) = \{A_1, \ldots, A_k, B_1, -B_1, \ldots, B_l, -B_l\},$$

with $A_i = -A_i$ for $1 \leq i \leq k$, then

$$\widetilde{\Omega}(\tau) = \{A_1 \cup \cdots \cup A_k, B_1, -B_1, \ldots, B_l, -B_l\}.$$

3^o *The set $NC^{(B)}(p, q)$ of* annular non-crossing partitions of type B *is defined as*

$$NC^{(B)}(p, q) := \left\{ \widetilde{\Omega}(\tau) \,\middle|\, \tau \in \mathcal{S}_{nc}^{(B)}(p, q) \right\}. \tag{2.6}$$

Remark 2.3. 1^o The set $NC^{(B)}(p, q)$ is defined in such a way that the map $\widetilde{\Omega}$: $\mathcal{S}_{nc}^{(B)}(p, q) \to NC^{(B)}(p, q)$ is surjective. It is remarkable that this map is in fact a *poset isomorphism*, where $\mathcal{S}_{nc}^{(B)}(p, q)$ is partially ordered as an interval of B_{p+q} (and where B_{p+q} is partially ordered as in Definition 2.1.2), while $NC^{(B)}(p, q)$ is partially ordered by reverse refinement. This is the content of [10, Theorem 1.4].

2^o From Definition 2.2 it is clear that a partition $\pi \in NC^{(B)}(p, q)$ can never have more than one inversion-invariant block (if such a block exists, then it is unique).

3^o Let $\widehat{0}$ be the partition of $\{\pm 1, \ldots, \pm(p+q)\}$ into $2(p+q)$ singletons, and let $\widehat{1}$ be the partition of $\{\pm 1, \ldots, \pm(p+q)\}$ that has only one block. Then, $\widehat{0}, \widehat{1} \in NC^{(B)}(p,q)$, as it is clear that $\widehat{0} = \widetilde{\Omega}(\varepsilon)$ and $\widehat{1} = \widetilde{\Omega}(\gamma_{p,q})$ (where ε is the unit of B_{p+q}, while $\gamma_{p,q}$ is as in Equation (2.5)). The partitions $\widehat{0}$ and $\widehat{1}$ are the minimal and maximal elements respectively of the poset $NC^{(B)}(p,q)$.

Remark 2.4. (Rank and connectivity for a partition in $NC^{(B)}(p,q)$)

1^o It is immediate that $\mathcal{S}_{nc}^{(B)}(p,q)$ is a ranked poset, where the rank of a permutation $\tau \in \mathcal{S}_{nc}^{(B)}(p,q)$ is given by the length $\ell_B(\tau)$ from Definition 2.1.1. It is moreover not hard to see that $\ell_B(\tau)$ can be alternatively described in terms of the cycle structure of τ, by the formula

$$\ell_B(\tau) = (p+q) - \frac{1}{2} \cdot \Big(\# \text{ of orbits } A \text{ of } \tau \text{ such that } A \neq -A\Big). \tag{2.7}$$

As a consequence, we see that $NC^{(B)}(p,q)$ is a ranked poset as well, where the rank of a partition $\pi \in NC^{(B)}(p,q)$ is given by the formula

$$\text{rank}(\pi) = (p+q) - \frac{1}{2} \cdot \Big(\# \text{ of blocks of } \pi \text{ that are not inversion-invariant}\Big). \tag{2.8}$$

2^o Another important statistic for partitions in $NC^{(B)}(p,q)$ is the connectivity. For $\pi \in NC^{(B)}(p,q)$, the *connectivity* of π is the number

$$c := \frac{1}{2} \left(\begin{array}{c} \# \text{ of blocks } A \text{ of } \pi \text{ such that } A \neq -A \\ \text{and such that } A \text{ intersects both sets} \\ \{\pm 1, \ldots, \pm p\} \text{ and } \{\pm(p+1), \ldots, \pm(p+q)\} \end{array} \right). \tag{2.9}$$

An important fact concerning the concept of connectivity is

$$\begin{cases} \text{if } \pi \in NC^{(B)}(p,q) \text{ has connectivity } c > 0, \\ \text{then } \pi \text{ has no inversion-invariant blocks} \end{cases} \tag{2.10}$$

(see [10, Proposition 3.4]). Thus the blocks of a partition π with connectivity $c > 0$ all come in pairs $A, -A$ with $A \neq -A$; there are c pairs of blocks as in (2.9), while each of the remaining pairs is either "exterior" $\big(A, -A \subseteq \{\pm 1, \ldots, \pm p\}\big)$ or "interior" $\big(A, -A \subseteq \{\pm(p+1), \ldots, \pm(p+q)\}\big)$. Note moreover that if $c > 0$ and if e and i denote the number of exterior and of interior pairs of blocks of π respectively, then one has the inequalities:

$$\begin{cases} 1 \leq c \leq \min\{p,q\}, \text{ and} \\ 0 \leq e \leq p - c, \;\; 0 \leq i \leq q - c. \end{cases} \tag{2.11}$$

Remark 2.5. It is instructive at this point to give a brief discussion, based on connectivity, about how the adjusted orbit map $\widetilde{\Omega}\colon \mathcal{S}_{nc}^{(B)}(p,q) \to NC^{(B)}(p,q)$ works. Let π be a partition in $NC^{(B)}(p,q)$, and let c be the connectivity of π. There are two possible cases.

(a) $c > 0$. Then by the fact in (2.10) above we have $\pi = \Omega(\tau) = \widetilde{\Omega}(\tau)$, where τ is a (uniquely determined) permutation in $\mathcal{S}_{nc}^{(B)}(p, q)$, and τ has no inversion-invariant orbits.

(b) $c = 0$. Let τ denote the unique permutation in $\mathcal{S}_{nc}^{(B)}(p, q)$ such that $\widetilde{\Omega}(\tau) = \pi$. Then every orbit of τ is contained either in $\{\pm 1, \ldots, \pm p\}$ or in $\{\pm(p+1), \ldots, \pm(p+q)\}$ (see [10, Lemma 3.3]). Moreover, τ can have at most one inversion-invariant orbit contained in $\{\pm 1, \ldots, \pm p\}$, and at most one inversion-invariant orbit contained in $\{\pm(p+1), \ldots, \pm(p+q)\}$ (this is due to the fundamental fact from [11] that partitions in $NC^{(B)}(p)$ or $NC^{(B)}(q)$ can have at most one zero-block). If τ has two inversion-invariant orbits, then π is obtained from the orbit partition $\Omega(\tau)$ by joining together these two orbits; otherwise (if τ has at most one inversion-invariant orbit) we just have $\pi = \Omega(\tau)$.

The case (b) of the discussion was the more complicated one to describe, but one should keep in mind that typically this is the simpler case to handle. Indeed, the case (b) can be summarized as follows: If $\pi \in NC^{(B)}(p, q)$ has connectivity equal to 0, then π is obtained by "putting together" a partition $\pi_{ext} \in NC^{(B)}(p)$ and a partition $\pi_{int} \in NC^{(B)}(q)$, with a special rule for what to do when both π_{ext} and π_{int} have zero-blocks.

Remark 2.6. We conclude this section with a comment on "how to draw pictures" of partitions in $NC^{(B)}(p, q)$. In fact, what one does is to draw (equivalently) pictures of permutations in $\mathcal{S}_{nc}^{(B)}(p, q)$. In order to do this, one starts by representing the elements of $\{\pm 1, \ldots, \pm(p+q)\}$ as points on the boundary of an annulus: On the outside circle of the annulus we mark $2p$ points which we label clockwise as $1, \ldots, p, -1, \ldots, -p$ (in this order), and on the inside circle of the annulus we mark $2q$ points which we label counterclockwise as $p+1, \ldots, p+q, -(p+1), \ldots, -(p+q)$ (in this order). In terms of pictures drawn in this annulus, the fact that a permutation $\tau \in B_{p+q}$ belongs to $\mathcal{S}_{nc}^{(B)}(p, q)$ corresponds then to the following prescription: One can draw a closed contour for each of the cycles of τ, such that

(i) each of the contours does not self-intersect, and goes clockwise around the region it encloses;
(ii) the region enclosed by each of the contours is contained in the annulus;
(iii) regions enclosed by different contours are mutually disjoint.

Two concrete examples of such drawings are given in Figure 1 below, in the particular case when $p = 4$ and $q = 2$. On the left we have the drawing of the permutation

$$\tau_1 = (1, 2, 5)(-1, -2, -5)(3, -6)(-3, 6)(4)(-4) \in \mathcal{S}_{nc}^{(B)}(4, 2);$$

the partition corresponding to it is

$$\pi_1 = \Omega(\tau_1) = \widetilde{\Omega}(\tau_1) = \{\{1, 2, 5\}, \{-1, -2, -5\}, \{3, -6\}, \{-3, 6\}, \{4\}, \{-4\}\},$$

which has connectivity $c = 2$. On the right of Figure 1 we have the drawing of the permutation

$$\tau_2 = (1, -1)(2, 3, 4)(-2, -3, -4)(5, -5)(6)(-6) \in \mathcal{S}_{nc}^{(B)}(4, 2);$$

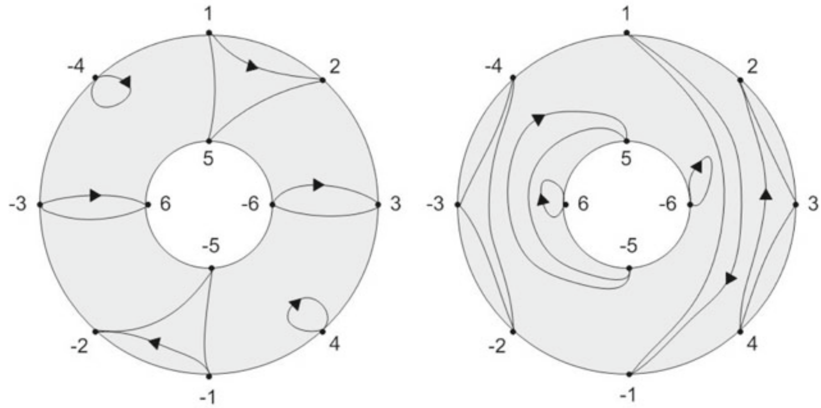

Figure 1: Examples of pictures of permutations in $\mathcal{S}_{nc}^{(B)}(4,2)$.

the partition corresponding to it is

$$\pi_2 = \widetilde{\Omega}(\tau_2) = \{\{1,-1,5,-5\},\{2,3,4\},\{-2,-3,-4\},\{6\},\{-6\}\},$$

which has connectivity $c = 0$. Note that in the latter example we have $\widetilde{\Omega}(\tau_2) \neq \Omega(\tau_2)$, since the inversion-invariant block $\{1,-1,5,-5\}$ of $\widetilde{\Omega}(\tau_2)$ is obtained by joining together the two inversion-invariant orbits of τ_2.

