Asymptotics for a Class of Meandric Systems, via the Hasse Diagram of NC(n)

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Abstract

We consider the framework of closed meandric systems, and its equivalent description in terms of the Hasse diagrams of the lattices of non-crossing partitions $\mathrm{NC}(n)$. In this equivalent description, considerations on the number of components of a random meandric system of order n translate into considerations about the distance between two random partitions in $\mathrm{NC}(n)$. We put into evidence a class of couples $(\pi, \rho) \in \mathrm{NC}(n)^2$ – namely the ones where π is conditioned to be an interval partition – for which it turns out to be tractable to study distances in the Hasse diagram. As a consequence, we observe a non-trivial class of meanders (i.e. connected meandric systems), which we call "meanders with shallow top", and which can be explicitly enumerated. Moreover, denoting by c_n the expected number of components for the corresponding notion of "meandric system with shallow top", we find the precise asymptotic $c_n \approx (9n + 28)/27$ for $n \to \infty$.

A variation of the methods used in the shallow-top case yields the existence of constants $0 < \alpha < \beta < 1$ such that the expected number of components of a general (unconditioned) random meandric system of order n falls in the interval $[\alpha n, \beta n]$ for all $n \in \mathbb{N}$. Our calculations concerning expected number of components are related to the idea of taking the derivative at $t = 1$ in a semigroup for the operation $\boxplus$ of free probability (but the underlying considerations are presented in a self-contained way, and can be followed without assuming a free probability background).

Another variation of the considerations described above goes by fixing a "base-point" λ_n, where λ_n is an interval partition in $\mathrm{NC}(n)$, and by focusing on distances in the Hasse diagram of $\mathrm{NC}(n)$ which are measured from λ_n. We illustrate this by determining precise formulas for the average distance to λ_n and for the cardinality of $\{\rho \in \mathrm{NC}(n) \mid \rho$ at maximal distance from $\lambda_n\}$ in the case when n is even and λ_n is the partition with blocks $\{1, 2\}, \{3, 4\}, \ldots, \{n - 1, n\}$.

Contents

1   Introduction

In this paper we consider the framework of closed meandric systems, and its equivalent description in terms of Hasse diagrams (that is, graphs of covers) of the lattices of non-crossing partitions NC(n). In this equivalent description, considerations on the number of components of a random meandric system of order n translate into considerations about the distance between two random partitions $\pi, \rho \in \mathrm{NC}(n)$, where π, ρ are viewed as vertices in the Hasse diagram. The relevant definitions and statements of facts concerning these frameworks will be reviewed precisely in Sections 2 and 3 below, here we only record the following important point: denoting the set of closed meandric systems of order n by $\mathfrak{M}(n)$, one has a natural bijection $\mathrm{NC}(n)^2 \ni (\pi, \rho) \mapsto M(\pi, \rho) \in \mathfrak{M}(n)$, with the property that

$$\left(\begin{array}{c} \text{number of components} \\ \text{of } M(\pi, \rho) \end{array} \right) \; = \; n - d_H(\pi, \rho), \quad \forall \, \pi, \rho \in \mathrm{NC}\,(n), \tag{1.1}$$

where d_H is the distance function in the Hasse diagram of NC (n). The picture of the meandric system $M(\pi, \rho)$ is obtained by "doubling" the partitions π and ρ, and by drawing the resulting doublings above and respectively below a horizontal line with $2n$ points marked on it. (See Section 3 for a precise description of this.)

When looking at meandric systems, two appealing questions that immediately arise are:

- What is the average number of components of a random meandric system in $\mathfrak{M}(n)$?

- What is the probability that a random meandric system in $\mathfrak{M}(n)$ is connected (that is, it has only one component)?

The latter question is a rather celebrated one: a connected meandric system goes under the short name of *meander*, and since the cardinality of $\mathfrak{M}(n)$ is easily calculated as the square Catalan number Cat_n^2, this question actually asks what is the number of (closed) meanders of order n. This is known to be a hard problem, which is open even at the level of finding the asymptotic growth rate as $n \to \infty$ (see [DF00, LZ04]). The question about average number of components seems to have received less attention in the literature, but is very natural; we suspect it to be rather hard as well. As explained below, there are reasons to believe that the average mentioned in this question must behave like $c \cdot n$ for a constant $c \in (0.17, 0.50)$ (which, if exists, is yet to be determined).

When converted to the framework of the Hasse diagram of NC(n), the two questions stated above become:

- What is the average distance between two random partitions in NC(n)?

- What is the probability that two random partitions $\pi, \rho \in \mathrm{NC}(n)$ have $d_H(\pi, \rho) = n - 1$ (i.e. that π and ρ "achieve a diameter" in the Hasse diagram of NC(n))?

Since the conversion from meandric systems to the Hasse diagrams of NC(n) is rather straightforward, the new questions are about as tractable (or intractable) as the original ones.

In this paper we study distances between $\pi, \rho \in NC(n)$ where π is conditioned to be an *interval* partition – that is, every block of π is of the form $\{i \in \mathbb{N} \mid p \leq i \leq q\}$ for some $p \leq q$ in $\{1, \ldots, n\}$. We denote the set of interval partitions of order n by Int (n). As motivation for why one might look

$\mathfrak{M}(n)$

Int (n)

at interval partitions, we can invoke the state of mind which comes from using partitions in the combinatorial development of non-commutative probability. When looking for non-commutative analogues for classical probability theory, a case can be made (cf. [Spe97]) that there are two main possibilities which can be pursued: free probability, whose combinatorics is based on the lattices $\mathrm{NC}(n)$, and the lighter theory of Boolean probability, whose combinatorics is based on the lattices $\mathrm{Int}(n)$ of interval partitions. There is no doubt that meandric systems have connections to free probability, and the specialization "$\pi \in \mathrm{Int}(n)$" can be construed as "going Boolean in the first variable". The cost of this specialization is that the cardinality of $\mathrm{Int}(n)$ is roughly only the square root of that of $\mathrm{NC}(n)$: $|\mathrm{Int}(n)| = 2^{n-1}$, while $|\mathrm{NC}(n)| = \mathrm{Cat}_n$, with magnitude of roughly 4^n. But on the bright side, one has the precise results stated in Theorems 1.1 and 1.3 below.

In the statement of Theorem 1.1, the "1" in the notation $\overline{\mathfrak{M}_1}(n)$ marks the fact that a meandric system corresponding to a couple $(\pi, \rho) \in \overline{\mathfrak{M}_1}(n)$ has 1 component. We are also using the notation $|\pi|$ for the number of blocks in a partition $\pi \in NC(n)$.

$|\pi|$

$\overline{\mathfrak{M}_1}(n)$

Theorem 1.1. *For every $n \in \mathbb{N}$, denote*

$$\overline{\mathfrak{M}_1}(n) = \{(\pi, \rho) \in \mathrm{Int}(n) \times \mathrm{NC}(n) \mid d_H(\pi, \rho) = n - 1\}. \tag{1.2}$$

Then one has

$$\left| \overline{\mathfrak{M}_1}(n) \right| = \sum_{m=1}^{n} \frac{1}{n} \binom{n}{m-1} \binom{n+m-1}{n-m}. \tag{1.3}$$

Moreover, in the latter sum, the term indexed by every $m \in \{1, \ldots, n\}$ gives precisely the number of couples (π, ρ) counted on the right-hand side of (1.2) and where $|\pi| = m$.

Remark 1.2. We say that $\rho \in \mathrm{NC}(n)$ is a "meandric partner" of $\pi \in \mathrm{NC}(n)$ if $M(\pi, \rho)$ is a meander, or, equivalently, if $d_H(\pi, \rho) = n - 1$. The exponential growth rate of the meanders counted in Theorem 1.1 is roughly 5.22, namely, $\overline{\lim}_{n \to \infty} \sqrt[n]{\left| \overline{\mathfrak{M}_1}(n) \right|} \approx 5.22$ (see Corollary 5.5). Since there are 2^{n-1} interval partitions, this means that the average number of meandric partners of an interval partition is roughly 2.61^n. This is less than the average for all non-crossing partitions, which has exponential growth rate of at least 2.845 [AP05, Theorem 1.1] and conjectured to be roughly 3.066 [JG00]. In Conjecture 1.8 below we make a conjecture as for the type of non-crossing partitions with the maximal number of meandric partners.

d_n

Theorem 1.3. *For every $n \in \mathbb{N}$, consider the average distance*

$$d_n = \frac{1}{2^{n-1} \cdot \mathrm{Cat}_n} \sum_{\substack{\pi \in \mathrm{Int}(n), \\ \rho \in NC(n)}} d_H(\pi, \rho). \tag{1.4}$$

Then one has

$$\lim_{n \to \infty} \left(d_n - \frac{2}{3} n \right) = -28/27. \tag{1.5}$$

We mention that the limit stated in Equation (1.5) is found by making $n \to \infty$ in an explicit expression which holds for a fixed value of n, and which is indicated precisely in Proposition 8.5 below.

A suggestive name for the meandric systems of order n that correspond to couples $(\pi, \rho) \in \mathrm{Int}(n) \times \mathrm{NC}(n)$ is "meandric systems with shallow top". Indeed, the standard drawing of the meander $M(\pi, \rho)$ with a horizontal infinite line L, is shallow, in the sense that every point on L has at most two lines above it, if and only if π is an interval partition (see Sections 3 and 5). When we convert back from Hasse diagrams to meandric systems, Theorem 1.1 will thus give us the number

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of meanders (i.e. connected meandric systems) of order n and with shallow top. On the other hand, denoting by c_n the expected number of components of a random meandric system of order n with shallow top, we have $c_n = n - d_n$ by (1.1), hence Theorem 1.3 gets converted into the statement that

$$\lim_{n\to\infty} \left(c_n - \frac{n}{3} \right) = 28/27. \tag{1.6}$$

In connection to Theorem 1.3, it is relevant to ask: if we were to average the distances $d_H(\pi, \rho)$ for all $\pi, \rho \in NC(n)$ (rather than conditioning π to be in $\text{Int}(n)$), would that average still follow a regime of "constant times n", where the constant is contained in $(0,1)$? Or, at the very least, does that average admit some lower and upper bounds of the form $\alpha\,n$ and $\beta\,n$, with $\alpha, \beta \in (0,1)$? The existence of a lower bound $\alpha\,n$ is in fact immediate, because it is easy to prove (c.f. Corollary 4.7) that

$$\frac{1}{\text{Cat}_n^2} \sum_{\pi, \rho \in NC(n)} d_H(\pi, \rho) \geq \frac{n-1}{2}, \quad \forall\, n \in \mathbb{N}.$$

The methods of the present paper allow us to also prove the existence of an upper bound $\beta\,n$. In order to obtain it, we use the following inequality:

$$d_H(\pi, \rho) \leq |\pi| + |\rho| - 2|\pi \vee \rho|, \quad \forall\, \pi, \rho \in \text{NC}(n), \tag{1.7}$$

where "$\vee$" is the join operation in $\text{NC}(n)$. This is just a triangle inequality, where the right-hand side is $d_H(\pi, \pi \vee \rho) + d_H(\pi \vee \rho, \rho)$; but it is nevertheless very useful, due to the following proposition, which is obtained along similar lines to the proof of Theorem 1.3 – details are given in Section 8 below (cf. Proposition 8.6 there).

Proposition 1.4.

$$\lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{\text{Cat}_n^2} \sum_{\pi, \rho \in \text{NC}(n)} |\pi| + |\rho| - 2|\pi \vee \rho| = \frac{3\pi - 8}{8 - 2\pi} < 0.83.$$

By using Proposition 1.4 and inequality (1.7), one immediately obtains the needed upper bound for average distance $d_H(\pi, \rho)$, and hence the following corollary.

Corollary 1.5. *(1) Denote by d_n' the expected distance between two randomly chosen non-crossing partitions in $\text{NC}(n)$. Then for any $\varepsilon > 0$ and large enough n*

$$(0.5 - \varepsilon)\, n \leq d_n' \leq 0.83n.$$

(2) Denote by c_n' the expected number of components of a random meandric system of order n. Then for any $\varepsilon > 0$ and large enough n

$$0.17n \leq c_n' \leq (0.5 + \varepsilon)\, n.$$

A variation of the framework used in Theorems 1.1 and 1.3 is obtained by fixing a "base-point" $\lambda_n \in NC(n)$, and by focusing on distances in the Hasse diagram of $NC(n)$ which are measured from λ_n. In this paper we analyze in detail the case when one picks n to be even and chooses

$$\lambda_n = \big\{ \{1, 2\}, \{3, 4\}, \ldots, \{n-1, n\} \big\}. \tag{1.8}$$

By using methods similar to those in the proofs of Theorems 1.1 and 1.3, we find that the following holds.

Proposition 1.6. *For every even $n \in \mathbb{N}$, let $\lambda_n = \{\{1, 2\}, \{3, 4\}, \ldots, \{n-1, n\}\} \in \text{NC}(n)$.*

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1. *One has*
$$|\{\rho \in \text{NC}(n) \,|\, d_H(\lambda_n, \rho) = n - 1\}| = 2^{(n-2)/2}\text{Cat}_{n/2}, \quad \forall\, n \in 2\mathbb{N}.$$

2. *For every even $n \in \mathbb{N}$, consider the average distance*
$$\widetilde{d_n} := \frac{1}{\text{Cat}_n} \sum_{\rho \in \text{NC}(n)} d_H(\lambda_n, \rho).$$

Then one has the explicit formula
$$\widetilde{d_n} = \frac{2^{n-1} \cdot \binom{n}{n/2}}{\text{Cat}_n} - \frac{3}{2}, \quad n \in 2\mathbb{N}.$$

As a consequence of this formula, it follows that:
$$\lim_{n \to \infty} \left(\widetilde{d_n} - \frac{\sqrt{2}}{2}n\right) = \frac{7\sqrt{2}}{16} - \frac{3}{2}.$$

The formulas found in Proposition 1.6 admit obvious conversions into formulas about the meandric systems of order n which have the top part arising from the doubling of λ_n. In particular, denoting by $\widetilde{c_n}$ the expected number of components of such a meandric system, one finds that

$$\lim_{n \to \infty} \left(\widetilde{c_n} - \left(1 - \frac{\sqrt{2}}{2}\right)n\right) = \frac{3}{2} - \frac{7\sqrt{2}}{16}. \tag{1.9}$$

Remark 1.7. It would be interesting to consider the kind of problem treated in Proposition 1.6, but where one picks other selections of λ_n's. We find it likely that the methods used in the present paper will still work for other situations where the λ_n's are picked to be interval partitions. However, things become complicated when the λ_n's start to have nestings. The hardest example along these lines appears to be the one where λ_n is the "rainbow" partition of $\{1, \ldots, n\}$: this is the partition with blocks $\{1, n\}, \{2, n-1\}, \ldots$ including a possible singleton block at $(n+1)/2$ when n is odd. In relation to this, we note that numerical experiments seem to support the following conjecture. Recall from (1.1) and Remark 1.2 that $\pi \in \text{NC}(n)$ is a meandric partner of $\lambda \in \text{NC}(n)$ if $d_H(\lambda, \rho) = n - 1$.