3. Rank Cardinalities and Möbius Function for $NC^{(B)}(n-1,1)$

Whereas the poset $NC^{(B)}(p,q)$ isn't a lattice in general, it is nevertheless true that $NC^{(B)}(n-1,1)$ is a lattice for every $n \geq 2$; and moreover, the meet operation on $NC^{(B)}(n-1,1)$ coincides with the usual "intersection meet" for partitions — the blocks of the meet $\pi \wedge \rho \in NC^{(B)}(n-1,1)$ are precisely the non-empty intersections $A \cap B$ where A is a block of π and B is a block of ρ. For a proof of these facts, see [10, Theorem 1.5]. The present section is devoted to this special "lattice" case, when the formulas for both the rank generating function and the Möbius function are nicer, and can be easily derived from known facts about $NC(n)$ and $NC^{(B)}(n)$.

The rank cardinalities for $NC^{(B)}(n-1,1)$ will be presented in Theorem 3.2. We first record a few known facts that will be used in the proof of this theorem.

Remark 3.1. 1^o We will use the well-known binomial identity

$$\sum_{k=0}^{n-r} \binom{n}{k}\binom{n}{k+r} = \binom{2n}{n-r}, \tag{3.1}$$

for any integers $0 \leq r \leq n$. This is a special case of the Chu-Vandermonde identity (see for instance, [1, Corollary 2.2.3, p. 67]).

2^o We will use the rank generating functions for the posets $NC(n)$ $\left(= NC^{(A)}(n)\right)$ and $NC^{(B)}(n)$.

(A) The rank of a partition $\pi \in NC^{(A)}(n)$ is given by the formula

$$\mathrm{rank}(\pi) = n - (\#\text{ of blocks of } \pi).$$

For every $0 \leq k \leq n-1$, we have (see [8, Corollary 4.1]) that

$$\left| \left\{ \pi \in NC^{(A)}(n) \,\middle|\, \mathrm{rank}(\pi) = k \right\} \right| = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}. \tag{3.2}$$

The numbers appearing on the right-hand side of (3.2) are called *Narayana numbers*. The total number of partitions in $NC^{(A)}(n)$ is the Catalan number

$$\left| NC^{(A)}(n) \right| = \frac{1}{n+1} \binom{2n}{n}. \tag{3.3}$$

(B) The rank of a partition $\pi \in NC^{(B)}(n)$ is given by the formula

$$\mathrm{rank}(\pi) = n - \frac{1}{2} \begin{pmatrix} \#\text{ of blocks } A \text{ of } \pi \\ \text{such that } A \neq -A \end{pmatrix}.$$

For every $0 \leq k \leq n$, we have (see [11, Proposition 6]) that

$$\left| \left\{ \pi \in NC^{(B)}(n) \,\middle|\, \mathrm{rank}(\pi) = k \right\} \right| = \binom{n}{k}^2. \tag{3.4}$$

The total number of partitions in $NC^{(B)}(n)$ is

$$\left| NC^{(B)}(n) \right| = \binom{2n}{n}. \tag{3.5}$$

3^o We will use a natural "absolute value map" that sends $NC^{(B)}(n)$ to $NC^{(A)}(n)$. We start with the map Abs: $\{\pm 1, \ldots, \pm n\} \to \{1, \ldots, n\}$ that sends $\pm i$ to i, for every $1 \leq i \leq n$. Note that for every $\pi \in NC^{(B)}(n)$ it makes sense to consider the partition of $\{1, \ldots, n\}$ into blocks of the form Abs(B), with B a block of π; this partition of $\{1, \ldots, n\}$ will be denoted by "Abs(π)". It turns out that Abs$(\pi) \in NC^{(A)}(n)$ for every $\pi \in NC^{(B)}(n)$, and moreover, the map

$$NC^{(B)}(n) \ni \pi \mapsto \mathrm{Abs}(\pi) \in NC^{(A)}(n) \tag{3.6}$$

defined in this way is an $(n+1)$-to-1 map (see [4, Section 1.3]). In the proof of the next theorem, we will use the following property (also observed in [4, Section 1.3]) of the map Abs from (3.6):

$$\begin{cases} \text{Given a partition } \pi_o \in NC^{(A)}(n) \text{ and a block } A \text{ of } \pi_o, \\ \text{there exists a unique } \pi \in NC^{(B)}(n) \text{ with a zero-block } Z, \\ \text{such that } \mathrm{Abs}(\pi) = \pi_o \text{ and } \mathrm{Abs}(Z) = A. \end{cases} \tag{3.7}$$

Theorem 3.2. *Let $n \geq 2$ be an integer. Then*

$$\left|NC^{(B)}(n-1,\,1)\right| = \binom{2n}{n}, \tag{3.8}$$

and for every $0 \leq k \leq n$ we have

$$\left|\left\{\pi \in NC^{(B)}(n-1,\,1) \,\middle|\, rank(\pi) = k\right\}\right| = \binom{n}{k}^2. \tag{3.9}$$

Proof. Equation (3.8) follows from (3.9) and (3.1), hence it will suffice to verify (3.9). We fix a k, for which we will prove (3.9). We will assume $k \neq 0$ (the case $k = 0$ is obvious).

From the first inequality (2.11) in Remark 2.4, it is clear that every partition in $NC^{(B)}(n-1,\,1)$ has connectivity equal to 0 or 1. Let us denote

$$\begin{aligned} \mathcal{C} &:= \left\{\pi \in NC^{(B)}(n-1,\,1) \,\middle|\, \pi \text{ has rank } k \text{ and connectivity } 1\right\}, \\ \mathcal{D} &:= \left\{\pi \in NC^{(B)}(n-1,\,1) \,\middle|\, \pi \text{ has rank } k \text{ and connectivity } 0\right\}. \end{aligned} \tag{3.10}$$

We note that every partition $\pi \in \mathcal{D}$ must be of the form $\pi = \widetilde{\Omega}(\tau)$, where τ is a permutation in $\mathcal{S}_{nc}^{(B)}(n-1,\,1)$ that leaves the set $\{n,\,-n\}$ invariant. Clearly, there are only two possibilities for how τ can act on $\{n,\,-n\}$: Either $\tau(n) = n$ and $\tau(-n) = -n$, or $\tau(n) = -n$ and $\tau(-n) = n$. We will denote by $\mathcal{D}_+$ and $\mathcal{D}_-$ respectively the set of partitions $\pi \in \mathcal{D}$ for which the first (the second, respectively) of these possibilities occurs. Thus we have $\mathcal{D} = \mathcal{D}_+ \cup \mathcal{D}_-$, disjoint, and it is clear that

$$\left|\left\{\pi \in NC^{(B)}(n-1,\,1) \,\middle|\, \text{rank}(\pi) = k\right\}\right| = |\mathcal{C}| + |\mathcal{D}_+| + |\mathcal{D}_-|. \tag{3.11}$$

It is immediate to see that $\mathcal{D}_+$ and $\mathcal{D}_-$ are in bijection with the sets of partitions in $NC^{(B)}(n-1)$ that have rank equal to k, and $k-1$, respectively. (For instance, for $\mathcal{D}_-$ we observe that every $\pi \in \mathcal{D}_-$ is canonically obtained from a partition π_o of rank $k-1$ in $NC^{(B)}(n-1)$, as follows: If π_o has no zero-block then we add to it a 2-element block $\{n,\,-n\}$, while if π_o has a zero-block Z then we replace Z by $Z \cup \{n,\,-n\}$.) By taking (3.4) into account, we thus find that

$$|\,\mathcal{D}_+\,| = \binom{n-1}{k}^2 \quad \text{and} \quad |\,\mathcal{D}_-\,| = \binom{n-1}{k-1}^2.$$

Let us now count the partitions in the set $\mathcal{C}$ from (3.10). Let π be in $\mathcal{C}$, and let us denote by A the block of π that contains n. We know that $A \neq -A$, and that $A \cap \{\pm 1, \ldots, \pm(n-1)\} \neq \emptyset$. Let π_o be the partition of $\{\pm 1, \ldots, \pm(n-1)\}$ that is obtained from π by taking its blocks A and $-A$ and replacing them with just one block, $Z := (A \cup (-A)) \setminus \{n,\,-n\}$. It is immediately seen that $\pi_o \in NC^{(B)}(n-1)$, and that the rank of π_o in $NC^{(B)}(n-1)$ is equal to k. The partition π we started with cannot be uniquely retrieved from π_o, but a moment's thought shows that π *can* be uniquely retrieved from the pair $(\pi_o,\,\tau(n))$, where $\tau \in \mathcal{S}_{nc}^{(B)}(n-1,\,1)$ is the permutation that corresponds to π. (The number $m = \tau(n) \in \{\pm 1, \ldots, \pm(n-1)\}$ could simply be described as "the point of A that follows n", when we move around A in clockwise order.)

The observations made in the preceding paragraph give us a one-to-one map

$$\mathcal{C} \ni \pi \mapsto (\pi_o, \tau(n)) \in \left\{ (\pi_o, m) \,\middle|\, \begin{array}{l} \pi_o \in NC^{(B)}(n-1) \text{ of rank } k \text{ and} \\ \text{with zero-block } Z, \text{ and } m \in Z \end{array} \right\}. \tag{3.12}$$

It is quite easy to see that the map in (3.12) is surjective as well. In pictorial terms: Given $\pi_o \in NC^{(B)}(n-1)$ with zero-block Z, and given an element $m \in Z$, we always know how to deform the convex polygon enclosed by Z so that it becomes a union of three regions — a small disc (which is part of a newly created annulus), and two regions enclosed by blocks A, $-A$ of a partition $\pi \in NC^{(B)}(n-1, 1)$. The role of the element $m \in Z$ in this geometric construction is to determine what side of the convex polygon enclosed by Z has to be deformed, and to indicate where on the emerging small disc we should place the labels n and $-n$.

Let us next observe that by using the "Abs" map and its property reviewed in (3.7) of Remark 3.1.2, we get another bijection

$$(\pi_o, m) \mapsto (\text{Abs}(\pi_o), m), \tag{3.13}$$

which sends the set $\left\{ (\pi_o, m) \,\middle|\, \begin{array}{l} \pi_o \in NC^{(B)}(n-1) \text{ of rank } k \text{ and} \\ \text{with zero-block } Z, \text{ and } m \in Z \end{array} \right\}$ onto the Cartesian product $\left\{\rho \in NC^{(A)}(n-1) \,\middle|\, \text{rank}(\rho) = k-1\right\} \times \{\pm 1, \ldots, \pm(n-1)\}$. By using the bijections (3.12) and (3.13) we thus find that

$$\begin{aligned} |\mathcal{C}| &= \left| \left\{ \rho \in NC^{(A)}(n-1) \,\middle|\, \text{rank}(\rho) = k-1 \right\} \right| \cdot 2(n-1) \\ &= \frac{1}{n-1} \binom{n-1}{k-1} \binom{n-1}{k} \cdot 2(n-1) \quad \text{(by (3.2))} \\ &= 2 \binom{n-1}{k-1} \binom{n-1}{k}. \end{aligned}$$

We finally return to (3.11) and substitute on its right-hand side the values found for the cardinalities of $\mathcal{C}$, $\mathcal{D}_+$, and $\mathcal{D}_-$, thus (3.9) immediately follows. ∎

Remark 3.3. We note the somewhat surprising fact that $NC^{(B)}(n-1, 1)$ has exactly the same rank generating function as the lattice $NC^{(B)}(n)$. For $n = 2$, we have in fact $NC^{(B)}(1, 1) = NC^{(B)}(2)$ (equality of sets of partitions of $\{1, 2\} \cup \{-1, -2\}$). But already for $n = 3$, it is no longer true that $NC^{(B)}(2, 1) = NC^{(B)}(3)$; moreover, by comparing the Hasse diagrams of $NC^{(B)}(2, 1)$ and of $NC^{(B)}(3)$, one easily sees that $NC^{(B)}(2, 1) \not\simeq NC^{(B)}(3)$. (The Hasse diagram for $NC^{(B)}(2, 1)$ is drawn in Figure 2, while the one for $NC^{(B)}(3)$ appears in Reiner's paper [11, p. 199]. In order to establish that $NC^{(B)}(2, 1) \not\simeq NC^{(B)}(3)$, one can for instance count edges in the Hasse diagrams — the Hasse diagram for $NC^{(B)}(2, 1)$ has 46 edges, while the one for $NC^{(B)}(3)$ has 44 edges.)

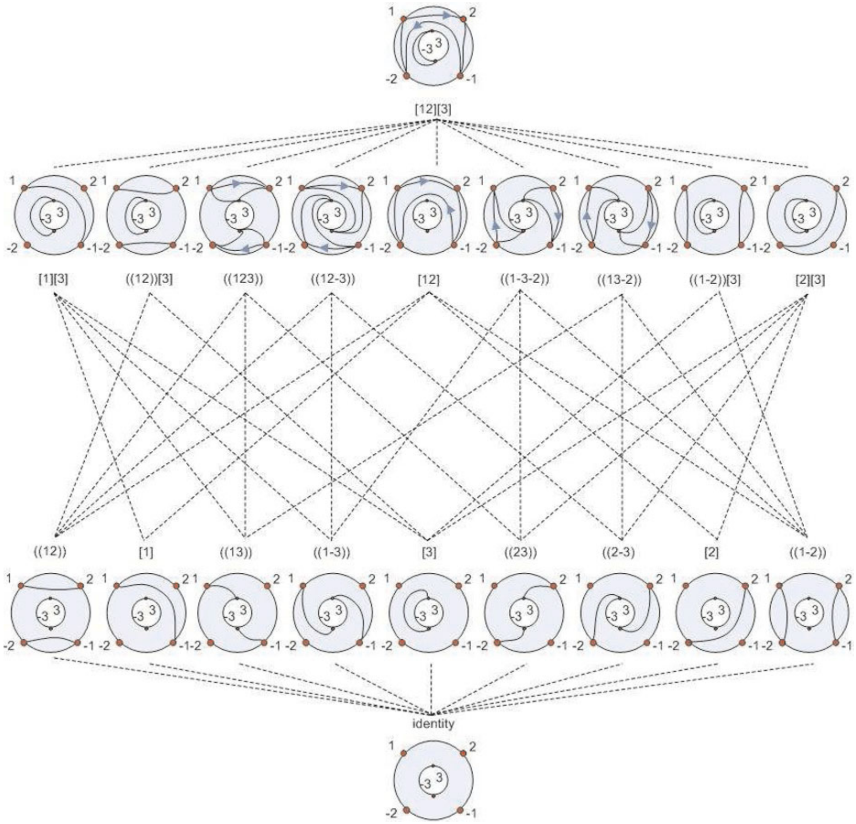

Figure 2: The Hasse diagram for $NC^{(B)}(2,1)$. The bracket notations $((\cdots))$ and $[\cdots]$ refer to the cycles of the corresponding permutations (e.g., $((1,2,-3))$ and $((1,-2))[3]$ are shorthand notations for the permutations $(1,2,-3)(-1,-2,3)$ and $(1,-2)(-1,2)(3,-3)$, respectively).

By comparing the specific formulas that give the Möbius functions for $NC^{(B)}(n)$ and for $NC^{(B)}(n-1,1)$, we will find in fact that $NC^{(B)}(n-1,1) \not\simeq NC^{(B)}(n)$ for all $n \geq 3$; see Remark 3.7 below.

So let us now consider the Möbius function of $NC^{(B)}(n-1,1)$. Its calculation will be presented in Theorem 3.6, and will be based on a partial Möbius inversion formula that is described as follows.

Lemma 3.4. *Let P be a finite lattice, let $\widehat{0}$ and $\widehat{1}$ denote the minimal and the maximal element of P respectively, and let ω be a fixed element of P, where $\omega \neq \widehat{1}$. Then,*

$$\sum_{\substack{\pi \in P \\ \pi \wedge \omega = \widehat{0}}} \mu_P\left(\pi, \widehat{1}\right) = 0. \tag{3.14}$$

For a proof of Lemma 3.4, see [12, Corollary 3.9.3]. A few other facts needed in the proof of Theorem 3.6 are collected in the next remark.

Remark 3.5. 1^o We will use the explicit formulas known for the Möbius functions of the posets $NC^{(A)}(n)$ and $NC^{(B)}(n)$.

(A) For every $n \geq 1$, we have

$$\mu_{NC^{(A)}(n)}\left(\widehat{0}, \widehat{1}\right) = (-1)^{n+1} \frac{(2n-2)!}{(n-1)!\,n!}, \tag{3.15}$$

where $\mu_{NC^{(A)}(n)}$ is the Möbius function of $NC^{(A)}(n)$, and $\widehat{0}, \widehat{1}$ are the minimal and maximal element of $NC^{(A)}(n)$, respectively. (See [8, Theorem 6].)

(B) For every $n \geq 1$, we have

$$\mu_{NC^{(B)}(n)}\left(\widehat{0}, \widehat{1}\right) = (-1)^n \cdot \binom{2n-1}{n}, \tag{3.16}$$

where $\mu_{NC^{(B)}(n)}$ is the Möbius function of $NC^{(B)}(n)$, and $\widehat{0}, \widehat{1}$ now stand for the minimal and maximal element of $NC^{(B)}(n)$, respectively. (See [11, Proposition 7].)

2^o Let p, q be positive integers. It is an easy exercise (left to the reader) to check that the formula

$$C(\tau) := \tau^{-1}\gamma_{p,q}, \quad \tau \in \mathcal{S}_{nc}^{(B)}(p,q) \tag{3.17}$$

defines a bijection $C\colon \mathcal{S}_{nc}^{(B)}(p,q) \to \mathcal{S}_{nc}^{(B)}(p,q)$, that is order-reversing — for $\sigma, \tau \in \mathcal{S}_{nc}^{(B)}(p,q)$ one has that $\sigma \leq \tau \Leftrightarrow C(\sigma) \geq C(\tau)$, where the partial order on $\mathcal{S}_{nc}^{(B)}(p,q)$ is as in Definition 2.1.2.