Conjecture 1.8. *For every $n \in \mathbb{N}$, the rainbow partition $\{\{1, n\}, \{2, n-1\}, \ldots\}$ is the partition in $\text{NC}(n)$ which has the largest number of meandric partners.*

More precisely, consider the orbit of the rainbow partition under the Kreweras complementation map (see Section 2). By symmetry, all n partitions in this orbit have the same number of meandric partners. We conjecture that this orbit constitutes the exact set of partitions in $\text{NC}(n)$ with the largest number of meandric partners. We also note here that counting the number of meandric partners for the rainbow partition amounts to counting some diagrams known in the literature under the name of *semi-meanders* (see e.g. [DFGG97, Section 2.2] or [LC03, Section 2.2.1]). This is a well-known problem, which is believed to be hard. The conjecture stated above is not about precisely counting semi-meanders, but about proving an inequality between the number of semi-meanders and the cardinality of other sets of meanders with prescribed top.

Remark 1.9. Another conjecture that is suggested by the results of this paper concerns the asymptotics for expected number of components, c'_n, of a general meandric system of order n. It is natural to ask, in the spirit of Corollary 1.5:

$$\text{Does } \lim_{n \to \infty} \frac{c'_n}{n} \text{ exist? (If yes, what is it?)}$$

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Computational experiments done for some fairly large values of n [VP] suggest that the limit exists and is ≈ 0.23. This is smaller than the constants of ≈ 0.33 and ≈ 0.29 arising out of Equations (1.6) and respectively (1.9), which suggests the idea that the number of components of a random meandric system $M(\pi, \rho)$ increases when one conditions π to be an interval partition. This agrees in spirit with the fact, stated in Remark 1.2, that interval partitions have, on average, fewer meandric partners than general non-crossing partitions.

Paper organization

We conclude this introduction by explaining how the paper is organized, and by giving a few highlights on the content of the various sections. In Sections 2, 3 and 4 we introduce the framework used in the paper, and we discuss some necessary background. Specifically, Section 2 reviews basic facts about $(\mathrm{NC}(n), \leq)$ and its Hasse diagram $\mathcal{H}_n$; Section 3 discusses the connection between these Hasse diagrams and meandric systems; and Section 4 presents several equivalent formulas for distances in $\mathcal{H}_n$.

In Section 5 we discuss interval partitions and meanders with shallow top, and in Section 6, relying on the facts from Sections 4 and 5, we prove there is a bijection between the set of meanders with shallow top and the set of certain finite trees. This leads to explicit enumerative results, which are presented separately in Section 7, and include in particular Theorem 1.1 and Proposition 1.6(1).

Finally, Section 8 discusses averages of distances and presents the proofs of Theorem 1.3 and of Propositions 1.4 and 1.6(2). Our calculations in this section are related to the idea of taking the derivative at $t = 1$ in a semigroup for the operation $\boxplus$ of free probability (but the underlying considerations are presented in a self-contained way, and can be followed without assuming a free probability background).

2 Background on $(\mathrm{NC}(n), \leq)$ and its Hasse diagram

Definition 2.1. Let n be a positive integer.

1. We will work with partitions of the set $\{1, \ldots, n\}$. Our typical notation for such a partition is $\pi = \{V_1, \ldots, V_k\}$, where $V_1, \ldots, V_k$ (the *blocks* of π) are non-empty, pairwise disjoint sets with $\bigcup_{i=1}^{k} V_i = \{1, \ldots, n\}$. We will occasionally use the notation "$V \in \pi$" to mean that V is one of the blocks of the partition π. The number of blocks of π is denoted by $|\pi|$.

2. We say that a partition π of $\{1, \ldots, n\}$ is *non-crossing* when it is not possible to find two distinct blocks $V, W \in \pi$ and numbers $a < b < c < d$ in $\{1, \ldots, n\}$ such that $a, c \in V$ and $b, d \in W$. Equivalently, π is non-crossing if it can be depicted in a diagram with n vertices arranged on an invisible horizontal line, so that the blocks of π are the connected components of a planar diagram drawn in the upper-half plane: see Figure 1 and also e.g. [NS06, Lecture 9]. The set of all non-crossing partitions of $\{1, \ldots, n\}$ is denoted by $\mathrm{NC}(n)$. This is one of the many combinatorial structures counted by Catalan numbers, namely

$$|\mathrm{NC}(n)| = \mathrm{Cat}_n := \frac{(2n)!}{n!(n+1)!} \quad \text{(the } n\text{th Catalan number)}$$

(see e.g. [NS06, Proposition 9.4] and also Remark 3.1 below).

3. On $\mathrm{NC}(n)$ we will use the partial order given by *reverse refinement*: for π, ρ we put

$$\left(\pi \leq \rho \right) \Leftrightarrow \left(\begin{array}{c} \text{for every } V \in \pi \text{ there} \\ \text{exists } W \in \rho \text{ such that } V \subseteq W \end{array} \right). \tag{2.1}$$
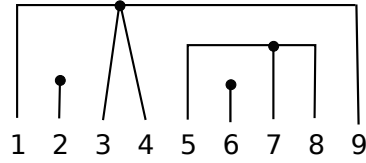
Figure 1: A non-crossing geometric realization of the non-crossing partition $\{\{1,3,4,9\},\{2\},\{5,7,8\},\{6\}\} \in \mathrm{NC}(9)$

4. We denote by 0_n the partition of $\{1,\ldots,n\}$ into n blocks of one element each, and we denote by 1_n the partition of $\{1,\ldots,n\}$ into one block of n elements. These are the minimum and the maximum elements in $(\mathrm{NC}(n),\leq)$, that is, one has $0_n \leq \pi \leq 1_n$ for every $\pi \in \mathrm{NC}(n)$.

5. The partially ordered set $(\mathrm{NC}(n),\leq)$ turns out to be a *lattice*, which means that every $\pi,\rho \in \mathrm{NC}(n)$ have a least common upper bound $\pi \vee \rho$ and a greatest common lower bound $\pi \wedge \rho$. The partitions $\pi \vee \rho$ and $\pi \wedge \rho$ are called the *join* and *meet*, respectively, of π and ρ. It is easily verified that $\pi \wedge \rho$ can be explicitly described as

$$\pi \wedge \rho := \{V \cap W \mid V \in \pi, W \in \rho, V \cap W \neq \emptyset\}, \tag{2.2}$$

but there is no such simple explicit formula for $\pi \vee \rho$. (For the proof that $\pi \vee \rho$ does indeed exist, see e.g. [Sta12, Proposition 3.3.1].)

Definition 2.2. Let n be a positive integer.

1. Let π,ρ be in $\mathrm{NC}(n)$. We will say that ρ *covers* π to mean that $\pi \leq \rho$, $\pi \neq \rho$, and there exists no $\theta \in \mathrm{NC}(n) \setminus \{\pi,\rho\}$ such that $\pi \leq \theta \leq \rho$.

$\mathcal{H}_n$

2. The Hasse diagram of $(\mathrm{NC}(n),\leq)$ is the undirected graph $\mathcal{H}_n$ described as follows: the vertex set of $\mathcal{H}_n$ is $\mathrm{NC}(n)$, and the edge set of $\mathcal{H}_n$ consists of subsets $\{\pi_1,\pi_2\} \subseteq \mathrm{NC}(n)$ where one of π_1,π_2 covers the other. This is illustrated in Figure 2.
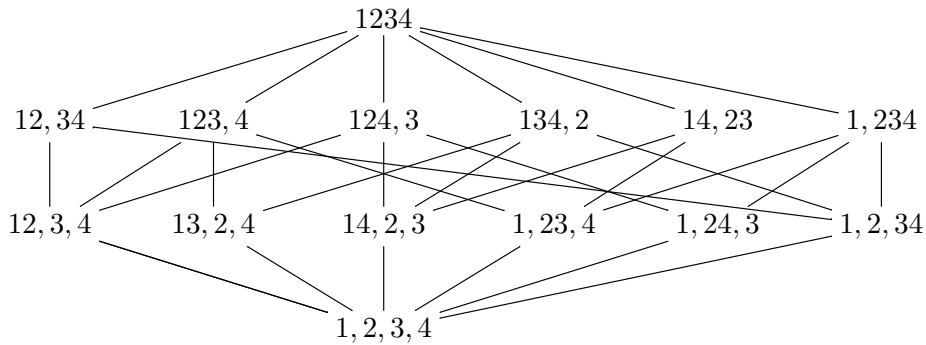

Figure 2: The Hasse diagram $\mathcal{H}_4$ of the lattice $\mathrm{NC}(4)$. Note that all partitions of $\{1,2,3,4\}$ are non-crossing except for $\{\{1,3\},\{2,4\}\}$.

Remark 2.3. It is not hard to see that for $\pi,\rho \in \mathrm{NC}(n)$, one has

$$\Big(\rho \text{ covers } \pi\Big) \Leftrightarrow \Big(\rho \geq \pi \text{ and } |\rho| = |\pi| - 1\Big).$$

So if one draws the vertices of $\mathcal{H}_n$ (that is, the partitions in $NC(n)$) arranged on horizontal levels according to number of blocks, then every edge of $\mathcal{H}_n$ will connect two vertices situated on adjacent

levels – see Figure 2. More precisely, an edge connects a partition π on level k to a partition ρ on level $k+1$ precisely when π can be obtained from ρ by taking a block $W \in \rho$ and breaking it into two pieces, in a non-crossing way.

As announced in the Introduction, the main object of concern for the present paper is the structure of *distances* in the Hasse diagram $\mathcal{H}_n$.

$d_H(\pi, \rho)$ **Notation 2.1.** For every positive integer n and for every $\pi, \rho \in \mathrm{NC}(n)$ we will use the notation $d_H(\pi, \rho)$ for the distance between π and ρ in the graph $\mathcal{H}_n$. That is, $d_H(\pi, \rho)$ is the length of the shortest path in $\mathcal{H}_n$ which connects π with ρ.

The next proposition records some easy observations about d_H:

Proposition 2.4. *Let n be a positive integer.*

1. *The diameter of $\mathcal{H}_n$ is $n-1$.*

2. *If $\pi, \rho \in \mathrm{NC}(n)$ are such that $\pi \leq \rho$, then $d_H(\pi, \rho) = |\pi| - |\rho|$.*

3. *If $\pi, \rho \in \mathrm{NC}(n)$ are such that $d_H(\pi, \rho) = n-1$, then it follows that $\pi \wedge \rho = 0_n$ and $\pi \vee \rho = 1_n$, and also that $|\pi| + |\rho| = n+1$.*

Proof. In (2), the inequality "$\geq$" is an immediate consequence of the fact that every edge in $\mathcal{H}_n$ connects partitions on adjacent levels of the graph. In order to prove "$\leq$", we take a saturated increasing chain in $\mathrm{NC}(n)$ which goes from π to ρ and we observe that it gives a path of length $|\pi| - |\rho|$ which connects the two partitions.

For (1), note that $d_H(0_n, 1_n) = n-1$ so the diameter is at least $n-1$. On the other hand, for every $\pi, \rho \in \mathrm{NC}(n)$ we have $\pi \wedge \rho \leq \pi$ and $\pi \leq \pi \vee \rho$, so part (2) assures us that

$$d_H(\pi \wedge \rho, \pi) = |\pi \wedge \rho| - |\pi| \text{ and } d_H(\pi, \pi \vee \rho) = |\pi| - |\pi \vee \rho|.$$

Similar inequalities hold with ρ featured in the place of π. But then we can write

$$
\begin{aligned}
d_H(\pi, \rho) &\leq \frac{d_H(\pi, \pi \wedge \rho) + d_H(\pi \wedge \rho, \rho) + d_H(\pi, \pi \vee \rho) + d_H(\pi \vee \rho, \rho)}{2} \\
&= |\pi \wedge \rho| - |\pi \vee \rho| \leq n-1,
\end{aligned}
\tag{2.3}
$$

where the latter inequality simply holds because $|\pi \wedge \rho| \leq n$ and $|\pi \vee \rho| \geq 1$.

To show (3), note that from (2.3) it is clear that the equality $d_H(\pi, \rho) = n-1$ forces the equalities $|\pi \wedge \rho| = n$, $|\pi \vee \rho| = 1$, i.e. $\pi \wedge \rho = 0_n$ and $\pi \vee \rho = 1_n$. Finally, if π, ρ are such that $d_H(\pi, \rho) = n-1$, then by writing

$$n - 1 = d_H(\pi, \rho) \leq d_H(\pi, 1_n) + d_H(1_n, \rho) = |\pi| + |\rho| - 2$$

we obtain that $|\pi| + |\rho| \geq n+1$, and by writing

$$n - 1 = d_H(\pi, \rho) \leq d_H(\pi, 0_n) + d_H(0_n, \rho) = 2n - (|\pi| + |\rho|)$$

we obtain that $|\pi| + |\rho| \leq n+1$. Hence the equality $d_H(\pi, \rho) = n-1$ implies that $|\pi| + |\rho| = n+1$. $\square$

Remark 2.5. The combination of necessary conditions $\pi \wedge \rho = 0_n$, $\pi \vee \rho = 1_n$ and $|\pi| + |\rho| = n+1$ found in Proposition 2.4(3) is not sufficient to ensure that $d_H(\pi, \rho) = n-1$. For example, the following piece (subgraph) from $\mathcal{H}_6$ shows two non-crossing partitions at distance 3 which satisfy these three properties:

$$1346, 2, 5 \qquad 16, 25, 34$$

$$13, 2, 46, 5 \qquad 16, 2, 34, 5$$
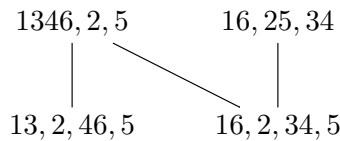
Embedding $\mathrm{NC}(n)$ **in** $\mathcal{S}_n$ **and the Kreweras complementation map**

Definition 2.6. Let n be a positive integer and let $\mathcal{S}_n$ be the group of permutations of $\{1, \ldots, n\}$. For every $\pi \in \mathrm{NC}(n)$ we construct an associated permutation $P_\pi \in \mathcal{S}_n$ as follows: the blocks of π become orbits of P_π, and P_π performs an increasing cycle on every such block; that is, if $V = \{i_1, i_2, \ldots, i_k\} \in \pi$ with $i_1 < i_2 < \cdots < i_k$, then we have $P_\pi(i_1) = i_2, \ldots, P_\pi(i_{k-1}) = i_k$, $P_\pi(i_k) = i_1$. This description includes the fact that if $\{i\} = V \in \pi$ is a singleton, then $P_\pi(i) = i$.

The map $\mathrm{NC}(n) \ni \pi \mapsto P_\pi \in \mathcal{S}_n$ is obviously injective. It is also clear that $\#(P_\pi) = |\pi|$ for every $\pi \in \mathrm{NC}(n)$, where for $\sigma \in \mathcal{S}_n$ we denote by $\#(\sigma)$ the number of cycles in σ. This embedding was introduced in [Bia97], and it has additional nice properties, some of which are mentioned in Section 4.

Remark 2.7. The Hasse diagram $\mathcal{H}_n$ displays top-down symmetry. This is explained by the existence of a natural bijection $\mathrm{Kr}_n \colon \mathrm{NC}(n) \to \mathrm{NC}(n)$ which reverses partial refinement order: for $\pi, \rho \in \mathrm{NC}(n)$ one has

$$(\pi \le \rho) \;\Leftrightarrow\; (\mathrm{Kr}_n(\rho) \le \mathrm{Kr}_n(\pi)).$$

The bijection Kr_n was introduced by Kreweras [Kre72] and is called the *Kreweras complementation map*. It can be defined in terms of permutations:

$$P_{\mathrm{Kr}_n(\pi)} = P_\pi^{-1} P_{1_n}, \quad \forall \, \pi \in NC(n). \tag{2.4}$$

Note that in (2.4), P_{1_n} is the long cycle $(1 \; 2 \; 3 \; \ldots \; n)$. For a discussion about this and a proof that for $\pi \in \mathrm{NC}(n)$, $P_\pi^{-1} P_{1_n}$ is equal indeed to P_λ for some $\lambda \in \mathrm{NC}(n)$, consult e.g. [NS06, Exercises 18.25 and 18.26]. For an alternative definition of the map Kr see e.g. [NS06, Pages 147-148] and also Remark 3.1 below. The fact that Kr_n is indeed a top-down symmetry follows from (2.4), together with Remark 3.1 and Theorem 4.4 below.

3 Meandric Systems and their Relation to NC (n)

Let L be a fixed oriented line in the Euclidean plane $\mathbb{E} := \mathbb{R}^2$ with $2n$ marked points $p_1, \ldots, p_{2n}$. Consider a non-intersecting (not necessarily connected) closed curve $C \subseteq \mathbb{E}$, which transversely intersects the line L at precisely the points $p_1, \ldots, p_{2n}$. Say that two such curves C_1 and C_2 are equivalent if they can be deformed into each other by an isotopy of $\mathbb{E}$ which fixes the line L pointwise. The equivalence class $M = [C]$ is called a *meandric system* of order n, or simply *a meander* of order n if C is connected. We denote the set of all meandric systems of order n by $\mathfrak{M}(n)$.

The notoriously difficult problem of enumerating meanders was first introduced by Lando and Zvonkin [LZ92]. It emerges in a variety of different areas inside and outside Mathematics – see e.g. [Arn88, DFGG96, LZ04]. Figure 3 shows all meanders of order 3 and Figure 4 illustrates a meandric system of order 4 with two connected components.

Consider the two open half-planes defined by L. The intersection of a meander with each half plane is a collection of n disjoint, self-avoiding arcs, each of which connects two of the points $p_1, \ldots, p_{2n}$. Such a configuration is called a *non-crossing pairing (or arch-diagram) of order n*. The set of non-crossing pairings of order n, considered up to homeomorphism as above, is denoted $\mathrm{NCP}(n)$, and it is standard that the cardinality of $\mathrm{NCP}(n)$ is Cat_n, the n-th Catalan * number. Evidently, there is a bijection between pairs of non-crossing pairings of order n and meandric systems of order n, given by

$$\mathrm{NCP}(n)^2 \ni (\pi, \rho) \mapsto M(\pi, \rho) \in \mathfrak{M}(n), \tag{3.1}$$

---

*Of course, $\mathrm{NCP}(n)$ is also a subset of $\mathrm{NC}(2n)$ consisting of non-crossing partitions with all blocks having size 2, but we do not use this point of view here.

P_π

Kr
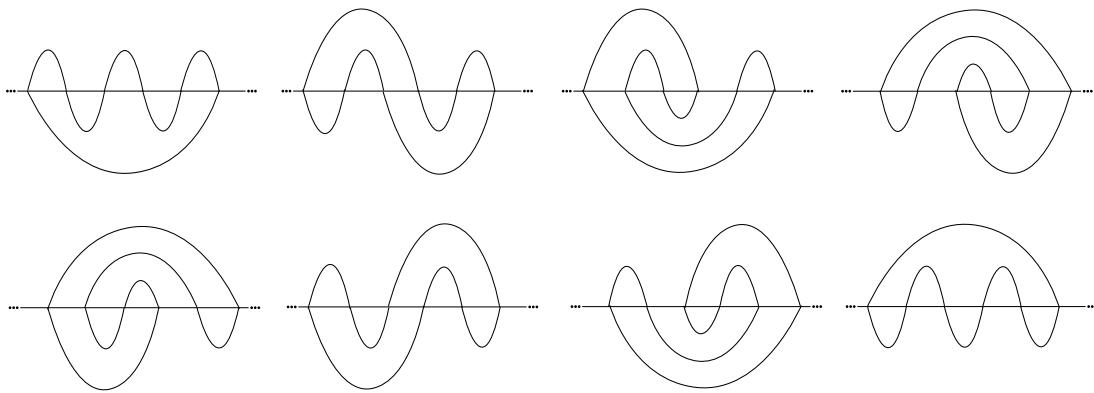
$\mathfrak{M}(n)$

$\mathrm{NCP}(n)$
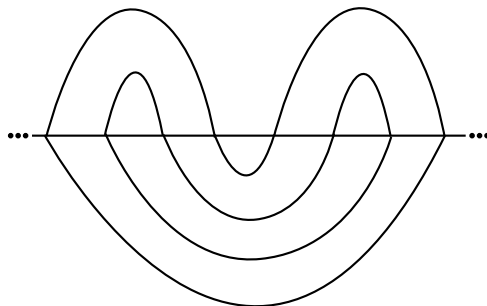
Figure 3: All meanders of order 3

Figure 4: A meandric system of order 4 with two connected components

where, by convention, the line L is assumed to be horizontal, π corresponds to the pairing in the upper half-plane, and ρ in the lower half-plane. The points $p_1, \ldots, p_{2n}$ appear from left to right. For example, in the meandric system in Figure 4, π is the pairing $\{\{p_1, p_4\}, \{p_2, p_3\}, \{p_5, p_8\}, \{p_6, p_7\}\} \in \mathrm{NCP}\,(4)$, while ρ is $\{\{p_1, p_8\}, \{p_2, p_7\}, \{p_3, p_6\}, \{p_4, p_5\}\} \in \mathrm{NCP}\,(4)$.

$\#M\,(\pi, \rho)$ The number of curves, or components, of the meandric system $M\,(\pi, \rho)$ is denoted $\#M\,(\pi, \sigma)$, and satisfies $1 \leq \#M\,(\pi, \sigma) \leq n$. Of course, $\#M\,(\pi, \sigma) = 1$ if and only if $M\,(\pi, \rho)$ is a meander. The total number of meandric systems of order n is, therefore, Cat_n^2. However, very little is known about the distribution of $\#M\,(\pi, \sigma)$ when n is large. Some numerics can be found in [DFGG97].

A bijection between non-crossing partitions and non-crossing pairings

There is a well-known natural bijection between $\mathrm{NC}\,(n)$, the set of non-crossing partitions of order n, and $\mathrm{NCP}\,(n)$, the set of non-crossing pairings of order n. The most intuitive way to describe this bijection is geometrical: given a non-crossing partition $\pi \in \mathrm{NC}\,(n)$ drawn as in Figure 1, take an ε-neighborhood $N_\varepsilon\,(\pi)$ of the graph describing π for small enough $\varepsilon > 0$, and consider $\partial\,(\pi)$, defined to be the intersection of the boundary of $N_\varepsilon\,(\pi)$ with the upper half-plane. Observe that $\partial\pi$ is a non-crossing pairing of order n, with $2n$ points on the horizontal line, one on each side of each of the original n points. The passage from $\pi \in NC(n)$ to $\partial\,(\pi) \in \mathrm{NCP}\,(n)$ is commonly known as the "doubling" or "fattening" of the partition π. This is illustrated in Figure 5.

    Conversely, given a non-crossing pairing $\rho \in \mathrm{NCP}\,(n)$ consisting of n disjoint curves with endpoints $p_1, \ldots, p_{2n}$ as above, mark additional n points $q_1, \ldots, q_n$ so that q_i lies on the interval of L between p_{2i-1} and p_{2i}. The n curves of ρ cut the upper-half plane into $n+1$ connected components. The connected components containing q-points define a non-crossing partition $c\,(\rho) \in \mathrm{NC}\,(n)$, where every block consists (of the indices) of the q-points lying inside one of these components. This is illustrated in Figure 6. The fact that $c\,(\rho)$ is non-crossing is immediate. The fact that $\rho \mapsto c\,(\rho)$ is the inverse map to $\pi \mapsto \partial\,(\pi)$ follows easily from the observation that if p_i is connected to p_j then
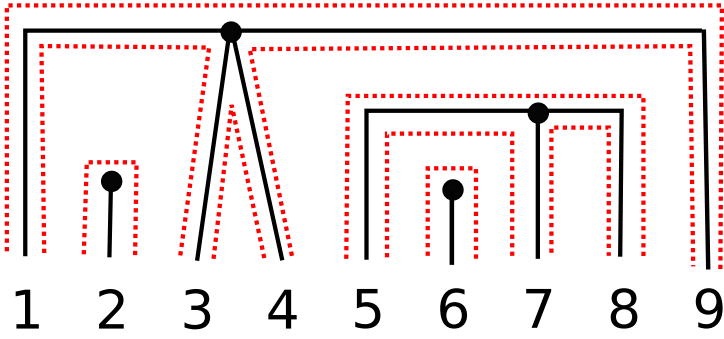
Figure 5: The non-crossing pairing (dashed lines) in $\mathrm{NCP}\,(9)$ corresponding to the non-crossing partition $\{\{1,3,4,9\}\,,\{2\}\,,\{5,7,8\}\,,\{6\}\} \in \mathrm{NC}\,(9)$
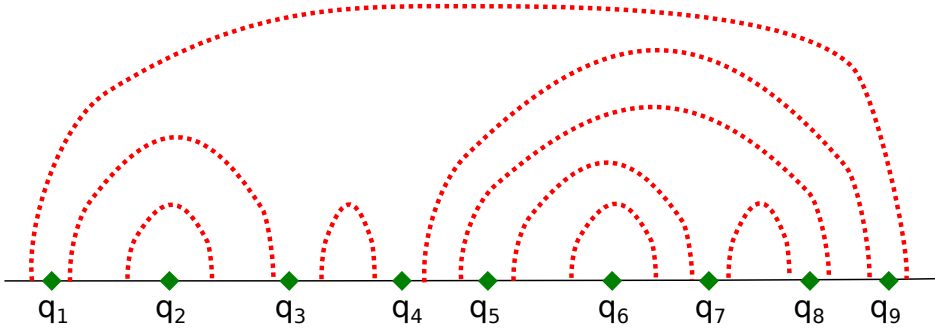

Figure 6: How a non-crossing pairing (dashed lines) in $\mathrm{NCP}\,(9)$ determines a non-crossing partitions of the same order. In this example the resulting non-crossing partition is $\{\{1,3,4,9\}\,,\{2\}\,,\{5,7,8\}\,,\{6\}\} \in \mathrm{NC}\,(9)$.

one of i and j is odd and the other is even.

Remark 3.1.   1. The bijection between $\mathrm{NC}\,(n)$ and $\mathrm{NCP}\,(n)$ yields a proof of the fact mentioned in Definition 2.1 that $|\mathrm{NC}\,(n)| = \mathrm{Cat}_n$.

2. In the description of $c\,(\rho)$, if the point q_i is taken to lie in the interval (p_{2i}, p_{2i+1}), instead of in the interval (p_{2i-1}, p_{2i}) (with q_n lying to the right of p_{2n}), then the resulting non-crossing partition $c'\,(\rho)$ is precisely the image of $c\,(\rho)$ under the Kreweras complementation map: $c'\,(\rho) = \mathrm{Kr}\,(c\,(\rho))$. Because the n curves of ρ cut the upper half-place into $n+1$ connected components, it follows that $|\pi| + |\mathrm{Kr}\,(\pi)| = n+1$ for every $\pi \in \mathrm{NC}\,(n)$. It is also easy to infer from this graphical definition of Kr why (2.4) holds.

$M\,(\pi,\rho)$ **Definition 3.2.** For $\pi,\rho \in \mathrm{NC}\,(n)$, we denote by

$$M\,(\pi,\rho) := M\,(\partial\,(\pi)\,,\partial\,(\rho))$$

the meandric system corresponding to $\partial\,(\pi)\,,\partial\,(\rho) \in \mathrm{NCP}\,(n)$. As before, $\#M\,(\pi,\rho)$ denotes the number of components of the meandric system.

Note the abuse of notation here: we use the same notation for the bijection $\mathrm{NCP}\,(n)^2 \to \mathfrak{M}\,(n)$ and for the bijection $\mathrm{NC}\,(n)^2 \to \mathfrak{M}\,(n)$.

4   Equivalent Expressions for $d_H(\pi,\rho)$

In this section we review some equivalent definitions, expressions and formulas for the quantity $d_H\,(\pi,\rho)$, as well as different criteria for $M\,(\pi,\rho)$ being a meander. All these equivalences can be
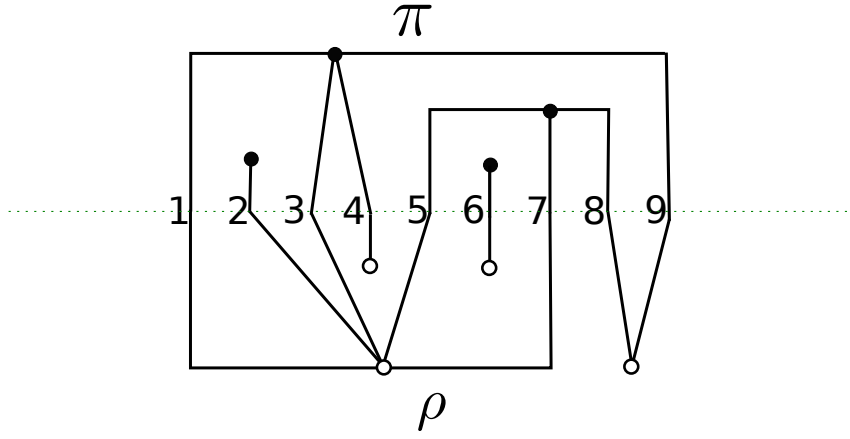
11

Figure 7: The planar, edge-labeled and vertex-colored graph $\Gamma(\pi,\rho)$ (in solid lines) for two partitions in NC (9): $\pi = \{\{1,3,4,9\},\{2\},\{5,7,8\},\{6\}\}$ and $\rho = \{\{1,2,3,5,7\},\{4\},\{6\},\{8,9\}\}$

found in the literature, although to the best of our knowledge, not in a single reference. We state the equivalences, refer to proofs in the literature and describe a few of the arguments.