Now, by using the canonical isomorphism $\widetilde{\Omega}\colon \mathcal{S}_{nc}^{(B)}(p,q) \to NC^{(B)}(p,q)$ (see Remark 2.3.1), we can transport the map C from (3.17) to an anti-isomorphism $K\colon NC^{(B)}(p,q) \to NC^{(B)}(p,q)$, defined via the formula

$$K\left(\widetilde{\Omega}(\tau)\right) = \widetilde{\Omega}\left(\tau^{-1}\gamma_{p,q}\right), \quad \tau \in \mathcal{S}_{nc}^{(B)}(p,q). \tag{3.18}$$

This anti-isomorphism K is the $NC^{(B)}(p, q)$-analogue for an anti-isomorphism of the lattice $NC^{(A)}(n)$ introduced by Kreweras in [8], which is commonly called the Kreweras complementation map. Following this trend, we will also refer to the map K from (3.18) as the *Kreweras complementation map* of $NC^{(B)}(p, q)$. Note that, due to the fact that it is an anti-isomorphism, the Kreweras complementation map has the property that

$$\mu(\pi, \rho) = \mu(K(\rho), K(\pi)), \quad \forall \pi, \rho \in NC^{(B)}(p, q) \text{ such that } \pi \leq \rho, \tag{3.19}$$

where μ is the Möbius function of $NC^{(B)}(p, q)$.

Theorem 3.6. *Let $n \geq 2$ be an integer, let $\mu_{NC^{(B)}(n-1,1)}$ be the Möbius function of $NC^{(B)}(n-1, 1)$, and let $\widehat{0}, \widehat{1}$ be the minimal and maximal element of $NC^{(B)}(n-1, 1)$, respectively. Then*

$$\mu_{NC^{(B)}(n-1,1)}\left(\widehat{0}, \widehat{1}\right) = (-1)^n \cdot \binom{2n-1}{n} \cdot \frac{5n-4}{4n-2}. \tag{3.20}$$

Proof. Throughout the proof we will write "μ" instead of "$\mu_{NC^{(B)}(n-1,1)}$", for compactness. We will apply Lemma 3.4 to the particular case when $P = NC^{(B)}(n-1, 1)$ and

$$\omega := \{\{\pm 1, \ldots, \pm(n-1)\}, \{n\}, \{-n\}\}. \tag{3.21}$$

By taking into account that the meet operation of $NC^{(B)}(n-1, 1)$ is just the usual "intersection" meet, one immediately sees that the partitions in $\left\{\pi \in NC^{(B)}(n-1, 1) \mid \pi \wedge \omega = \widehat{0}\right\}$ can be listed explicitly as $\widehat{0}, \pi_0, \pi_1, \ldots, \pi_{n-1}, \pi_{-1}, \ldots, \pi_{-(n-1)}$, where

$$\pi_0 := \{\{n, -n\}, \{1\}, \{-1\}, \ldots, \{n-1\}, \{-(n-1)\}\},$$

and where for every $i \in \{\pm 1, \ldots, \pm(n-1)\}$ we put

$$\pi_i := \{\{i, n\}, \{-i, -n\}\} \cup \{\{j\} \mid j \in \{\pm 1, \ldots, \pm(n-1)\}, |j| \neq |i|\}.$$

When applied to this particular situation, Lemma 3.4 thus implies that

$$0 = \mu\left(\widehat{0}, \widehat{1}\right) + \mu\left(\pi_0, \widehat{1}\right) + \sum_{i=1}^{n-1} \mu\left(\pi_i, \widehat{1}\right) + \sum_{i=1}^{n-1} \mu\left(\pi_{-i}, \widehat{1}\right). \tag{3.22}$$

It is convenient to consider the equivalent restatement of (3.22) that is obtained by taking Kreweras complements and by invoking formula (3.19) from Remark 3.5.2:

$$0 = \mu\left(\widehat{0}, \widehat{1}\right) + \mu\left(\widehat{0}, \rho_0\right) + \sum_{i=1}^{n-1} \mu\left(\widehat{0}, \rho_i\right) + \sum_{i=1}^{n-1} \mu\left(\widehat{0}, \rho_{-i}\right), \tag{3.23}$$

where we denoted $\rho_i := K(\pi_i)$, for $i \in \{0\} \cup \{\pm 1, \ldots, \pm(n-1)\}$.

Let us now determine explicitly the partitions ρ_0 and $\rho_{\pm 1}, \ldots, \rho_{\pm(n-1)}$. We do this by using the corresponding permutations in $\mathcal{S}_{nc}^{(B)}(n-1, 1)$, and formula (3.18) from Remark 3.5.2. For $i \in \{\pm 1, \ldots, \pm(n-1)\}$, we write $\pi_i = \widetilde{\Omega}(\tau_i)$ with $\tau_i = (i, n)(-i, -n) \in B_n$, and we compute

$$\begin{aligned}\tau_i^{-1}\gamma_{n-1,1} &= ((i, n)(-i, -n))((1, \ldots, n-1, -1, \ldots, -(n-1))(n, -n))\\ &= ((1, \ldots, i-1, n, -i, \ldots, -(n-1))((-1, \ldots, -(i-1), -n, i, \ldots, n-1).\end{aligned}$$

Since $\rho_i = K\left(\widetilde{\Omega}(\tau_i)\right) = \widetilde{\Omega}\left(\tau_i^{-1}\gamma_{n-1,1}\right)$, we thus obtain that, for $i > 0$,

$$\rho_i = \{\{1, \ldots, i-1, n, -i, \ldots, -(n-1)\}, \{-1, \ldots, -(i-1), -n, i, \ldots, n-1\}\}, \tag{3.24}$$

with a similar formula (left to the reader) in the case $i < 0$. For ρ_0, one does a similar calculation, by writing $\pi_0 = \widetilde{\Omega}(\tau_0)$ for $\tau_0 = (n, -n) \in B_n$. The reader should have no difficulty in checking that this calculation simply leads to the equality $\rho_0 = \omega$, with ω taken from (3.21).

From the explicit form found in (3.24) for ρ_i with $i \in \{\pm 1, \ldots, \pm(n-1)\}$, one easily infers that the interval $\left[\widehat{0}, \rho_i\right]$ of $NC^{(B)}(n-1, 1)$ is a poset isomorphic with the lattice $NC^{(A)}(n)$. Indeed, the process of constructing a partition $\sigma \in NC^{(B)}(n-1, 1)$ such that $\sigma \leq \rho_i$ amounts precisely to breaking in a non-crossing way the block $\{1, \ldots, i-1, n, -i, \ldots, -(n-1)\}$ of ρ_i, where the cyclic order of the n elements of the block is as listed above. (This must be of course matched by the corresponding, uniquely determined, non-crossing breaking of the other block $\{-1, \ldots, -(i-1), -n, i, \ldots, n-1\}$ of ρ_i.) The isomorphism $\left[\widehat{0}, \rho_i\right] \simeq NC^{(A)}(n)$ and (3.15) thus give us that

$$\mu\left(\widehat{0}, \rho_i\right) = (-1)^{n+1}\frac{(2n-2)!}{(n-1)!\, n!}.$$

In a similar way, one finds that the interval $\left[\widehat{0}, \rho_0\right]$ of $NC^{(B)}(n-1, 1)$ is isomorphic with $NC^{(B)}(n-1)$, hence (by (3.16)) we have

$$\mu\left(\widehat{0}, \rho_0\right) = (-1)^{n-1}\binom{2n-3}{n-1}.$$

Finally, by substituting in (3.23) the concrete values obtained above for the $\mu\left(\widehat{0}, \rho_i\right)$, we find that

$$-\mu\left(\widehat{0}, \widehat{1}\right) = (-1)^{n-1}\binom{2n-3}{n-1} + (2n-2)\cdot(-1)^{n-1}\cdot\frac{(2n-2)!}{(n-1)!\, n!},$$

and the required formula for $\mu\left(\widehat{0}, \widehat{1}\right)$ follows by straightforward calculation. ∎

Remark 3.7. By comparing formula (3.20) in Theorem 3.6 with the corresponding formula (3.16) that holds for $NC^{(B)}(n)$, we see that $\mu_{NC^{(B)}(n)}\left(\widehat{0}, \widehat{1}\right)$ is different from $\mu_{NC^{(B)}(n-1,1)}\left(\widehat{0}, \widehat{1}\right)$ for all $n \geq 3$. This implies, of course, that $NC^{(B)}(n-1, 1) \not\simeq NC^{(B)}(n)$ for $n \geq 3$.

4. Rank Generating Function for $NC^{(B)}(p,q)$

In this section, we determine the rank generating function for $NC^{(B)}(p,q)$. Our results follow directly from a bijection, in Proposition 4.2 below, which is similar to [7, Lemma 2.1] and [11, Proposition 6]. As a preliminary, we have the following discussion of strings of parentheses.

Remark 4.1. We let $\{(,)\}^*$ be the set of strings of left parentheses "(" and right parentheses ")". With multiplication given by concatenation, this set forms a monoid, with the empty string acting as identity element.

If $s = s_1 \cdots s_n \in \{(,)\}^*$, $n \geq 1$, then the nontrivial left-substrings of s are given by $u_i := s_1 \cdots s_i$, $i = 1, \ldots, n$. If all nontrivial left-substrings of s have (strictly) more left parentheses than right parentheses, then we will say that s is *legal from the left*.