The following two definitions are central to some of the equivalences, and the first one plays a crucial role in the subsequent sections:

$\Gamma(\pi,\rho)$ **Definition 4.1.** Given $\pi,\rho \in$ NC (n), let $\Gamma(\pi,\rho)$ denote the planar, edge-labeled and vertex-colored graph obtained by drawing π in the upper half-plane as in Figure 1, and ρ in the lower half plane, with the same n points labeled $1,\ldots,n$. More precisely, mark n points along an invisible horizontal line and label them, from left to right, by $1,\ldots,n$, draw π in the upper half-plane with a vertex for every block $V \in \pi$, and ρ in the lower half-plane with a vertex for every block $W \in \rho$. These are the $|\pi|+|\rho|$ vertices of the graph. Its n edges consist of the n lines incident with the marked points, and are labeled accordingly. That is, for every $i = 1,\ldots,n$, there is an edge labeled i connecting the block in π containing i with the block in ρ containing i. By convention, the vertices of π are black and the vertices of ρ are white*. This is illustrated in Figure 7. To the best of our knowledge, this graph was first introduced in [Fra98].

$\pi\widetilde{\vee}\rho$ *Remark* 4.2. For $\pi,\rho \in$ NC (n), we let $\pi\widetilde{\vee}\rho$ denote the smallest common upper bound of π and ρ in the lattice of all partitions of $\{1,\ldots,n\}$, crossing and non-crossing alike. It is easy to see that two numbers $i,j \in \{1,\ldots,n\}$ belong to the same block of $\pi\widetilde{\vee}\rho$ if and only if there exist $k \in \mathbb{N}$ and $i_0,i_1,\ldots,i_{2k} \in \{1,\ldots,n\}$ such that $i = i_0 \overset{\pi}{\sim} i_1 \overset{\rho}{\sim} i_2 \overset{\pi}{\sim} \cdots \overset{\pi}{\sim} i_{2k-1} \overset{\rho}{\sim} i_{2k} = j$, where $i \overset{\pi}{\sim} j$ means i and j belong to the same block of π. Evidently, the blocks of $\pi\widetilde{\vee}\rho$ are in one-to-one correspondence with the connected components of $\Gamma(\pi,\rho)$: the elements of a block correspond to the edge-labels in a component.

$\|\sigma\|$ **Definition 4.3.** For a permutation $\sigma \in \mathcal{S}_n$, denote by $\|\sigma\|$ its norm, defined as the length of the shortest product of transpositions in $\mathcal{S}_n$ giving σ. Namely,

$$\|\sigma\| \overset{\mathrm{def}}{=} \min\{g \,|\, \sigma = t_1\cdots t_g, \text{ where } t_1,\ldots,t_g \text{ are transpositions}\}.$$

It is standard that $\|\sigma\| = n - \#\mathrm{cycles}(\sigma)$ (e.g. [Hal06, Proposition 2.4]).

We can now state the equivalent expressions for the distance $d_H(\pi,\rho)$ between two non-crossing partitions in the Hasse diagram $\mathcal{H}_n$. The equivalence of 1, 2 and 3 from Theorem 4.4 appears in [Hal06, Theorem 3.3] and independently in [Sav09]. The equivalence of 2, of 4 and, in a slightly different language of 5, is shown in [Fra98].

---

*This convention will be useful only in subsequent sections.

Theorem 4.4. *Let $\pi, \rho \in \mathrm{NC}(n)$ be two non-crossing partitions of order n. Then the following quantities are equal:*

1. $d_H(\pi, \rho)$

2. $n - \#M(\pi, \rho)$

3. $\left\| P_\pi P_\rho^{-1} \right\| = n - \#\mathrm{cycles}\left(P_\pi P_\rho^{-1} \right)$

4. $|\pi| + |\rho| - 2\left| \pi \widetilde{\vee} \rho \right|$

5. $n - \displaystyle\sum_{\substack{C:\text{ connected} \\ \text{component of } \Gamma(\pi,\rho)}} \#\mathrm{faces}(C)$

Proof. For a short and elegant proof of the equivalence 1$\Longleftrightarrow$3 we refer the reader to [Hal06, Theorem 3.3]. It is an easy observation that the different components of the meandric system $M(\pi, \rho)$ are in one-to-one correspondence with the cycles of $P_\pi P_\rho^{-1}$: starting at a point on the horizontal invisible line just left to some edge i in $\Gamma(\pi, \rho)$, the permutation P_ρ^{-1} takes i along the arc in the lower half-plane, and then P_π takes the resulting point through an arc in the upper half-plane. Consult, for example, Figure 8. This shows the equivalence 2$\Longleftrightarrow$3.

It is clear that the components of $M(\pi, \rho)$ are curves tracing faces (in the sense of planar graphs) of $\Gamma(\pi, \rho)$, which shows 2$\Longleftrightarrow$5. Recall Euler's formula for a connected planar graph C:

$$v(C) - e(C) + f(C) = 2,$$

where $v(C)$ is the number of vertices of a graph, $e(C)$ the number of edges and $f(C)$ the number of faces. Thus, recalling the fact from Remark 4.2 that $\left| \pi \widetilde{\vee} \rho \right|$ is the number of components of $\Gamma(\pi, \rho)$, we obtain:

$$
\begin{aligned}
n - \sum_{\substack{C:\text{connected} \\ \text{component of } \Gamma(\pi,\rho)}} f(C) &= e(\Gamma(\pi, \rho)) - \sum_{\substack{C:\text{connected} \\ \text{component of } \Gamma(\pi,\rho)}} [2 + e(C) - v(C)] \\
&= v(\Gamma(\pi, \rho)) - \sum_{\substack{C:\text{connected} \\ \text{component of } \Gamma(\pi,\rho)}} 2 \\
&= |\pi| + |\rho| - 2\left| \pi \widetilde{\vee} \rho \right|,
\end{aligned}
$$

whence 4$\Longleftrightarrow$5. $\square$

1. An equivalent way of defining the norm of a permutation is through a Cayley graph of $\mathcal{S}_n$, as follows. Let T_n be the set of $\binom{n}{2}$ transpositions in $\mathcal{S}_n$. The Cayley graph $\mathrm{Cay}(\mathcal{S}_n, \mathrm{T}_n)$ is a graph with set of vertices corresponding to the elements of $\mathcal{S}_n$, and an edge between the permutations σ and τ whenever $\sigma\tau^{-1} \in \mathrm{T}_n$. Denote by $d_C(\sigma, \tau)$ the distance between two permutations in this graph. It is obvious that $\|\sigma\| = d_C(e, \sigma)$ where $e \in \mathcal{S}_n$ is the identity, and, more generally, that $d_C(\sigma, \tau) = \left\| \sigma\tau^{-1} \right\|$. This means that the equivalence 1$\Longleftrightarrow$3 from Theorem 4.4 can be interpreted as saying that the embedding $\mathrm{NC}(n) \hookrightarrow \mathcal{S}_n$ from Definition 2.6 is an isometry: $d_H(\pi, \rho) = d_C(P_\pi, P_\rho)$.
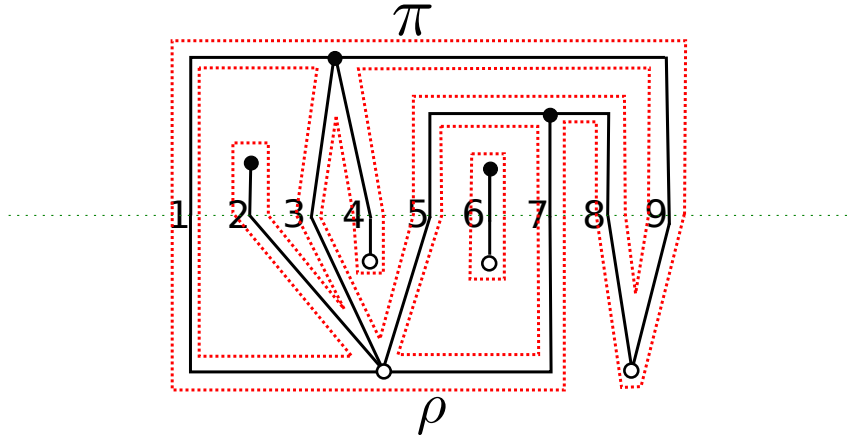
Figure 8: The meandric system $M(\pi, \rho)$ (in dashed lines) as obtained from $\Gamma(\pi, \rho)$, the graph from Figure 7 (in solid lines); $\#M(\pi, \rho) = 5$

(a) A sixth equivalent quantity can be added to the list in Theorem 4.4: a natural distance defined on the set $\mathrm{NCP}(n)$ of non-crossing pairings by saying that $\pi, \rho \in \mathrm{NCP}(n)$ are neighbors if and only if one can be obtained from the other by a simple switch $\{a, b\}, \{c, d\} \mapsto \{a, d\}, \{b, c\}$. See [Sav09] for more details.

As an immediate corollary from Theorem 4.4 we get the following equivalent criteria for $M(\pi, \rho)$ being a meander:

Corollary 4.5. *Let $\pi, \rho \in \mathrm{NC}(n)$ be two non-crossing partitions of order n. Then the following are equivalent:*

1. *$M(\pi, \rho)$ is a meander*

2. *$d_H(\pi, \rho) = n - 1$ namely, π and ρ define a diameter in $\mathcal{H}_n$*

3. *$\left\| P_\pi P_\rho^{-1} \right\| = n - 1$, namely, $P_\pi P_\rho^{-1}$ is an n-cycle*

4. *$|\pi| + |\rho| = n + 1$ and $\left| \pi \widetilde{\vee} \rho \right| = 1$*

5. *$\Gamma(\pi, \rho)$ is a tree*

Proof. The only equivalence which is not completely obvious from Theorem 4.4 is that of 4 with the others, so we prove here 4$\Longleftrightarrow$5. Indeed, since the graph $\Gamma(\pi, \rho)$ has exactly n edges, it is a tree if and only if it is connected ($\left| \pi \widetilde{\vee} \rho \right| = 1$) and has exactly $n + 1$ vertices ($|\pi| + |\rho| = n + 1$). $\qquad\square$

The following corollary shows that every non-crossing partition has a meandric partner[*] which is, in some sense, canonical. The statement of this corollary may be known, but we are not aware of a reference.

Corollary 4.6. *For every $\pi \in \mathrm{NC}(n)$, $M(\pi, \mathrm{Kr}(\pi))$ is a meander, or, equivalently $d_H(\pi, \mathrm{Kr}(\pi)) = n - 1$.*

Proof. We show that Criterion 4 from Corollary 4.5 holds. Indeed, the well-known fact that $|\pi| + |\mathrm{Kr}(\pi)| = n + 1$ was explained in Remark 3.1. To see that $\left| \pi \widetilde{\vee} \mathrm{Kr}(\pi) \right| = 1$, note that for every $i = 2, \ldots, n$,

$$i \overset{\pi}{\sim} P_\pi^{-1}(i) \overset{\mathrm{Kr}(\pi)}{\sim} i - 1$$

---

[*]The mere fact that every $\pi \in \mathrm{NC}(n)$ has a meandric partner is not hard: for example, one can easily find $\rho \in \mathrm{NC}(n)$ whose diagram in the lower-half plane completes the diagram of π in the upper half-plane into a tree.

because, by (2.4), $P_{\mathrm{Kr}(\pi)}(i-1) = \left[P_\pi^{-1}(1\ 2\ \ldots\ n) \right](i-1) = P_\pi^{-1}(i)$. Hence i and $i-1$ belong to the same block of $\pi \widetilde{\vee} \mathrm{Kr}(\pi)$. $\qquad \square$

Corollary 4.7. *For every $n \in \mathbb{N}$ one has that*

$$\frac{1}{\mathrm{Cat}_n^2} \sum_{\pi, \rho \in \mathrm{NC}(n)} d_H(\pi, \rho) \geq \frac{n-1}{2}. \tag{4.1}$$

Proof. We fix a partition $\rho \in \mathrm{NC}(n)$ and we use the triangle inequality and Corollary 4.6 to infer that:

$$d_H(\pi, \rho) + d_H(\mathrm{Kr}_n(\pi), \rho) \geq d_H(\pi, \mathrm{Kr}_n(\pi)) = n - 1, \quad \forall\, \pi \in \mathrm{NC}(n).$$

Summing over π gives

$$\left(\sum_{\pi \in \mathrm{NC}(n)} d_H(\pi, \rho) \right) + \left(\sum_{\pi \in \mathrm{NC}(n)} d_H(\mathrm{Kr}_n(\pi), \rho) \right) \geq (n-1) \cdot \mathrm{Cat}_n.$$

However, the two sums on the left-hand side of the preceding equation are equal to each other (because Kr_n is bijective), so what we have obtained is that, for our fixed ρ:

$$\sum_{\pi \in \mathrm{NC}(n)} d_H(\pi, \rho) \geq \frac{n-1}{2} \cdot \mathrm{Cat}_n.$$

Finally, we let ρ vary in $\mathrm{NC}(n)$ and sum over it, and the required formula follows. $\qquad \square$

5 Interval Partitions and Meanders with Shallow Top

Definition 5.1. A partition π of $\{1, \ldots, n\}$ is said to be an *interval partition* when every block of π is of the form $\{i \in \mathbb{N} \,|\, p \leq i \leq q\}$ for some $p \leq q$ in $\{1, \ldots, n\}$. We denote the set of all interval partitions of $\{1, \ldots, n\}$ by $\mathrm{Int}(n)$. It is obvious that $\mathrm{Int}(n) \subseteq \mathrm{NC}(n)$, and it is easy to count that $|\mathrm{Int}(n)| = 2^{n-1}$. Some of the arguments and results below work, in fact, for the larger collection $\widetilde{\mathrm{Int}}(n)$ which includes all cyclic permutations of interval partitions. This is still a subset of $\mathrm{NC}(n)$, with $\left| \widetilde{\mathrm{Int}}(n) \right| = 2^n - n$.

$\widetilde{\mathrm{Int}}(n)$

    If $\pi \in \mathrm{Int}(n)$, we say that the associated non-crossing pairing $\partial\pi \in \mathrm{NCP}(n)$ is *shallow*. We say that the meandric system $M(\pi, \rho)$ with $\pi, \rho \in \mathrm{NC}(n)$ has a *shallow top* if $\pi \in \mathrm{Int}(n)$ is an interval partition.

shallow top

Note that $\partial\pi \in \mathrm{NCP}(n)$ being shallow is equivalent to the fact that its diagram, as in Figure 6, can be drawn with at most two curves above every point on the infinite horizontal line. This is the rationale behind the term "shallow".

Proposition 5.2. *Let $n \in \mathbb{N}$ and consider partitions $\pi \in \widetilde{\mathrm{Int}}(n)$ and $\rho \in \mathrm{NC}(n)$. Then $\pi \vee \rho = \pi \widetilde{\vee} \rho$, and consequently*

$$d_H(\pi, \rho) = |\pi| + |\rho| - 2|\pi \vee \rho|. \tag{5.1}$$

Proof. The equality (5.1) follows from $\pi \vee \rho = \pi \widetilde{\vee} \rho$ together with the equivalence $1 \Longleftrightarrow 4$ in Theorem 4.4, so it is enough to prove that $\pi \vee \rho = \pi \widetilde{\vee} \rho$. Assume first that π consists of $n-1$ blocks: a block of size two and $n-2$ singletons. By symmetry, we can assume the block of size two is $\{1, n\}$. We claim that for every $\rho \in \mathrm{NC}(n)$, the partition $\pi \widetilde{\vee} \rho$ is non-crossing. Indeed, $\pi \widetilde{\vee} \rho$ is the partition obtained from ρ by merging the block containing 1 with the one containing n. If $1 \overset{\rho}{\sim} n$, then $\pi \widetilde{\vee} \rho = \rho$ and

the claim is clear. Otherwise, assume the block of 1 in ρ is $B = \{a_1 = 1, a_2, \ldots, a_k\}$ and the block of n is $B' = \{c_1, \ldots, c_{\ell-1}, c_\ell = n\}$ with

$$1 < a_2 < a_3 < \ldots < a_k < c_1 < c_2 < \ldots < c_{\ell-1} < n.$$

Now assume $a < b < c < d$ in $\{1, \ldots, n\}$ with a, c in one block of $\pi \widetilde{\vee} \rho$ and b, d in another. Since ρ is non-crossing, one of these blocks has to be the new block $B \cup B'$ and the other B'' – a third, different block of ρ. Without loss of generality,

$$a, c \in \{a_1 = 1, a_2, \ldots, a_k, c_1, \ldots, c_{\ell-1}, c_\ell = n\}$$

and $b, d \in B''$. Moreover, we must have $a = a_i$ and $c = c_j$ for some i and j, again because ρ is non-crossing. But then the four numbers $b < c = c_j < d < n$ contradict the fact that ρ is non-crossing.

For a general $\pi \in \widetilde{\mathrm{Int}}\,(n)$, let $\mathrm{Pairs}\,(\pi) = \left\{ \{i, i+1\} \,\middle|\, i \in \{1, \ldots, n\} \ \& \ i \overset{\pi}{\sim} i+1 \right\}$, where the addition is modulo n so if $i = n$, then $i + 1 = 1$. For $i \neq j$ in $\{1, \ldots, n\}$, let $\sigma_{i,j} \in \mathrm{NC}\,(n)$ be the partition consisting of the block $\{i, j\}$ together with $n - 2$ singletons. It is clear that $\pi = \widetilde{\bigvee}_{\{i,i+1\} \in \mathrm{Pairs}(\pi)} \sigma_{i,i+1}$, so $\rho \widetilde{\vee} \pi = \rho \widetilde{\vee} \left(\widetilde{\bigvee}_{\{i,i+1\} \in \mathrm{Pairs}(\pi)} \sigma_{i,i+1} \right)$. Since the operator $\widetilde{\vee}$ is associative, we are done by the special case. $\qquad\square$

Remark 5.3. As a converse to the preceding proposition, we observe that if $\pi \in \mathrm{NC}(n)$ has the property that $\pi \vee \rho = \pi \widetilde{\vee} \rho$ for every $\rho \in \mathrm{NC}\,(n)$, then it follows that $\pi \in \widetilde{\mathrm{Int}}\,(n)$. Here is the simple argument: Assume by contradiction that, up to a cyclic shift, there exists a block $B \in \pi$ and numbers $1 \leq a < b < c < d \leq n$ such that $a, c \in B$ yet $b, d \notin B$. Consider the non-crossing partition $\rho \in \mathrm{NC}\,(n)$ consisting of the block $\{b, d\}$ together with $n - 2$ singletons. It is easy to see that $\pi \widetilde{\vee} \rho$ is the partition obtained from π by merging the block containing b with the block containing d. Clearly, this partition is not non-crossing, hence it cannot equal $\pi \vee \rho$.

There is a version of Theorem 1.1 for the more generalized notion of interval partitions:

Theorem 5.4. *For every* $n \in \mathbb{N}$, *the number of pairs* $(\pi, \rho) \in \widetilde{\mathrm{Int}}\,(n) \times \mathrm{NC}\,(n)$ *which define a meander and such that* $|\pi| = m$ *is*

$$\begin{cases} 1 & m = 1 \\ \dfrac{1}{m} \begin{pmatrix} n \\ m-1 \end{pmatrix} \begin{pmatrix} n+m-1 \\ n-m \end{pmatrix} & m \geq 2 \end{cases}. \tag{5.2}$$

Compare this with the quantity $\dfrac{1}{n} \begin{pmatrix} n \\ m-1 \end{pmatrix} \begin{pmatrix} n+m-1 \\ n-m \end{pmatrix}$ from Theorem 1.1, where π is conditioned to be in the smaller set $\mathrm{Int}\,(n)$ rather than in $\widetilde{\mathrm{Int}}\,(n)$. The proof of Theorem 5.4 follows the same lines as the one of Theorem 1.1, with minor adaptations, and we omit it, but do consult Remark 6.5 below.

In Remark 1.2 we mentioned the exponential growth rate of the number of meanders with shallow top, and how it is compared with the conjectural exponential growth rate of the total number of meanders. We now describe the computation leading from Theorem 1.1 to this quantity:

Corollary 5.5. *The exponential growth rate of the number of meanders with shallow top of order* n *is roughly* 5.22.

Proof. Recall that $\overline{\mathfrak{M}_1}\,(n)$ denotes the set of meanders of order n with shallow top. The exponential growth rate we compute is, by definition, $\varlimsup_{n \to \infty} \sqrt[n]{\left| \overline{\mathfrak{M}_1}\,(n) \right|}$. Since exponential growth rate does

not see polynomial factors, for the sake of computing it we may replace the summation in (1.3) by the following quantities:

$$
\begin{aligned}
\left|\overline{\mathfrak{M}_1}\left(n\right)\right| &= \sum_{m=1}^{n} \frac{1}{n}\binom{n}{m-1}\binom{n+m-1}{n-m} \\
&\asymp \max_{1 \le m \le n} \binom{n}{m}\binom{n+m}{n-m} \\
&\asymp \max_{\alpha \in (0,1)} \binom{n}{\alpha n}\binom{(1+\alpha)\,n}{(1-\alpha)\,n},
\end{aligned}
$$

(here $\asymp$ means "up to polynomial factors in n"). By Stirling's formula, for $\alpha > \beta$ one has $\binom{\alpha n}{\beta n} \asymp \left[\frac{\alpha^\alpha}{\beta^\beta (\alpha-\beta)^{\alpha-\beta}}\right]^n$ so

$$
\left|\overline{\mathfrak{M}_1}\left(n\right)\right| \asymp \left[\max_{\alpha \in (0,1)} \frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha\,(1-\alpha)^{2(1-\alpha)}\,(2\alpha)^{2\alpha}}\right]^n. \tag{5.3}
$$

Simple (numerical) analysis shows that the maximum is obtained for $\alpha \approx 0.4694$, for which the fraction in (5.3) gives, roughly, 5.21914.

1. Interestingly, the value of m which asymptotically generates the most meanders $M\left(\pi, \rho\right)$ with $\pi \in \mathrm{Int}\left(n\right)$ and $|\pi| = m$ is, therefore, $m \approx 0.47n$. This is in contrast to the case of general non-crossing partitions, where it is expected that the maximum is obtained for $m \approx 0.5n$. At least, by the top-down symmetry of $\mathcal{H}_n$, if the maximum is obtained for αn, it is also obtained for $(1-\alpha)\,n$. We also comment that the most common number of blocks in an interval partition, or in a general non-crossing partition, is $m \approx 0.5n$. See also Conjecture 1.8 in this regard.

2. The statement of Corollary 5.5 holds just as well, and with the same proof, for the somewhat larger set of meanders with top partition in $\widetilde{\mathrm{Int}}\left(n\right)$.

$\square$

6 A Bijection for Meanders with Shallow Top

Recall the graph $\Gamma\left(\pi, \rho\right)$ from Definition 4.1, which, by Corollary 4.5, is a tree whenever $M\left(\pi, \rho\right)$ is a meander. When π and ρ have order n, the graph has n edges labeled by the elements $\{1, \ldots, n\}$, whence π and ρ can be completely recovered from $\Gamma\left(\pi, \rho\right)$. This section shows that when $M\left(\pi, \rho\right)$ is a meander with shallow top, one can "forget" the edge-labels of the tree $\Gamma\left(\pi, \rho\right)$, record, instead, smaller amount of information, and still recover π and ρ. Moreover, there is a bijection between trees with this type of information and the set $\overline{\mathfrak{M}_1} = \bigcup_{n=1}^{\infty} \overline{\mathfrak{M}_1}\left(n\right)$ of meanders with shallow top.

$\overline{\mathfrak{M}_1}$

Recall that the vertices of $\Gamma\left(\pi, \rho\right)$ are colored black for the blocks of π (in the upper half-plane) and white for the blocks of ρ (in the lower half-plane). It is also a *fat-tree*, in the sense that it comes with a particular embedding in the plane, which means that in every vertex there is a cyclic order on the edges emanating from it (graphs with this extra information of cyclic orders at every vertex are called *ribbon graphs*, *fat graphs*, or *cyclic graphs*). Of course, the cyclic order can be derived from the labels of the edges: if the edges around a black vertex are labeled $i_1 < i_2 < \ldots < i_k$, this is exactly the cyclic order, counterclockwise; if the vertex is white, this is the cyclic order clockwise.

fat-tree

Now assume that π is an interval partition and $M\left(\pi, \rho\right)$ a meander. We then keep the following data from $\Gamma\left(\pi, \rho\right)$:

- We record the block of π containing the element 1. We do so by marking the corresponding black vertex as the root of the tree.

- We record the cyclic order at every vertex.

- We also record, for every black vertex, the edge with the smallest label. Namely, there is a special edge for every black vertex.

As for the last piece of data, recall that every black vertex of $\Gamma(\pi, \rho)$ is a block of π, so the labels of the edges emanating from it form an interval in $\{1, \ldots, n\}$. The edge with the smallest label is, therefore, the beginning point of the interval.

Formally, the resulting tree belongs to the following set of fat-trees:

$\mathfrak{T}$ **Definition 6.1.** Let $\mathfrak{T}$ denote the set of finite rooted fat-trees with the following additional data:

- The vertices of every $T \in \mathfrak{T}$ are properly colored by black and white, with the root colored black[*].

- For every black vertex, one of the edges emanating from it is marked as special.

forget   The forgetful map

$$\mathsf{forget} \colon \overline{\mathfrak{M}_1} \to \mathfrak{T}$$

is defined as above: for $M(\pi, \rho) \in \overline{\mathfrak{M}_1}$, construct the tree $\Gamma(\pi, \rho)$ with its embedding in the plane, mark the black vertex containing 1 as the root, record the edge with smallest label at every black vertex, and "forget" the labels of the edges.

Theorem 6.2. *The map* $\mathsf{forget} \colon \overline{\mathfrak{M}_1} \to \mathfrak{T}$ *is a bijection.*

In the proof of the theorem we use the following two key lemmas which capture the crucial property of the tree $\mathsf{forget}(M(\pi, \rho))$ when π is an interval partition. In the lemmas, we use the notion of a *subtree* of a tree T spanned by an oriented edge: If $\vec{e} = (u, v)$ is an edge e of T with
$T_{\vec{e}}$ some orientation, we let $T_{\vec{e}}$ denote the subtree which is the connected component of v, the head of $\vec{e}$, in $T \setminus \{e\}$. Note that $T_{\vec{e}}$ does not contain the edge e itself.

Lemma 6.3. *Let* $(\pi, \rho) \in \widetilde{\mathrm{Int}}(n) \times \mathrm{NC}(n)$ *define a meander, so* $T := \Gamma(\pi, \rho)$ *is a tree. Let b be some black vertex in T with emanating edges, in counterclockwise order,* $\vec{e}_1, \ldots, \vec{e}_k$. *Then for $i = 1, \ldots, k$, the edge-labels in the subtree $T_{\vec{e}_i}$ form a cyclic interval[†] in $\{1, \ldots, n\}$, and up to cyclic shift,*

$$\{e_1, \ldots, e_k\} < E(T_{\vec{e}_k}) < E(T_{\vec{e}_{k-1}}) < \ldots < E(T_{\vec{e}_1}), \tag{6.1}$$

where $E(G)$ marks the set of edges of the graph G, two edges are compared according to their labels, and an inequality between sets $A < B$ means that $a < b$ for every $a \in A$, $b \in B$.

Proof. It is enough to prove that if $\vec{e}_i$ and $\vec{e}_{i+1}$ emanate from b with labels j and $(j+1) \mod n$, respectively, then up to a cyclic shift,

$$\{e_1, \ldots, e_k\} < E(T_{\vec{e}_{i+1}}) < E(T_{\vec{e}_i}). \tag{6.2}$$

Indeed, the cyclic order in (6.1) easily follows from (6.2). But the sets of edges in (6.1) are disjoint sets which exhaust the entire edges of the tree, so every subset in (6.1) must form a cyclic interval.

To prove the claim about $\vec{e}_i$ and $\vec{e}_{i+1}$, consider π', the non-crossing partition in $\widetilde{\mathrm{Int}}(n)$ obtained from π by breaking the block b into singletons. The edge-labels in $T_{\vec{e}_i} \cup \{e_i\}$ and those in $T_{\vec{e}_{i+1}} \cup \{e_{i+1}\}$ form two distinct blocks B_1 and B_2, respectively, in the partition $\pi' \widetilde{\vee} \rho$. By Proposition 5.2, $\pi' \vee \rho =$

---

[*]Namely, every edge of the tree has one black endpoint and one white endpoint. Of course, there is only one way to properly color by black and white the vertices of a tree with a black root, so there is no real information here. The coloring is used merely to facilitate the reference to one of the two subsets of vertices.

[†]A cyclic interval is a cyclic shift of an interval.