For $s = s_1 \cdots s_n \in \{(,)\}^*$, $n \geq 1$, the *cyclic shifts* of s are the n strings

$$s^{(1)} = s_2 \cdots s_n s_1,\, s^{(2)} = s_3 \cdots s_n s_1 s_2,\, \ldots,\, s^{(n-1)} = s_n s_1 s_2 \cdots s_{n-1},\;\; s^{(n)} = s.$$

Suppose that s has m more left parentheses than right parentheses, for some $m \geq 1$. Then the well-known Cycle Lemma (see for instance, the discussion in [13, p. 67]) says that exactly m of the cyclic shifts of s are legal from the left.

For example, if s is the string $()(()(($, which has 5 left parentheses and 2 right parentheses, then the 3 cyclic shifts of s that are legal from the left are

$$s^{(2)} = (()((()(), \quad s^{(5)} = ((()((), \quad s^{(6)} = (()(()(.$$

Symmetrically, if all nontrivial right-substrings of s have more right parentheses than left parentheses, then we say that s is *legal from the right*. For this case, suppose that s has m more right parentheses than left parentheses, for some $m \geq 1$. Then the Cycle Lemma says that exactly m of the cyclic shifts of s are legal from the right.

Proposition 4.2. *Let p, q be positive integers. Suppose that c, e, i are integers satisfying the inequalities stated in (2.11) of Remark 2.4, that is, $1 \leq c \leq \min\{p,q\}$, $0 \leq e \leq p-c$, $0 \leq i \leq q-c$. Then there exists a bijection between the set*

$$\left\{ \left(d, L^E, R^E, L^I, R^I\right) \;\middle|\; \begin{array}{l} 1 \leq d \leq 2c, \\ L^E, R^E \subseteq \{1,\ldots,p\},\ \left|L^E\right| = e+c,\ \left|R^E\right| = e, \\ L^I, R^I \subseteq \{p+1,\ldots,p+q\},\ \left|L^I\right| = i,\ \left|R^I\right| = i+c \end{array} \right\} \tag{4.1}$$

and the set of partitions in $NC^{(B)}(p,q)$ that have connectivity equal to c, have e exterior pairs of blocks, and have i interior pairs of blocks.

Proof. We will describe explicitly the constructions for two maps $\left(d, L^E, R^E, L^I, R^I\right) \mapsto \pi$ and $\pi \mapsto \left(d, L^E, R^E, L^I, R^I\right)$, and we will leave it as an exercise to the reader to check that these two maps are inverse to each other (thus giving together a bijection as stated). We recommend that the general descriptions given below for the two maps are read in parallel with Remark 4.3, which illustrates how the maps work on a concrete example.

A. Description of the map $\left(d, L^E, R^E, L^I, R^I\right) \mapsto \pi$. Given $\left(d, L^E, R^E, L^I, R^I\right)$ as in (4.1), insert left and right parentheses into the string

$$1, \dots, p, -1, \dots, -p$$

by placing a left (right, respectively) parenthesis before (after, respectively) each occurrence of j and $-j$, for each value j in L^E (R^E, respectively). In this way we obtain the string u of length $2(p+2e+c)$, consisting of numbers and parentheses. In u, there are $2c$ more left parentheses than right parentheses, so the Cycle Lemma in Remark 4.1 implies that there are $2c$ cyclic shifts of u beginning with a left parenthesis such that the subsequence consisting of parentheses only is legal from the left. Suppose that these $2c$ cyclic shifts are given by $u^{(i_1)}, \dots, u^{(i_{2c})}$, ordered so that $i_1 < \cdots < i_{2c}$. Then let $t_1 = u^{(i_d)}$.

Similarly, insert left and right parentheses into the string

$$p+1, \dots, p+q, -(p+1), \dots, -(p+q)$$

by placing a left (right, respectively) parenthesis before (after, respectively) each occurrence of j and $-j$, for each value j in L^I (R^I, respectively), to obtain the string v of numbers and parentheses. In v there are $2c$ more right parentheses than left parentheses, so the Cycle Lemma in Remark 4.1 implies that there are $2c$ cyclic shifts of v ending with a right parenthesis such that the subsequence consisting of parentheses only is legal from the right. If these $2c$ cyclic shifts are $v^{(j_1)}, \dots, v^{(j_{2c})}$, ordered so that $j_1 < \cdots < j_{2c}$, then let $t_2 = v^{(j_{2c})}$.

Now consider the concatenation $t_1 t_2$ of the two strings t_1 and t_2 found above. From the string $t_1 t_2$ we read off a unique partition π in $NC^{(B)}(p,q)$ in the following way: The numbers inside a lowest-level pair of matching parentheses form a block of π; remove these numbers and this pair of parentheses from the string, and iterate until the string is empty. (See part A of Remark 4.3 below, for a concrete example of how this works.)

B. Description of the map $\pi \mapsto \left(d, L^E, R^E, L^I, R^I\right)$. Let π be a partition in $NC^{(B)}(p,q)$ that has connectivity equal to c, has e pairs of external blocks and has i pairs of internal blocks. A significant fact we will use here is that (even though π is drawn on a circular picture) every block of π that is either an external block or an internal block comes with a canonical total order on it, and thus has a *first element* and a *last element*.

Indeed, say for instance, that A is an external $\big($i.e., such that $A \subseteq \{\pm 1, \dots, \pm p\}\big)$ block of π. Let us choose an element $i \in -A$ and, by starting from this i, let us travel around the external circle of the annulus (in the sense that we always use for this circle, that is, clockwise). When we do this, we encounter the elements of A in a certain order, and this order does not depend on our choice of the starting point $i \in -A$. (The latter fact is an immediate consequence of the fact that the blocks A and $-A$ of π do not cross.) We thus end with a "canonical" total order for the elements of A. Clearly, a similar argument can be given when we deal with an internal block of π. Moreover, the same argument can also be used to introduce a total order on each of the sets $A \cap \{\pm 1, \dots, \pm p\}$ and $A \cap \{\pm(p+1), \dots, \pm(p+q)\}$, in the case when A is a connecting block of π.

So then, starting from the given $\pi \in NC^{(B)}(p,q)$, let us draw some parentheses on the picture representing π, according to the following recipe:

(a) For every block A of π that is either an external block or an internal block, we draw a left parenthesis immediately before the first element of A, and a right parenthesis immediately after the last element of A.
(b) For every connecting block A of π we draw a left parenthesis immediately before the first element of $A \cap \{\pm 1, \ldots, \pm p\}$, and a right parenthesis immediately after the last element of $A \cap \{\pm(p+1), \ldots, \pm(p+q)\}$.

But now, if the parentheses added to the picture of π are read starting from 1 on the outside circle and starting from $p+1$ on the inside circle, then one gets two strings of numbers and parentheses u and v, that are exactly of the same kind as the strings denoted by "u" and "v" in part A of the proof. Furthermore, it is immediate that the strings u and v obtained here correspond to some subsets

$$L^E, R^E \subseteq \{1, \ldots, p\}, \qquad L^I, R^I \subseteq \{p+1, \ldots, p+q\},$$

which have the properties required in (4.1).

In order to complete the description of the map $\pi \mapsto \left(d, L^E, R^E, L^I, R^I\right)$, we are thus left to explain how we obtain the number $d \in \{1, \ldots, 2c\}$. It is immediate that determining d (in the context where we have already identified the strings u and v) is equivalent to choosing one of the $2c$ cyclic shifts of u that are legal from the left. Expressed directly in terms of the partition π, this in turn amounts to choosing one of the $2c$ connecting blocks of π. (To be precise: If a connecting block A is chosen, then we pick the cyclic shift of u that starts with the left parenthesis placed immediately before the first element of $A \cap \{\pm 1, \ldots, \pm p\}$.) So what we have to do is to give a procedure for canonically selecting a connecting block of π. To do so, we look at how the connecting blocks intersect the interior circle of the annulus: We start from $p+1$ and move counterclockwise around the interior circle, and stop the first time that we meet an element belonging to a connecting block. (See part B of Remark 4.3 below for a concrete example of how this works.) ∎

Remark 4.3. Let us illustrate how the two maps described in the proof of the preceding proposition work on a concrete example. Consider the situation when the integers p, q, c, e, i given in Proposition 4.2 are $p = 5$, $q = 3$, $c = 1$, $e = 2$, $i = 1$.

A. Let us determine explicitly the partition $\pi \in NC^{(B)}(5,3)$ that corresponds (by the first of the two maps described in the proof of Proposition 4.2) to the tuple $\left(d, L^E, R^E, L^I, R^I\right)$ where

$$d = 2, \quad L^E = \{2,4,5\}, \quad R^E = \{1,2\}, \quad L^I = \{7\}, \quad R^I = \{6,7\}. \tag{4.2}$$

By inserting parentheses in $1, \ldots, 5, -1, \ldots, -5$, we obtain the following string of length 20, consisting of numbers and parentheses:

$$u = 1\,)\,(\,2\,)\,3\,(\,4\,(\,5\;-1\,)\,(\,-2\,)\,-3\,(\,-4\,(\,-5. \tag{4.3}$$

The two cyclic shifts of u that begin with a left parenthesis and are legal from the left are $u^{(6)}$ and $u^{(16)}$. Since we have $d = 2$, the string t_1 from the description of the above bijection is hence

$$t_1 = u^{(16)} = (-4(-5\,1)(2)3(4(5\,-1)(-2)-3.$$

In a similar way, by inserting parentheses in 6, 7, 8, −6, −7, −8 we get

$$v = 6)(7)8\ -6)(-7)-8, \tag{4.4}$$

and then have

$$t_2 = v^{(8)} = (-7)-8\,6)(7)8-6\,).$$

Finally, we concatenate t_1 and t_2, and from the string t_1t_2 we read off the desired partition $\pi \in NC^{(B)}(5,3)$, which is

$$\pi = \{\{1,-5\},\{-1,5\},\{2\},\{-2\},\{3,-4,-6,8\},\{-3,4,6,-8\},\{7\},\{-7\}\}. \tag{4.5}$$

B. Conversely, let us now start from the partition $\pi \in NC^{(B)}(5,3)$ that appeared in (4.5) above, and determine explicitly the tuple $\left(d, L^E, R^E, L^I, R^I\right)$ that corresponds to π by the second map described in the proof of Proposition 4.2.

The annular picture for π and the parentheses that have to be added to it are shown in Figure 3 below. When looking at Figure 3, the reader should keep in mind that the placing of a parenthesis “immediately before” (or “immediately after”) a labelled point on one of the circles of the annulus must be always done in agreement with the chosen running direction on that particular circle. Thus for instance the parenthesis sitting next to −5 in Figure 3 is a “left parenthesis placed immediately before −5”, since the outside circle is run clockwise; while next to 6 we have a “right parenthesis placed immediately after 6”, as the running direction on the inside circle is counterclockwise.

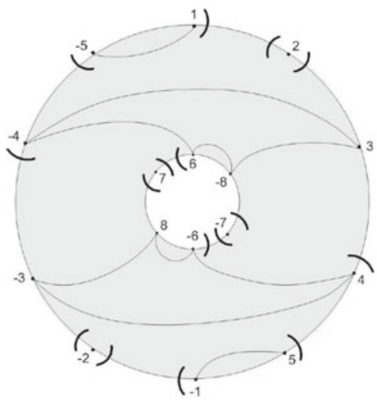

Figure 3. Adding parentheses to the picture of a partition in $NC^{(B)}(p,q)$.

If we read Figure 3 starting with 1 on the outside circle and starting with 6 on the inside circle, we find the strings u and v displayed in (4.3) and (4.4), respectively, and from these u and v we clearly get back to the sets L^E, R^E, L^I, R^I indicated in (4.2).

Finally, let us also follow on Figure 3 the procedure for finding the value of d. What we have to do is to start from $p+1\,(=6)$ and move counterclockwise around the interior circle of the annulus, and stop the first time that we meet an element belonging to a connecting block. But in this example the number 6 belongs to the connecting block $A = \{3, -4, 6, -8\}$ of π; so this is the connecting block of π that is chosen. The first element of $A \cap \{\pm 1, \ldots, \pm p\}$ is -4, hence we choose the cyclic shift of u that starts with "(-4", and this corresponds to the value $d = 2$.

Corollary 4.4. *Let p, q, c, e, i be integers such that $1 \le c \le \min\{p, q\}$ and $0 \le e \le p-c$, $0 \le i \le q-c$. Then there are exactly*

$$2c \binom{p}{e} \binom{p}{e+c} \binom{q}{i} \binom{q}{i+c} \tag{4.6}$$

partitions in $NC^{(B)}(p, q)$ that have connectivity equal to c, e exterior pairs of blocks, and i interior pairs of blocks.

Proof. This follows by taking cardinalities in the bijection from Proposition 4.2. ∎

As an immediate consequence of the above corollary, one can enumerate the partitions in $NC^{(B)}(p, q)$ by their connectivity.

Theorem 4.5. *Let p, q be positive integers.*

1^o *For every $1 \le c \le \min\{p, q\}$, there are exactly*

$$2c \binom{2p}{p-c} \binom{2q}{q-c} \tag{4.7}$$

partitions in $NC^{(B)}(p, q)$ that have connectivity equal to c.

2^o *There are exactly*

$$\binom{2p}{p} \binom{2q}{q} \tag{4.8}$$

partitions in $NC^{(B)}(p, q)$ that have connectivity equal to 0.

3^o *The total number of partitions in $NC^{(B)}(p, q)$ is*

$$\left| NC^{(B)}(p, q) \right| = \frac{p+q+pq}{p+q} \cdot \binom{2p}{p} \binom{2q}{q}. \tag{4.9}$$

Proof. 1^o From Proposition 4.2, the number of partitions of connectivity c in the set $NC^{(B)}(p,q)$ equals

$$\begin{aligned} 2c \sum_{e,i \ge 0} \binom{p}{e} \binom{p}{e+c} \binom{q}{i} \binom{q}{i+c} &= 2c \left(\sum_{e=0}^{p-c} \binom{p}{e} \binom{p}{e+c} \right) \left(\sum_{i=0}^{q-c} \binom{q}{i} \binom{q}{i+c} \right) \\ &= 2c \binom{2p}{p-c} \binom{2q}{q-c}, \end{aligned}$$

where for the second equality we used identity (3.1).

2^o As observed in Remark 2.5, the partitions with connectivity 0 in $NC^{(B)}(p,q)$ are given by the direct product of $NC^{(B)}(p)$ with $NC^{(B)}(q)$; hence their number is

$$\left|NC^{(B)}(p)\right| \cdot \left|NC^{(B)}(q)\right| = \binom{2p}{p} \cdot \binom{2q}{q} \qquad \text{(by using (3.5))}.$$

3^o From the above arguments it follows that

$$\left|NC^{(B)}(p,q)\right| = \binom{2p}{p}\binom{2q}{q} + \sum_{c\geq 1} 2c \binom{2p}{p-c}\binom{2q}{q-c}. \tag{4.10}$$

In the summation over c in (4.10), we observe that the ratio of two consecutive terms is a rational function of c, hence we are dealing with a hypergeometric summation. Referring to the standard notations for hypergeometric series one sees, more precisely, that

$$\sum_{c\geq 1} 2c \binom{2p}{p-c}\binom{2q}{q-c} = 2\binom{2p}{p-1}\binom{2q}{q-1} \cdot {}_3F_2\left(\begin{matrix} 2, -(p-1), -(q-1) \\ p+2, q+2 \end{matrix}; 1\right). \tag{4.11}$$

(For the precise definition of ${}_3F_2$ see for instance formula (2.1.2) on [1, p. 62].)

Now, it turns out that the special ${}_3F_2$ series on the right-hand side of (4.11) can be summed in closed form; this is by a theorem of Dixon (see formula (2.2.11) on [1, p. 72]), which gives us that

$${}_3F_2\left(\begin{matrix} 2, -(p-1), -(q-1) \\ p+2, q+2 \end{matrix}; 1\right) = \frac{(p+1)(q+1)}{2(p+q)}. \tag{4.12}$$

By substituting this expression into (4.11), and then by plugging the result into (4.10), we obtain the stated formula for the cardinality of $NC^{(B)}(p,q)$. ∎

From Corollary 4.4, one can also infer a formula for the rank generating function of $NC^{(B)}(p,q)$.

Theorem 4.6. *Let p,q be positive integers and let $F(x)$ denote the rank generating function for $NC^{(B)}(p,q)$. Then*

$$F(x) = \sum_{i,j\geq 0} \binom{p}{i}^2 \binom{q}{j}^2 x^{i+j} + \sum_{c\geq 1}\sum_{e,i\geq 0} 2c \binom{p}{e}\binom{p}{e+c}\binom{q}{i}\binom{q}{i+c} x^{p+q-e-i-c}.$$

Proof. The first summation on the right-hand side of the result gives the contribution to $F(x)$ from partitions $\pi \in NC^{(B)}(p,q)$ that have connectivity equal to 0. Indeed, we saw in Remark 2.5 how such a partition π is obtained by putting together a partition $\pi_{ext} \in NC^{(B)}(p)$ and a partition $\pi_{int} \in NC^{(B)}(q)$; it is moreover immediate that when this is done, the rank of π in $NC^{(B)}(p,q)$ is the sum of the ranks of π_{ext} and π_{int} in $NC^{(B)}(p)$ and in $NC^{(B)}(q)$, respectively. Thus when summing over partitions $\pi \in$

$NC^{(B)}(p,q)$ with connectivity equal to 0, we get

$$\sum_{\pi} x^{\mathrm{rank}(\pi)} = \left(\sum_{\pi_{ext} \in NC^{(B)}(p)} x^{\mathrm{rank}(\pi_{ext})} \right) \left(\sum_{\pi_{int} \in NC^{(B)}(q)} x^{\mathrm{rank}(\pi_{int})} \right)$$
$$= \left(\sum_{i=0}^{p} \binom{p}{i}^2 x^i \right) \left(\sum_{j=0}^{q} \binom{q}{j}^2 x^j \right) \qquad \text{(by (3.4))}.$$

On the other hand, let us observe that if $\pi \in NC^{(B)}(p,q)$ has connectivity $c \geq 1$, e pairs of exterior blocks, and i pairs of internal blocks, then from (2.8) it follows that

$$\mathrm{rank}(\pi) = (p+q) - (c+e+i).$$

Hence in view of Corollary 4.4, the contribution to $F(x)$ from the partitions $\pi \in NC^{(B)}(p,q)$ that have connectivity different from 0 is given precisely by the second summation on the right-hand side of the result. ∎

Remark 4.7. It can be shown that the second summation on the right-hand side of Theorem 4.6 can be reexpressed with only two summation indices instead of three, in the form:

$$\frac{2pq}{p+q} \sum_{i,j\geq 1} \left(\binom{p}{i}\binom{q}{j-1} + \binom{p}{i-1}\binom{q}{j} \right) \binom{p-1}{i-1}\binom{q-1}{j-1} x^{i+j-1}. \tag{4.13}$$

The proof of this fact is technical, and is omitted.