$\pi'\widetilde{\vee}\rho$, so, in particular, $\pi'\widetilde{\vee}\rho$ is non-crossing. As $j \in B_1$ and $j + 1 \in B_2$, this means that up to a cyclic shift,

$$\{j, j+1\} < B_2 \setminus \{j+1\} < B_1 \setminus \{j\}.$$

Since the labels of $\{e_1, \ldots, e_k\}$ form a cyclic interval containing $\{j, j+1\}$, we deduce (6.2). $\quad\square$

Lemma 6.4. *Let* $(\pi, \rho) \in \widetilde{\mathrm{Int}}(n) \times \mathrm{NC}(n)$ *define a meander, so* $T := \Gamma(\pi, \rho)$ *is a tree. Let* w *be some black vertex in* T *with emanating edges, in clockwise order,* $\vec{e}_1, \ldots, \vec{e}_k$. *Then for* $i = 1, \ldots, k$, *the edge-labels of* $E(T_{\vec{e}_i}) \cup \{e_i\}$ *form a cyclic interval in* $\{1, \ldots, n\}$, *and up to a cyclic shift,*

$$E(T_{\vec{e}_1}) \cup \{e_1\} < E(T_{\vec{e}_2}) \cup \{e_2\} < \ldots < E(T_{\vec{e}_k}) \cup \{e_k\}. \tag{6.3}$$

Proof. For every $\vec{e}$ with white tail, the labels of $E(T_{\vec{e}}) \cup \{e\}$ form a cyclic interval because they are the complement in $\{1, \ldots, n\}$ of the labels of $E(T_{\overleftarrow{e}})$ which form a cyclic interval by Lemma 6.3. But by the assumptions, the cyclic order on $e_1, \ldots, e_k$ means that up to a cyclic shift $e_1 < e_2 < \ldots < e_k$, which yields (6.3). $\quad\square$

Proof of Theorem 6.2. We need to show that forget is injective and surjective.

Injectivity

We start with injectivity, namely, we need to show that if the pair $(\pi, \rho) \in \mathrm{Int}(n) \times \mathrm{NC}(n)$ corresponds to a meander with shallow top, then we can recover π and ρ from $T := \mathsf{forget}(M(\pi, \rho))$. What we need to recover is the labels of the edges. Of course, n is the number of edges of T. The edges emanating from every black vertex form an interval, and the order inside the interval is known thanks to the marked edge in every black vertex in T. Hence, it is enough to recover the order in which the different intervals lie in $\{1, \ldots, n\}$, namely, to recover the order induced on the black vertices. We let $B(T)$ denote the set of black vertices, and $\prec$ denote the order on $B(T)$, so $(B(T), \prec)$ is an ordered set.

Lemmas 6.3 and 6.4 yield that for every oriented edge $\vec{e}$, be its head black or white, the set $B(T_{\vec{e}})$ of black vertices in $T_{\vec{e}}$ form a cyclic interval in $(B(T), \prec)$. Moreover, if $\vec{e}$ points away from the root of T, then $B(T_{\vec{e}})$ forms an actual interval (rather than a cyclic interval), as $B(T_{\vec{e}})$ does not contain the smallest black vertex which is the root.

Thus, to fully recover the order on the black vertices of T, it is enough to determine the following:

- for every black vertex b with emanating edges away from the root $\vec{e}_1, \ldots, \vec{e}_k$, the relative order among $\{b\}, B(T_{\vec{e}_1}), \ldots, B(T_{\vec{e}_k})$

- for every white vertex w with emanating edges away from the root $\vec{e}_1, \ldots, \vec{e}_k$, the relative order among $B(T_{\vec{e}_1}), \ldots, B(T_{\vec{e}_k})$

The above lemmas yield the following solution to this task:

- If b is the root of T, and the edges emanating from it in order are $\vec{e}_1, \ldots, \vec{e}_k$, then by Lemma 6.3, the required order is

$$\{b\} \prec B(T_{\vec{e}_k}) \prec B(T_{\vec{e}_{k-1}}) \prec \ldots \prec B(T_{\vec{e}_1}).$$

- If b is a non-root black vertex of T and the edges emanating from it in the order of the interval b represents are $\vec{e}_1, \ldots, \vec{e}_k$, with $\vec{e}_j$ pointing to the root, the required order is

$$B(T_{\vec{e}_{j-1}}) \prec B(T_{\vec{e}_{j-2}}) \prec \ldots \prec B(T_{\vec{e}_1}) \prec \{b\} \prec B(T_{\vec{e}_k}) \prec B(T_{\vec{e}_{k-1}}) \prec \ldots \prec B(T_{\vec{e}_{j+1}}),$$

because Lemma 6.3 gives the order up to a cyclic shift and $B(T_{\vec{e}_j})$ contains the root which is the smallest black vertex.

- If w is a white vertex of T and the edges emanating from it clockwise cyclic order are $\vec{e}_1, \ldots, \vec{e}_k$, with $\vec{e}_1$ pointing to the root, the required order is

$$B\left(T_{\vec{e}_2}\right) \prec B\left(T_{\vec{e}_3}\right) \prec \ldots \prec B\left(T_{\vec{e}_k}\right),$$

because Lemma 6.4 translates to the cyclic order $B\left(T_{\vec{e}_1}\right) \prec B\left(T_{\vec{e}_2}\right) \prec \ldots \prec B\left(T_{\vec{e}_k}\right)$, and again $B\left(T_{\vec{e}_1}\right)$ contains the root which is the smallest black vertex.

We illustrate this procedure of recovering the order $\Gamma\left(\pi, \rho\right)$ in Figure 9.

Surjectivity

To prove surjectivity of forget, take an arbitrary $T \in \mathfrak{T}$. Let n denote the number of edges of T. We need to show one can construct $\left(\pi, \rho\right) \in \text{Int}\left(n\right) \times \text{NC}\left(n\right)$ so that $M\left(\pi, \rho\right) \in \overline{\mathfrak{M}_1}\left(n\right)$ and forget $\left(M\left(\pi, \rho\right)\right) = T$. One can label the edges of T by $\{1, \ldots, n\}$ (in a bijection) by following the "recovery procedure" above, obtaining T_{colored}. The graph T_{colored} looks like a $\Gamma\left(\pi, \rho\right)$ in terms of the data it holds, but we still need to show it is actually equal to some $\Gamma\left(\pi, \rho\right)$.

The unique possible candidate for π (resp. ρ) is obvious: this is the partition of $\{1, \ldots, n\}$ defined by the black (resp. white) vertices of T_{colored}. It is clear by the procedure that $\pi \in \text{Int}\left(n\right)$. To see that $\rho \in \text{NC}\left(n\right)$, let w_1 and w_2 be two white blocks (vertices). We show that w_1 and w_2 do not cross. We separate into three cases:

- If w_1 and w_2 are both descendants in T of the black vertex b, but one is in $T_{\vec{e}_1}$ and one in $T_{\vec{e}_2}$ for two distinct edges emanating from b, then the edges of w_1 (resp. w_2) are contained in $T_{\vec{e}_1} \cup \{e_1\}$ (resp. $T_{\vec{e}_2} \cup \{e_2\}$), and the recovery procedure ensures that the sets $T_{\vec{e}_1} \cup \{e_1\}$ and $T_{\vec{e}_2} \cup \{e_2\}$ lie in two distinct cyclic intervals, whence they do not cross.

- If w_1 and w_2 are both proper descendants in T of the white vertex w but, one is in $T_{\vec{e}_1}$ and one in $T_{\vec{e}_2}$ for two distinct edges emanating from w, then the recovery procedure ensures that w_1 and w_2 are contained in two distinct intervals, whence they do not cross.

- If one of w_1, w_2 is a descendant of the other, say without loss of generality, that w_2 is a descendant of w_1, consider $\vec{e}$, the last edge in the geodesic from w_1 to w_2. The recovery procedure ensures that $\{e\} \cup E\left(T_{\vec{e}}\right)$, which contains w_2, is contained in an interval disjoint from w_1.

Hence $\left(\pi, \rho\right) \in \text{Int}\left(n\right) \times \text{NC}\left(n\right)$. Finally, it is easy to check that the procedure ensures that the cyclic order on the edges in every vertex of T matches the cyclic order induced by the labels in T_{colors}. Hence $\Gamma\left(\pi, \rho\right)$ is the exact same fat-tree (tree with an embedding in the plane) as T_{color}. Thus $M\left(\pi, \rho\right) \in \overline{\mathfrak{M}_1}\left(n\right)$ and forget $\left(M\left(\pi, \rho\right)\right) = T$. $\square$

Remark 6.5. There is an analogue of Theorem 6.2 for meanders in $\widetilde{\text{Int}}\left(n\right) \times \text{NC}\left(n\right)$. This time, the element 1 is not always the first element in its cyclic interval in π, so we also record the edge of the root corresponding to 1. So now the root of every tree in $\mathfrak{T}$ should have two marked edges: one marked as 1, and one, possibly the same one, marked as the smallest in the interval. The proof of the bijection is almost the same, with small adaptation to the change in the root.

7 Enumerative Consequences of the Tree Bijection

In this section, we determine the generating function for the set of fat-trees $\mathfrak{T}$ defined in Definition 6.1 and deduce the enumerative results of Theorem 1.1. In order to do so, we define two additional sets of bicolored fat-trees with additional data. These sets describe proper subtrees of trees in $\mathfrak{T}$, where a proper subtree here means the subtree of some $T \in \mathfrak{T}$ spanned by a non-root vertex, its **parent** and all its descendants. We let $\mathfrak{T}_w$ ($\mathfrak{T}_b$, respectively) denote the set of possible proper fat-
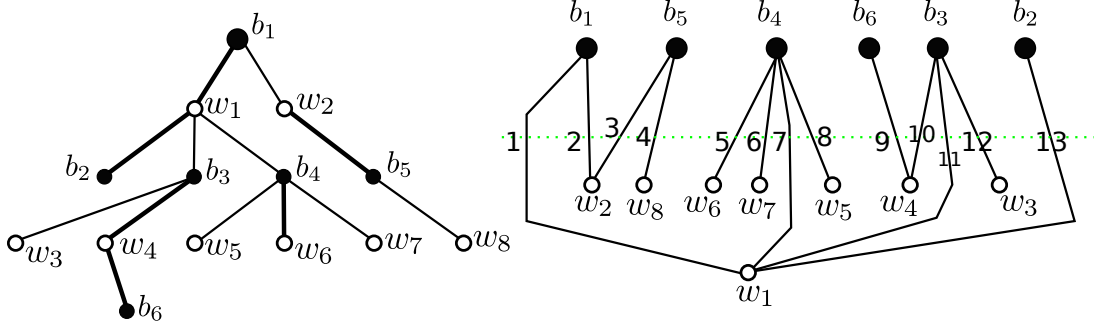
$\mathfrak{T}_w, \mathfrak{T}_b$

Figure 9: The left part is a tree $T \in \mathfrak{T}$ with root b_1 and a special edge for every black vertex marked by a thicker line. The recovery procedure from the proof of Theorem 6.2 gives, in this case, the following order on the black vertices: by the root b_1, we obtain that $\{b_1\} < \{b_5\} < \{b_2, b_3, b_4, b_6\}$, in w_1 we get $\{b_4\} < \{b_3, b_6\} < \{b_2\}$, and in b_3 we see that $\{b_6\} < \{b_3\}$. The recovered meander $M(\pi, \rho) \in \overline{\mathfrak{M}_1}(13)$, in the shape of the tree $\Gamma(\pi, \rho)$, is on the right. We stress that the vertex labels are not part of the data of T nor of $\Gamma(\pi, \rho)$: they are given here merely to clarify the right isomorphism between the two fat-trees.

subtrees spanned by a *white* (respectively, *black*) vertex, its parent and its descendants. So every $T \in \mathfrak{T}_w$ (resp. $T \in \mathfrak{T}_b$) is a fat-tree with some white (resp. black) vertex marked as the root and some black (resp. white) neighbor of the root, which is a leaf of a tree, marked as the parent of the root. In addition, every black vertex has one emanating edge marked as special.

We let Φ, W and B denote the generating functions of the sets $\mathfrak{T}$, $\mathfrak{T}_w$ and $\mathfrak{T}_b$, respectively. In the following generating functions, the exponent of the variable x is the number of edges (which is n), the exponent of the variable y is the number of black vertices (which is the number of blocks in $\pi \in \mathrm{Int}(n)$), the exponent of z_k is the number of black vertices of degree k, and the exponent of w_k is the number of white vertices of degree k. To obtain this, we think of every black vertex of degree k as marked by $yx^k z_k$ and every white vertex of degree k as marked by w_k.

Since the root of a tree $T \in \mathfrak{T}_w$ has some degree $\ell \geq 1$ and $(\ell - 1)$ black children, we obtain

$$W = \sum_{\ell \geq 1} w_\ell B^{\ell-1}. \tag{7.1}$$

Analogously, the root of a tree $T \in \mathfrak{T}_b$ has some degree $k \geq 1$ with $(k-1)$ white children and k choices for which edge is special, so

$$B = \sum_{k \geq 1} k y x^k z_k W^{k-1}. \tag{7.2}$$

Similarly, the root of a tree $T \in \mathfrak{T}$ has degree k for some $k \geq 1$, with k white children. Unlike the case of a subtree rooted at a black vertex, here there is no special edge at the root pointing to the parent, so all edges at the root look the same and there is no real choice of which one to mark as special. Thus, we obtain the following equation for Φ in terms of W:

$$\Phi = \sum_{k \geq 1} y x^k z_k W^k. \tag{7.3}$$

Now, substituting for W via (7.1) in (7.2) and (7.3), we get the pair of functional equations

$$\Phi = \sum_{k \geq 1} y x^k z_k \left(\sum_{\ell \geq 1} w_\ell B^{\ell-1} \right)^k, \qquad B = y \sum_{k \geq 1} k x^k z_k \left(\sum_{\ell \geq 1} w_\ell B^{\ell-1} \right)^{k-1}. \tag{7.4}$$

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i, j
$N(m, n; \mathbf{i}, \mathbf{j})$

Also, using the notation $\mathbf{i} = (i_1, i_2, \ldots)$ and $\mathbf{j} = (j_1, j_2, \ldots)$, let $N(m, n; \mathbf{i}, \mathbf{j})$ denote the number of meandric pairs (π, ρ) in which the interval partition $\pi \in \text{Int}(n)$ has m blocks, of which j_k blocks have size k, $k \geq 1$, and $\rho \in \text{NC}(n)$ has $n - m + 1$ blocks, of which i_ℓ blocks have size ℓ, $\ell \geq 1$. Of course, the elements of $\mathbf{i}$ and $\mathbf{j}$ are subject to the restrictions:

$$\sum_{\ell \geq 1} i_\ell = n - m + 1, \qquad \sum_{\ell \geq 1} \ell i_\ell = n, \qquad \sum_{k \geq 1} j_k = m, \qquad \sum_{k \geq 1} k j_k = n. \qquad (7.5)$$

$N_a(m, n; \mathbf{i}, \mathbf{j})$

Let $N_a(m, n; \mathbf{i}, \mathbf{j})$ denote the further restricted number of meandric pairs above such that the block of π containing the element 1 is of size a (so it is the block $\{1, \ldots, a\}$), where we have $j_a \geq 1$.

We now determine explicit formulas for the numbers $N(m, n; \mathbf{i}, \mathbf{j})$ and $N_a(m, n; \mathbf{i}, \mathbf{j})$ by using the bijection from Theorem 6.2.