5. Zeta Polynomial and Möbius Function for $NC^{(B)}(p,q)$

In this section, we determine the zeta polynomial and Möbius function for $NC^{(B)}(p,q)$. These follow immediately by extending the bijection given in Proposition 4.2 to count multichains in $NC^{(B)}(p,q)$, similar to [7, Theorem 3.2] and [11, Proposition 7].

Proposition 5.1. *For $p,q \geq 1$ and $m \geq 2$, the bijection given in Proposition 4.2 extends to a bijection between*

$$\left\{ \begin{array}{l|l} & 1 \leq c, \quad 1 \leq d \leq 2c, \\ (c,d; L^E, R_1^E, \ldots, R_{m-1}^E; & L^E, R_1^E, \ldots, R_{m-1}^E \subseteq \{1,\ldots,p\}, \\ \qquad L^I, R_1^I, \ldots, R_{m-1}^I) & |L^E| = |R_1^E| + \cdots + |R_{m-1}^E| + c, \\ & L^I, R_1^I, \ldots, R_{m-1}^I \subseteq \{p+1,\ldots,p+q\}, \\ & |L^I| = |R_1^I| + \cdots + |R_{m-1}^I| - c, \end{array} \right\} \tag{5.1}$$

and the set of multichains $\pi_1 \leq \cdots \leq \pi_{m-1}$ in $NC^{(B)}(p,q)$, in which

$$\mathrm{rank}(\pi_i) = p+q-\left(|R_i^E| + \cdots + |R_{m-1}^E| + |R_i^I| + \cdots + |R_{m-1}^I|\right), \quad 1 \leq i \leq m-1, \tag{5.2}$$

and at least one of the π_i has positive connectivity.

Proof. This proof is to a good extent a repetition of the one given earlier for Proposition 4.2 (which corresponds to the case $m = 2$ of the present proposition). Because of this, we will only give an outline of the argument, with emphasis on the points that are specific to the situation at hand.

Given a tuple $\left(c, d; L^E, R_1^E, \ldots, R_{m-1}^E; L^I, R_1^I, \ldots, R_{m-1}^I\right)$ as in (5.1), insert left and right parentheses into the string

$$1, \ldots, p, -1, \ldots, -p,$$

with $m-1$ types of right parentheses $)^k$ for $k = 1, \ldots, m-1$, as follows: Place a left parenthesis before each occurrence of j and $-j$, for each value j in L^E; for $k = 1, \ldots, m-1$, place a right parenthesis of type $)^k$ after each occurrence of j and $-j$, for each value j in R_k^E. $\big($If j occurs in both R_a^E and R_b^E, for $a < b$, then place the corresponding $)^b$ to the right of the $)^a$.$\big)$ In the resulting string of numbers and parentheses, there are $2c$ more left parentheses than right parentheses, so the Cycle Lemma in Remark 4.1 implies that there are $2c$ cyclic shifts of the string beginning with a left parenthesis such that the subsequence consisting of parentheses only is legal from the left. Order these $2c$ cyclic shifts in the canonical way (by the same method as in the proof of Proposition 4.2), and choose the dth of them to give the string t_1.

Similarly, insert left and right parentheses into the string

$$p+1, \ldots, p+q, -(p+1), \ldots, -(p+q),$$

by placing a left parenthesis before each occurrence of j and $-j$, for each value j in L^I; for $k = 1, \ldots, m-1$, place a right parenthesis of type $)^k$ after each occurrence of j and $-j$, for each value j in R_k^I. In the resulting string of numbers and parentheses, there are $2c$ more right parentheses than left parentheses, so the Cycle Lemma in Remark 4.1 implies that there are $2c$ cyclic shifts of the sequence ending with a right parenthesis such that the subsequence consisting of parentheses only is legal from the right. Let t_2 be the canonical choice (found in the same way as in the proof of Proposition 4.2) from among these $2c$ cyclic shifts.

Now, from the string $t_1 t_2$, we create a partition π_1 in $NC^{(B)}(p, q)$ in the following way: The numbers inside a lowest-level pair of matching parentheses form a block of π_1; remove these numbers and this pair of parentheses from the string, and iterate until the string is empty. Then for $j = 2, \ldots, m-1$, remove the right parentheses of types $)^1, \ldots,)^{j-1}$ from $t_1 t_2$, together with the left parentheses that pair with them, and read the remaining string as above to obtain π_j. This produces the multichain $\pi_1 \leq \cdots \leq \pi_{m-1}$ in $NC^{(B)}(p, q)$, and gives a bijection with the required properties.

To see that at least one of the π_i has positive connectivity, note that the string $t_1 t_2$ starts with $(a \cdots$ and ends with $\cdots b)^{\ell_1} \cdots)^{\ell_k}$, where $a \in \{1, \ldots, p, -1, \ldots, -p\}$, $b \in \{p+1, \ldots, p+q, -p-1, \ldots, -p-q\}$, $k \geq 1$, and $1 \leq \ell_1 < \cdots < \ell_k \leq m-1$. Then, in $t_1 t_2$, the initial left parenthesis (is paired with the terminal right parenthesis $)^{\ell_k}$, and therefore π_i must have positive connectivity when $\ell_{k-1} < i \leq \ell_k$, since for these choices of i, elements a and b must appear in the same block of π_i in the construction above. ∎

Remark 5.2. As a concrete example for how Proposition 5.1 works, suppose we have $p = 6, q = 3, m = 3$, with $c = 2$, $d = 1$, $L^E = \{1,2,3,5,6\}$, $R_1^E = \{1,3\}$, $R_2^E = \{3\}$, and $L^I = \{8,9\}$, $R_1^I = \{7,8,9\}$, $R_2^I = \{7\}$. By inserting parentheses in $1,\dots,6,-1,\dots,-6$, we obtain the string

$$(1)^1(2(3)^1)^2 4(5(6(-1)^1(-2(-3)^1)^2 - 4(-5(-6,$$

which has 4 cyclic shifts that we might consider. Since we are given that $d = 1$, the cyclic shift we select is the one that begins with "(5", thus getting

$$t_1 = (5(6(-1)^1(-2(-3)^1)^2 - 4(-5(-6(1)^1(2(3)^1)^2 4.$$

Similarly, we obtain

$$t_2 = (-8)^1(-9)^1 7)^1)^2(8)^1(9)^1 - 7)^1)^2,$$

and from the string t_1t_2, we obtain the partitions

$$\pi_1 = \{\{4,-6,7\},\{-4,6,-7\}\} \cup \{\{i\} \mid 1 \le |i| \le 9,\ |i| \ne 4,6,7\},$$

$$\pi_2 = \{\{1,4,-5,-6,7,-8,-9\},\{-1,-4,5,6,-7,8,9\},\{2,3\},\{-2,-3\}\}.$$

Note that $\pi_1 \le \pi_2$, and that both π_1 and π_2 have positive connectivity (both have connectivity equal to 1).

As an immediate enumerative consequence of Proposition 5.1, we obtain the zeta polynomial for $NC^{(B)}(p,q)$.

Theorem 5.3. *Let p,q be positive integers.*

1^o *The zeta polynomial of $NC^{(B)}(p,q)$ is given by*

$$Z_{NC^{(B)}(p,q)}(m) = \binom{mp}{p}\binom{mq}{q} + \sum_{c=1}^{p} 2c \binom{mp}{p-c}\binom{mq}{q+c}. \tag{5.3}$$

2^o *The number of maximal chains in $NC^{(B)}(p,q)$ is equal to*

$$\binom{p+q}{p} p^p q^q + \sum_{c\ge1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}. \tag{5.4}$$

Proof. 1^o The zeta polynomial Z_P of a partially ordered set P is defined via the condition that for every $m \ge 2$, the value $Z_P(m)$ is equal to the number of multichains $\pi_1 \le \cdots \le \pi_{m-1}$ in P (see [12, Section 3.11]). From Proposition 5.1, the number of such multichains in which at least one of the π_i has positive connectivity is given by

$$\sum_{c\ge1} 2c \sum_{\substack{a_1,\dots,a_{m-1},\\ b_1,\dots,b_{m-1}\ge0}} \binom{p}{a_1+\cdots+a_{m-1}+c}\binom{q}{b_1+\cdots+b_{m-1}-c} \prod_{j=1}^{m-1}\binom{p}{a_j}\binom{q}{b_j}. \tag{5.5}$$

The summation in (5.5) can be simplified if one invokes a well-known multinomial formula (which incidentally is a generalization of the binomial identity (3.1) used in the preceding sections), stating that for any integers $A_1, \dots, A_k, A_{k+1} \geq 0$ and $0 \leq b \leq A_{k+1}$, we have

$$\sum_{a_1,\dots,a_k \geq 0} \binom{A_1}{a_1} \cdots \binom{A_k}{a_k} \binom{A_{k+1}}{a_1 + \cdots + a_k + b} = \binom{A_1 + \cdots + A_{k+1}}{A_{k+1} - b}. \tag{5.6}$$

By applying (5.6) to the inner summation, we find that (5.5) can be simplified to

$$\sum_{c \geq 1} 2c \binom{mp}{p-c} \binom{mq}{q+c}. \tag{5.7}$$

On the other hand, we also have a simple formula for multichains $\pi_1 \leq \cdots \leq \pi_{m-1}$ in which all of the π_i have connectivity equal to 0. Indeed, these are simply multichains in the direct product of $NC^{(B)}(p)$ with $NC^{(B)}(q)$, and thus the number of these is given by

$$Z_{NC^{(B)}(p)}(m) \cdot Z_{NC^{(B)}(q)}(m) = \binom{mp}{p} \binom{mq}{q}, \tag{5.8}$$

from [11, Proposition 7].

The expression for $Z_{NC^{(B)}(p,q)}(m)$ now follows by adding together (5.7) and (5.8).