Proposition 7.1. *For $n \geq m \geq 1$, and $\mathbf{i}, \mathbf{j}$ satisfying (7.5), we have*

$$N(m, n; \mathbf{i}, \mathbf{j}) = m!(n - m)! \prod_{\ell \geq 1} \frac{1}{i_\ell!} \prod_{k \geq 1} \frac{k^{j_k}}{j_k!} \qquad (7.6)$$

and (where $j_a \geq 1$),

$$N_a(m, n; \mathbf{i}, \mathbf{j}) = j_a(m - 1)!(n - m)! \prod_{\ell \geq 1} \frac{1}{i_\ell!} \prod_{k \geq 1} \frac{k^{j_k}}{j_k!} \qquad (7.7)$$

Proof. For the first part of the result, if $m = 1$, the restrictions (7.5) yield that $\mathbf{j} = (0, \ldots 0, 1, 0, \ldots)$ with $j_n = 1$, and that $\mathbf{i} = (n, 0, 0, \ldots)$. Then the formula (7.6) gives 1 which is easily seen to be equal to $N(1, n; \mathbf{i}, \mathbf{j})$. Now assume $m \geq 2$. From the bijection from Theorem 6.2, $N(m, n; \mathbf{i}, \mathbf{j})$ is equal to the number of fat-trees in $\mathfrak{T}$ with i_ℓ white vertices of degree ℓ and j_ℓ black vertices of degree ℓ, $\ell \geq 1$. Thus, using the notation $\mathbf{w^i} = w_1^{i_1} w_2^{i_2} \cdots$ and $\mathbf{z^j} = z_1^{j_1} z_2^{j_2} \cdots$, we conclude that

$$N(m, n; \mathbf{i}, \mathbf{j}) = [y^m x^n \mathbf{w^i} \mathbf{z^j}] \Phi,$$

where we also use here the notation $[F]G$ to denote the *coefficient* of the monomial F in the expansion of the formal power series G. This is because Φ has been defined to be precisely the appropriate generating function.

Now we can determine $N := N(m, n; \mathbf{i}, \mathbf{j})$ by applying Lagrange's Implicit Function Theorem to solve the equations in (7.4) (see e.g. [GJ83, Theorem 1.2.4(1), Page 17]) via the following calculation. Recall that $m \geq 2$:

$$
\begin{aligned}
N &= \left[y^m x^n \mathbf{w^i} \mathbf{z^j}\right] \sum_{k \geq 1} y x^k z_k \left(\sum_{\ell \geq 1} w_\ell B^{\ell-1}\right)^k = \left[y^{m-1} x^n \mathbf{w^i} \mathbf{z^j}\right] \sum_{k \geq 1} x^k z_k \left(\sum_{\ell \geq 1} w_\ell B^{\ell-1}\right)^k \\
&= \frac{1}{m-1} \left[\lambda^{m-2} x^n \mathbf{w^i} \mathbf{z^j}\right] \frac{d}{d\lambda} \left[\sum_{k \geq 1} x^k z_k \left(\sum_{\ell \geq 1} w_\ell \lambda^{\ell-1}\right)^k\right] \left(\sum_{k \geq 1} k x^k z_k \left(\sum_{\ell \geq 1} w_\ell \lambda^{\ell-1}\right)^{k-1}\right)^{m-1} \\
&= \frac{1}{m-1} \left[\lambda^{m-2} x^n \mathbf{w^i} \mathbf{z^j}\right] \frac{d}{d\lambda} \left(\sum_{\ell \geq 1} w_\ell \lambda^{\ell-1}\right) \left(\sum_{k \geq 1} k x^k z_k \left(\sum_{\ell \geq 1} w_\ell \lambda^{\ell-1}\right)^{k-1}\right)^m. \qquad (7.8)
\end{aligned}
$$

We now compute the coefficient of $[x^n \mathbf{z^j}]$ in (7.8). There are exactly $\binom{m}{j_1, j_2, \ldots} = \frac{m!}{j_1! j_2! \cdots}$ ways to pick exactly j_r times the summand containing z_r for every $r \geq 1$, and then the coefficient of x is exactly

$\sum_{r\geq 1} r j_r = n$, as required. So

$$
\begin{aligned}
N &= \frac{m!}{m-1} \prod_{k\geq 1} \frac{k^{j_k}}{j_k!} \left[\lambda^{m-2}\mathbf{w^i}\right] \frac{d}{d\lambda} \left(\sum_{\ell\geq 1} w_\ell \lambda^{\ell-1}\right) \left(\sum_{\ell\geq 1} w_\ell \lambda^{\ell-1}\right)^{n-m} \\
&= \frac{m!}{m-1} \prod_{k\geq 1} \frac{k^{j_k}}{j_k!} \left[\lambda^{m-2}\mathbf{w^i}\right] \frac{1}{n-m+1} \cdot \frac{d}{d\lambda} \left(\sum_{\ell\geq 1} w_\ell \lambda^{\ell-1}\right)^{n-m+1} \\
&\overset{(*)}{=} \frac{m!}{n-m+1} \prod_{k\geq 1} \frac{k^{j_k}}{j_k!} \left[\lambda^{m-1}\mathbf{w^i}\right] \left(\sum_{\ell\geq 1} w_\ell \lambda^{\ell-1}\right)^{n-m+1} \\
&= \frac{m!}{n-m+1} \prod_{k\geq 1} \frac{k^{j_k}}{j_k!} \binom{n-m+1}{i_1, i_2, \ldots} = m! \, (n-m)! \prod_{k\geq 1} \frac{k^{j_k}}{j_k!} \prod_{\ell\geq 1} \frac{1}{i_\ell!}.
\end{aligned}
$$

where the equality $\overset{(*)}{=}$ follows from the fact that $[\lambda^{m-2}]\frac{d}{d\lambda}f(\lambda) = (m-1)[\lambda^{m-1}]f(\lambda)$ for $m \geq 2$ and any formal power series f. This gives (7.6).

For the second part of the result, it follows immediately from cyclic symmetry that $N_a(m, n; \mathbf{i}, \mathbf{j}) = \frac{j_a}{m} N(m, n; \mathbf{i}, \mathbf{j})$, and thus (7.7) follows from (7.6). $\qquad\square$

Now suppose that we are not interested in the distribution of block sizes in ρ, and thus let $N(m, n; \mathbf{j})$ denote the number of meandric pairs (π, ρ) in which the interval partition $\pi \in \mathrm{NC}(n)$ has m blocks, of which j_k blocks have size k, $k \geq 1$, and $\rho \in \mathrm{NC}(n)$ has $n - m + 1$ blocks, subject to the restrictions:

$$
\sum_{\ell\geq 1} j_\ell = m, \qquad \sum_{\ell\geq 1} \ell j_\ell = n. \tag{7.9}
$$

We also let $N_a(m, n; \mathbf{j})$ denote the further restricted number of meandric pairs above such that the block of π containing the element 1 is of size a (so it is the block $\{1, \ldots, a\}$), where we have $j_a \geq 1$.

We now determine explicit formulas for the numbers $N(m, n; \mathbf{j})$ and $N_a(m, n; \mathbf{j})$ by summing the results of Proposition 7.1.

Proposition 7.2. *For $n \geq m \geq 1$, and $\mathbf{j}$ satisfying (7.9), we have*

$$
N(m, n; \mathbf{j}) = \frac{m(n-1)!}{(n-m+1)!} \prod_{k\geq 1} \frac{k^{j_k}}{j_k!} \tag{7.10}
$$

and (where $j_a \geq 1$),

$$
N_a(m, n; \mathbf{j}) = \frac{j_a(n-1)!}{(n-m+1)!} \prod_{k\geq 1} \frac{k^{j_k}}{j_k!} \tag{7.11}
$$

Proof. We have

$$
N(m, n; \mathbf{j}) = \sum_{\substack{i_1, i_2, \ldots \geq 0 \\ i_1+i_2+\ldots=n-m+1 \\ i_1+2i_2+\ldots=n}} N(m, n; \mathbf{i}, \mathbf{j}) = m!(n-m)! \prod_{k\geq 1} \frac{k^{j_k}}{j_k!} \sum_{\substack{i_1, i_2, \ldots \geq 0 \\ i_1+i_2+\ldots=n-m+1 \\ i_1+2i_2+\ldots=n}} \prod_{\ell\geq 1} \frac{1}{i_\ell!},
$$

from (7.6). But elementary generating function methods yield

$$\sum_{\substack{i_1,i_2,\ldots\geq 0 \\ i_1+i_2+\ldots=n-m+1 \\ i_1+2i_2+\ldots=n}} \prod_{\ell\geq 1} \frac{1}{i_\ell!} = \frac{1}{(n-m+1)!}[x^n]\left(\sum_{\ell\geq 1}x^\ell\right)^{n-m+1} = \frac{1}{(n-m+1)!}[x^n]\left(x(1-x)^{-1}\right)^{n-m+1}$$

$$= \frac{1}{(n-m+1)!}\binom{n-1}{m-1},$$

and the first part of the result (7.10) follows immediately.

The second part of the result follows from the first part, together with the fact that $N_a(m,n;\mathbf{j}) = \frac{j_a}{m}N(m,n;\mathbf{j})$. $\qquad\square$

Proposition 1.6(1) is a special case of Theorem 7.2:

Proof of Proposition 1.6(1). For n even, the partition $\lambda_n = \{\{1,2\},\{3,4\},\ldots,\{n-1,n\}\}$ is the only interval partition with $\frac{n}{2}$ blocks of size 2 each. Thus, the number of meandric partners of λ_n is equal to $N\left(\frac{n}{2},n;\mathbf{j}\right)$ where $\mathbf{j} = \left(0,\frac{n}{2},0,0,\ldots\right)$, and

$$N\left(\frac{n}{2},n;\mathbf{j}\right) = \frac{\frac{n}{2}(n-1)!}{\left(n-\frac{n}{2}+1\right)!}\frac{2^{n/2}}{(n/2)!} = \mathrm{Cat}_{n/2}\cdot 2^{n/2-1}.$$

$\qquad\square$

Finally, suppose that in addition we are not interested in the distribution of block sizes in π, and thus let $N(m,n)$ denote the number of meandric pairs (π,ρ) in which the interval partition $\pi \in \mathrm{NC}(n)$ has m blocks (where $n \geq m \geq 1$) and $\rho \in \mathrm{NC}(n)$ has $n-m+1$ blocks. We now determine explicit formulas for the numbers $N(m,n)$ by summing the first part of Proposition 7.2. This is exactly the content of Theorem 1.1:

Proof of Theorem 1.1. We have

$$N(m,n) = \sum_{\substack{j_1,j_2,\ldots\geq 0 \\ j_1+j_2+\ldots=m \\ j_1+2j_2+\ldots=n}} N(m,n;\mathbf{j}) = \frac{m(n-1)!}{(n-m+1)!}\sum_{\substack{j_1,j_2,\ldots\geq 0 \\ j_1+j_2+\ldots=m \\ j_1+2j_2+\ldots=n}}\prod_{k\geq 1}\frac{k^{j_k}}{j_k!}$$

from (7.10). But elementary generating function methods yield

$$\sum_{\substack{j_1,j_2,\ldots\geq 0 \\ j_1+j_2+\ldots=m \\ j_1+2j_2+\ldots=n}}\prod_{k\geq 1}\frac{k^{j_k}}{j_k!} = \frac{1}{m!}[x^n]\left(\sum_{k\geq 1}kx^k\right)^m = \frac{1}{m!}[x^n]\left(x(1-x)^{-2}\right)^m = \frac{1}{m!}\binom{n+m-1}{n-m},$$

and the result follows immediately. $\qquad\square$

8 Distance Distributions in the Graphs $\mathcal{H}_n$

$b(\pi,\rho)$

In this section we study average distances in $\mathcal{H}_n$. For this sake, we define the number $b(\pi,\rho)$ for two non-crossing partitions $\pi,\rho \in \mathrm{NC}(n)$, by

$$b(\pi,\rho) = |\pi| + |\rho| - 2|\pi\vee\rho|. \tag{8.1}$$

Recall from Theorem 4.4(4) that $d_H(\pi,\rho) = |\pi|+|\rho|-2\left|\pi\widetilde{\vee}\rho\right|$, and as $|\pi\vee\rho| \leq \left|\pi\widetilde{\vee}\rho\right|$, the quantity $b(\pi,\rho)$ is in general an *upper bound* on the distance. However, when $\pi \in \widetilde{\mathrm{Int}}(n)$, we have the equality $b(\pi,\rho) = d_H(\pi,\rho)$ by Proposition 5.2. We consider the average value of b for three families:

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- Let $\widetilde{d_n}$, for $n \geq 1$ even, denote the average value of $b(\lambda_n, \rho) = d_H(\lambda_n, \rho)$ where $\lambda_n = \{\{1,2\},\{3,4\},\ldots,\{n-1,n\}\} \in \mathrm{Int}(n)$ is fixed and $\rho \in \mathrm{NC}(n)$, as in Proposition 1.6.

- Let d_n, $n \geq 1$, denote the average value of $b(\pi, \rho) = d_H(\pi, \rho)$ where $\pi \in \mathrm{Int}(n)$ and $\rho \in \mathrm{NC}(n)$, as in Theorem 1.3.

$\widetilde{b_n}$

- Let $\widetilde{b_n}$, $n \geq 1$, denote the average value of $b(\pi, \rho)$ where $\pi, \rho \in \mathrm{NC}(n)$. So $\widetilde{b_n}$ is an upper bound on the average distance in $\mathcal{H}_n$.

Now, to determine these average values, we begin with the simplest contributions. The number of non-crossing partitions $\rho \in \mathrm{NC}(n)$ with $|\rho| = k$ blocks is given by the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ (e.g. [Kre72, Corollaire 4.1]), and the number of interval partitions $\pi \in \mathrm{Int}(n)$ with k blocks is given by $\binom{n-1}{k-1}$. In both cases, these distributions are symmetric about the centre $\frac{1}{2}(n+1)$, which implies that the average values both for $|\rho|$, $\rho \in \mathrm{NC}(n)$ and for $|\pi|$, $\pi \in \mathrm{Int}(n)$, are given by

$$\tfrac{1}{2}(n+1). \tag{8.2}$$

To evaluate the averages for the three families above, we also need to consider the more complicated $|\pi \vee \rho|$ term in (8.1). To help with this term, we introduce the three generating functions

$$\Psi^{(1)}(z,t) = \sum_{n \geq 1} z^n \sum_{\rho \in \mathrm{NC}(2n)} t^{|\lambda_{2n} \vee \rho|},$$

$$\Psi^{(2)}(z,t) = \sum_{n \geq 1} y^n \sum_{\substack{\pi \in \mathrm{Int}(n) \\ \rho \in \mathrm{NC}(n)}} z^{|\pi|} t^{|\pi \vee \rho|},$$

$$\Psi^{(3)}(z,t) = \sum_{n \geq 1} z^n \sum_{\pi, \rho \in \mathrm{NC}(n)} t^{|\pi \vee \rho|},$$

and note that from elementary considerations we have

$$\Psi^{(1)}(z,1) = \sum_{n \geq 1} \mathrm{Cat}_{2n} z^n, \quad \Psi^{(2)}(z,1) = \sum_{n \geq 1} \mathrm{Cat}_n z(1+z)^{n-1} y^n, \quad \Psi^{(3)}(z,1) = \sum_{n \geq 1} \mathrm{Cat}_n^2 z^n. \tag{8.3}$$

Note that a third variable y appears in $\Psi^{(2)}(z,t)$: we consider $\Psi^{(2)}(z,t)$ to be a formal power series in the variables z and t, with coefficients that are formal power series in the variable y.