2^o For any partially ordered set P, the number of maximal chains is given by $d!$ times the coefficient of m^d in $Z_P(m)$, where the zeta polynomial $Z_P(m)$ has degree d (see [12, Proposition 3.11.1(a)]). From part 1^o of this result, we have $d = p+q$ in this case, and the result follows from part 1^o. ∎

Corollary 5.4. *For $p, q \geq 1$, $NC^{(B)}(p,q)$ has Möbius function*

$$\mu_{NC^{(B)}(p,q)}\left(\widehat{0}, \widehat{1}\right)$$
$$= (-1)^{p+q} \left(\binom{2p-1}{p} \binom{2q-1}{q} + \sum_{c=1}^{p} 2c \binom{2p-c-1}{p-1} \binom{2q+c-1}{q-1} \right).$$

Proof. This follows immediately from Theorem 5.3, using the fact that for any partially ordered set P, one has $\mu_P\left(\widehat{0}, \widehat{1}\right) = Z_P(-1)$ (see [12, Proposition 3.11.1(c)]). ∎

Remark 5.5. It is straightforward to specialize Corollary 5.4 to the case $p = n-1$, $q = 1$, either by directly evaluating the summation or by setting $p = 1$, $q = n-1$ and using the symmetry between p and q. Using either of these means, we obtain the expression given in Theorem 3.6. Note further that we can specialize Theorem 5.3 itself in the same way, to obtain that for every $n \geq 2$, the zeta polynomial of $NC^{(B)}(n-1,1)$ is given by the formula

$$Z_{NC^{(B)}(n-1,1)}(m) = \left(2 + \frac{mn}{(m-1)(n-1)}\right) \cdot \binom{m(n-1)}{n}. \tag{5.9}$$

6. The Case $NC^{(B)}(n_1,\ldots,n_k)$

In this section, we consider the extension to multiannular non-crossing partitions of type B. The main point of the section is to establish that, due to a topological restriction called "the genus inequality", the general multiannular case reduces in fact to the cases of non-crossing partitions of type B in a disc or an annulus.

We find it convenient to introduce the following notation.

Notation 6.1. 1^o For $\tau \in B_n$, let $\#(\tau)$ denote the number of orbits of τ, and let $ii(\tau)$ denote the number of inversion-invariant orbits of τ.

2^o For $\tau, \sigma \in B_n$, let $\#(\tau, \sigma)$ denote the number of orbits for the action of the subgroup of B_n generated by $\{\tau, \sigma\}$.

Remark 6.2. 1^o In terms of the notations introduced above, the formula (2.7) for the length of an element $\tau \in B_n$ is now written in the form

$$\ell_B(\tau) = n - \tfrac{1}{2}(\#(\tau) - ii(\tau)). \tag{6.1}$$

2^o The genus inequality mentioned at the beginning of the section is an inequality that arises in multiplying arbitrary permutations, and is stated as follows: For any permutations τ, σ of a finite set X, we have

$$\#(\sigma) + \#(\tau) + \#\left(\tau^{-1}\sigma\right) \leq |X| + 2\cdot\#(\tau,\sigma). \tag{6.2}$$

The name "genus inequality" comes from the fact that the difference of the right-hand side and left-hand side of (6.2) is a non-negative even integer $2g$, where g is the genus for a certain orientable surface constructed from τ and σ (see, e.g., [9, Proposition 1.5.3]).

Definition 6.3. *Throughout the rest of the section, we let γ be a fixed element of B_n with $\#(\gamma) = ii(\gamma) = k$, in which the k orbits of γ have sizes $2n_1,\ldots,2n_k$, where $n_1,\ldots,n_k \geq 1$ and $n_1+\cdots+n_k = n$. Analogously to* (2.4) *and* (2.5)*, we define*

$$\mathcal{S}_{nc}^{(B)}(n_1,\ldots,n_k) := \{\tau \in B_n \mid \tau \leq \gamma\}, \tag{6.3}$$

where the partial order $\leq$ was defined in (2.3)*. We then define $NC^{(B)}(n_1,\ldots,n_k)$ as in* (2.6)*, by putting*

$$NC^{(B)}(n_1,\ldots,n_k) := \left\{\widetilde{\Omega}(\tau) \,\middle|\, \tau \in \mathcal{S}_{nc}^{(B)}(n_1,\ldots,n_k)\right\}, \tag{6.4}$$

where the adjusted orbit map $\widetilde{\Omega}$ is as in Definition 2.2.

Proposition 6.4. *Let τ be a permutation in $\mathcal{S}_{nc}^{(B)}(n_1,\ldots,n_k)$. We denote $\#(\tau,\gamma) =: m$ $(1 \leq m \leq k)$. Let $Y_1,\ldots,Y_m$ denote the orbits counted by $\#(\tau,\gamma)$, with $\left|Y_j\right| = 2y_j$, $j = 1,\ldots,m$. Moreover, for every $1 \leq j \leq m$, let us denote the restrictions of τ and γ to Y_j by τ_j and γ_j, respectively. We then have*

$$\#(\gamma_j) = ii(\gamma_j) \leq 2, \qquad j = 1,\ldots,m. \tag{6.5}$$

Proof. Note that $Y_j = X_j \cup (-X_j)$, for $j = 1, \ldots, m$, where $\{X_1, \ldots, X_m\}$ is a partition of $\{1, \ldots, n\}$ into nonempty subsets, with $|X_j| = y_j$ for $1 \le j \le m$. Now, the triangle inequality (2.2) gives

$$\ell_B(\gamma_j) \le \ell_B(\tau_j) + \ell_B\left(\tau_j^{-1}\gamma_j\right), \qquad j = 1, \ldots, m.$$

Then these inequalities, together with (2.3), (6.3) and the facts that $\ell_B(\gamma) = \ell_B(\gamma_1) + \cdots + \ell_B(\gamma_m)$, $\ell_B(\tau) = \ell_B(\tau_1) + \cdots + \ell_B(\tau_m)$, give

$$\tau \in \mathcal{S}_{nc}^{(B)}(n_1, \ldots, n_k) \Rightarrow \ell_B(\gamma_j) = \ell_B(\tau_j) + \ell_B\left(\tau_j^{-1}\gamma_j\right), \qquad j = 1, \ldots, m. \tag{6.6}$$

But, from (6.1), rearranging the equation in (6.6), and using the fact that $\#(\gamma_j) = ii(\gamma_j)$, we obtain

$$\#(\gamma_j) + \#(\tau_j) + \#\left(\tau_j^{-1}\gamma_j\right) = 2y_j + ii(\gamma_j) + ii(\tau_j) + ii\left(\tau_j^{-1}\gamma_j\right), \qquad j = 1, \ldots, m. \tag{6.7}$$

On the other hand, the genus inequality (6.2) implies that

$$\#(\gamma_j) + \#(\tau_j) + \#\left(\tau_j^{-1}\gamma_j\right) \le 2y_j + 2, \tag{6.8}$$

since $\#(\tau_j, \gamma_j) = 1$. Thus (6.7) and (6.8) together imply that

$$ii(\gamma_j) + ii(\tau_j) + ii\left(\tau_j^{-1}\gamma_j\right) \le 2,$$

and, in particular, that $ii(\gamma_j) \le 2$, as required. ∎

Remark 6.5. When rephrased in terms of partitions, Proposition 6.4 says that every partition $\pi \in NC^{(B)}(n_1, \ldots, n_k)$ splits the k orbits of γ into groups of cardinality 1 or 2; thus π is obtained by putting together several separate "pieces", where each piece is either a non-crossing partition of type B in a disc, or a non-crossing partition of type B in an annulus — precisely the cases that were considered earlier in the paper. Due to this phenomenon, the enumerative properties of $NC^{(B)}(n_1, \ldots, n_k)$ are quickly reduced to what we know from the disc and the annular cases, where one also has to do a suitable summation over partial matchings for the k orbits of γ. For illustration, we finish with an example of how such a calculation is carried out, in the particular case when $k = 3$.

Example 6.6. Suppose $k = 3$, and we want to determine the total number of partitions in $NC^{(B)}(n_1, n_2, n_3)$. By invoking Proposition 6.4, taking into account that there are 4 possible partial matchings of the set of 3 orbits of γ, we find that

$$\begin{aligned}\left|NC^{(B)}(n_1, n_2, n_3)\right| = \left|NC^{(B)}(n_1)\right| \cdot \left|NC_+^{(B)}(n_2, n_3)\right| + \left|NC^{(B)}(n_2)\right| \cdot \left|NC_+^{(B)}(n_1, n_3)\right| \\ + \left|NC^{(B)}(n_3)\right| \cdot \left|NC_+^{(B)}(n_1, n_2)\right| \\ + \left|NC^{(B)}(n_1)\right| \cdot \left|NC^{(B)}(n_2)\right| \cdot \left|NC^{(B)}(n_3)\right|,\end{aligned}$$

where $NC_{+}^{(B)}(p,q)$ denotes the subset of $NC^{(B)}(p,q)$ with positive connectivity. But Theorems 4.5.2 and 4.5.3 give

$$\left|NC_{+}^{(B)}(p,q)\right| = \frac{pq}{p+q}\binom{2p}{p}\binom{2q}{q},$$

and from these results together with (3.5) we conclude that

$$\left|NC^{(B)}(n_1,n_2,n_3)\right| = \left(1+\frac{n_1n_2}{n_1+n_2}+\frac{n_1n_3}{n_1+n_3}+\frac{n_2n_3}{n_3+n_3}\right)\binom{2n_1}{n_1}\binom{2n_2}{n_2}\binom{2n_3}{n_3}.$$

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