Now in order to evaluate the numerator for the contribution of the $|\pi \vee \rho|$ term in the averages $\widetilde{d_n}$, d_n and $\widetilde{b_n}$, we need to apply the partial derivative $\frac{\partial}{\partial t}$ to $\Psi^{(i)}$, $i = 1, 2, 3$, and set $t = 1$. The following proposition will allow us to carry out this process in the three cases, by appropriate specialization. A comment related to the result of Proposition 8.1 is that it has a nice interpretation in terms of the operation of "free additive convolution" used in free probability; this is explained in Remark 8.2, following the proposition.

A notational detail: in Proposition 8.1 (and later on in this section) we use subscripts to denote partial derivatives, via the notation $F_i = \frac{\partial}{\partial x_i} F$, $i = 1, \ldots, m$, for a formal power series $F(x_1, \ldots, x_m)$ in $m \geq 1$ variables.

F_i

Proposition 8.1. *Let*

$$F(z,t) = \sum_{n \geq 1} z^n \sum_{\theta \in NC(n)} \prod_{\theta_i \in \theta} \left(t\, g_{|\theta_i|} \right), \tag{8.4}$$

where $g_1, g_2, \ldots$ are expressions which do not depend on z nor on t. Then

$$F_2(z,1) = F(z,1) + zF_1(z,1) - \frac{zF_1(z,1)}{1 + F(z,1)}.$$

Proof. By [NS06, Theorem 10.23], Equation (8.4) is equivalent to the functional equation

$$F(z,t) = t\,G\left(z\left(1 + F\left(z,t\right)\right)\right),\tag{8.5}$$

where $G(z) = \sum_{m \geq 1} g_m z^m$. Now, applying $\frac{\partial}{\partial t}$ to (8.5) and setting $t = 1$, we obtain

$$F_2(z,1) = G\big(z(1 + F(z,1))\big) + G'\big(z(1 + F(z,1))\big)zF_2(z,1)$$

and, applying $\frac{\partial}{\partial z}$ to (8.5) and setting $t = 1$, we obtain

$$F_1(z,1) = G'\big(z(1 + F(z,1))\big)\Big(1 + F(z,1) + zF_1(z,1)\Big).$$

We now eliminate $G'\big(z(1 + F(z,1))\big)$ between these two equations, and use $G\big(z(1 + F(z,1))\big) = F(z,1)$ (obtained by setting $t = 1$ in (8.5)), to give

$$\Big(F_2(z,1) - F(z,1)\Big)\Big(1 + F(z,1) + zF_1(z,1)\Big) = F_1(z,1)\,zF_2(z,1).$$

The result follows immediately by solving this linear equation for $F_2(z,1)$. $\qquad\square$

Remark 8.2. In this remark we discuss an interpretation that the preceding proposition has, when placed in the framework of free probability.

Let μ be a "probability distribution with finite moments of all orders", but construed in a purely algebraic sense, when it is simply a linear functional $\mu\colon \mathbb{C}[X] \to \mathbb{C}$ such that $\mu(1) = 1$. We use the notation

$$M_\mu(z) := \sum_{n=1}^{\infty} \mu(X^n)z^n \quad \text{(the \emph{moment series} of } \mu\text{)}.$$

In free probability one also associates with μ another power series, denoted by R_μ and called the *R-transform of* μ, which in many respects is the free probability analogue for the concept of "characteristic function of μ" used in classical probability. The two series M_μ and R_μ satisfy the functional equation

$$R_\mu\big(z(1 + M_\mu(z))\big) = M_\mu(z),\tag{8.6}$$

which can in fact be used as the definition of R_μ. That is, $R_\mu(z)$ is the unique power series of the form $\sum_{n=1}^{\infty} \alpha_n z^n$ which satisfies Equation (8.6) (for details, see e.g. Lecture 16 of [NS06]).

Now, an important concept of free probability is the operation of *free additive convolution* $\boxplus$; this is the operation with probability distributions which corresponds to the operation of adding freely independent elements of a non-commutative probability space (for details, see e.g. Lecture 12 of [NS06]). What is of interest for us here is that the probability distribution μ from the preceding paragraph defines a convolution semigroup $(\mu_t)_{t \in (0,\infty)}$ with respect to the operation $\boxplus$. The notation commonly used for μ_t is "$\mu^{\boxplus t}$", and in the case when t is an integer one really has $\mu_t = \mu \boxplus \cdots \boxplus \mu$ with t terms in the $\boxplus$-sum. The definition of μ_t for arbitrary (not necessarily integer) $t \in (0,\infty)$ is made via the equation

$$R_{\mu_t}(z) = t\,R_\mu(z);\tag{8.7}$$

that is, μ_t is the uniquely determined distribution whose R-transform equals the right-hand side of (8.7).

Let us next look at the moment series $M_{\mu_t}(z)$ of the distributions $\mu_t = \mu^{\boxplus t}$, and let us bundle all these moment series in one series F of two variables:

$$F(z,t) := M_{\mu_t}(z).$$

The functional equation of the R-transform (stated as Equation (8.6) above) then boils down precisely to the Equation (8.5) used in the proof of Proposition 8.1, where one puts $G := R_\mu$. So then,

Proposition 8.1 can be seen as stating the fact that the "time-derivative" of $M_{\mu_t}(z)$ at time $t = 1$ can be written in terms of M_μ and M'_μ. Indeed, as the reader can easily check, the formula in the conclusion of Proposition 8.1 takes now the form

$$\frac{\partial F(z,t)}{\partial t}\Big|_{t=1} = M_\mu(z) + \frac{zM_\mu(z)M'_\mu(z)}{1 + M_\mu(z)}. \tag{8.8}$$

Note that $M_\mu(z)$ and its derivative $M'_\mu(z)$ are series which are calculated at the exact time $t = 1$. The point in Equation (8.8) is that we can calculate $\frac{\partial F}{\partial t}$ at $t = 1$ just from information available at the exact time $t = 1$: there is no need to look at other times t near 1!

We now return to the framework of Proposition 8.1. In order to place the generating function $\Psi^{(i)}(z,t)$ in this framework, we need to prove that it satisfies condition (8.4), which we do in the next lemma for the three cases $i = 1, 2, 3$.

Some notational details used in the proof of Lemma 8.3: for a set α of positive integers, let $\mathrm{NC}(\alpha)$ denote the set of non-crossing partitions of the elements of α, ordered from smallest to largest. This is a lattice that is isomorphic to $\mathrm{NC}(n)$, where $n = |\alpha|$. The maximum element of this lattice is the partition of α with a single block, namely the set α itself, and we denote this maximum element by 1_α.

Lemma 8.3. *The generating function $\Psi^{(i)}(z,t)$ satisfies condition (8.4) for $i = 1, 2, 3$.*

Proof. For $i = 1$, we introduce some specialized notation for pairs of positive integers. For a positive integer i, let $P_i = \{2i - 1, 2i\}$. For a set α of positive integers, let $P(\alpha) = \cup_{i \in \alpha} P_i$, and let $\lambda(\alpha)$ denote the partition of $P(\alpha)$ in which the blocks are the pairs P_i for $i \in \alpha$. In other words, $\lambda(\alpha)$ is the interval partition of the even set $P(\alpha)$ in which the blocks are consecutive pairs of elements.

Using this notation, note that for every n and $\rho \in \mathrm{NC}(2n)$, we must have $\lambda_{2n} \vee \rho \geq \lambda_{2n}$ and so $\lambda_{2n} \vee \rho$ must have blocks of the form $P(\theta_1), P(\theta_2), \ldots$, where $\theta_1, \theta_2, \ldots$ are the blocks of some partition $\theta \in \mathrm{NC}(n)$. Thus,

$$\Psi^{(1)}(z,t) = \sum_{n \geq 1} z^n \sum_{\theta \in \mathrm{NC}(n)} \prod_{\theta_i \in \theta} \left(t \sum_{\substack{\tau \in \mathrm{NC}(P(\theta_i)) \\ \tau \vee \lambda(\theta_i) = 1_{P(\theta_i)}}} 1 \right),$$

where the inner sum depends only on the size of the set θ_i. We conclude that $\Psi^{(1)}(z,t)$ is of the form $F(z,t)$ as in (8.4) with

$$g_m = \left| \{ \tau \in \mathrm{NC}(2m) \mid \tau \vee \lambda_{2m} = 1_{2m} \} \right|,$$

proving the result for $i = 1$.

For $i = 2$, we introduce further notation for interval partitions. Each interval partition with k blocks is specified by a k-tuple of positive integers $\mathbf{a}_k = (a_1, \ldots, a_k)$, where $a_1, \ldots, a_k \geq 1$ specify the sizes of the blocks in order. For a positive integer i, let $B_i^{(\mathbf{a}_k)} = \{a_1 + \ldots + a_{i-1} + 1, \ldots, a_1 + \ldots + a_i\}$, which is the ith block in the interval partition. For a set α of positive integers, let $B^{(\mathbf{a}_k)}(\alpha) = \cup_{i \in \alpha} B_i^{(\mathbf{a}_k)}$, and let $\gamma^{(\mathbf{a}_k)}(\alpha)$ denote the partition of $B^{(\mathbf{a}_k)}(\alpha)$ in which the blocks are the intervals $B_i^{(\mathbf{a}_k)}$ for $i \in \alpha$. We let $\gamma_k^{(\mathbf{a}_k)} = \gamma^{(\mathbf{a}_k)}([k])$, the interval partition with block sizes $a_1, \ldots, a_k$, in order.

Using this notation, note that for any $a_1, \ldots, a_k \geq 1$ and $\rho \in \mathrm{NC}(a_1 + \ldots + a_k)$, we must have $\gamma_k^{(\mathbf{a}_k)} \vee \rho \geq \gamma_k^{(\mathbf{a}_k)}$, and so $\gamma_k^{(\mathbf{a}_k)} \vee \rho$ must have blocks of the form $B^{(\mathbf{a}_k)}(\theta_1), B^{(\mathbf{a}_k)}(\theta_2), \ldots$ where

$\theta_1, \theta_2, \ldots$ are the blocks of some partition $\theta \in \mathrm{NC}\,(k)$. Thus we have

$$
\Psi^{(2)}(z,t) = \sum_{\substack{k\geq 1}} z^k \sum_{\substack{\theta\in\mathrm{NC}(k)\\a_1,\ldots,a_k\geq 1}} \prod_{\theta_i\in\theta}\left(t \sum_{\substack{\tau\in\mathrm{NC}\left(B^{(\mathbf{a}_k)}(\theta_i)\right)\\ \gamma^{(\mathbf{a}_k)}(\theta_i)\vee\tau=1_{B^{(\mathbf{a}_k)}(\theta_i)}}} \prod_{j\in\theta_i} y^{a_j} \right)
$$

$$
= \sum_{k\geq 1} z^k \sum_{\theta\in\mathrm{NC}(k)} \prod_{\theta_i\in\theta}\left(t \sum_{\substack{a_1,\ldots,a_{|\theta_i|}\geq 1}} \sum_{\substack{\tau\in\mathrm{NC}\left(a_1+\ldots+a_{|\theta_i|}\right)\\ \gamma_m^{(\mathbf{a}_m)}\vee\tau=1_{a_1+\ldots a_{|\theta_i|}}}} \prod_{j=1}^{|\theta_i|} y^{a_j} \right), \qquad (8.9)
$$

where the inner sum in (8.9) depends only on the size of the set θ_i. We conclude that $\Psi^{(2)}(z,t)$ is of the form (8.4) with

$$
g_m = \sum_{\substack{a_1,\ldots,a_m\geq 1\\ \tau\in\mathrm{NC}(a_1+\ldots+a_m)\\ \gamma_m^{(\mathbf{a}_m)}\vee\tau=1_{a_1+\ldots+a_m}}} y^{a_1+\ldots+a_m}, \qquad m\geq 1,
$$

proving the result for $i=2$. (We repeat that in this case, g_m is a formal power series in the variable y, not simply a scalar; nonetheless, it is independent of the variables z and t, which is all we need in order to apply Proposition 8.1.)

Finally, for $i=3$ we have

$$
\Psi^{(3)}(z,t) = \sum_{n\geq 1} z^n \sum_{\theta\in\mathrm{NC}(n)} \prod_{\theta_i\in\theta}\left(t \sum_{\substack{\tau,\sigma\in\mathrm{NC}(\theta_i)\\ \tau\vee\sigma=1_{\theta_i}}} 1 \right),
$$

where the inner sum depends only on the size of the set θ_i. We conclude that $\Psi^{(3)}(z,t)$ is of the form (8.4) with

$$
g_m = \sum_{\substack{\tau,\sigma\in\mathrm{NC}(m)\\ \tau\vee\sigma=1_m}} 1 = \left| \{\pi,\rho\in\mathrm{NC}\,(m)\,|\,\pi\vee\rho=1_m\} \right|, \qquad m\geq 1,
$$

proving the result for $i=3$. $\qquad\square$

Remark 8.4. Each of the three verifications made in the proof of Lemma 8.3 has (on the lines of Remark 8.2) an interpretation in terms of a $\boxplus$-convolution semigroup $(\mu_t)_{t\in(0,\infty)}$. In each of the three cases, the relevant probability distribution μ turns out to be related to the Marchenko-Pastur distribution, which is the counterpart of the Poisson distribution in free probability. For illustration, we discuss below in a bit more detail the first of the three cases, concerning the series $\Psi^{(1)}(z,t)$.

The standard Marchenko-Pastur (or free Poisson) distribution is the absolutely continuous distribution on $[0,4]$ with density $(2\pi)^{-1}\sqrt{(4-t)/t}dt$. In the algebraic setting of Remark 8.2, where a compactly supported probability distribution is treated as a linear functional on $\mathbb{C}[X]$, the standard free Poisson distribution is the linear functional $\nu : \mathbb{C}[X] \to \mathbb{C}$ uniquely determined by the requirement that

$$
\nu(1) = 1 \ \text{ and } \ \nu(X^n) = \mathrm{Cat}_n, \quad \forall n \in \mathbb{N}. \qquad (8.10)
$$

The above formula for moments translates into a very simple formula for the R-transform of ν:

$$R_\nu(z) = \sum_{n=1}^\infty z^n = z/(1-z) \tag{8.11}$$

(see e.g. [NS06], pages 203-206 in Lecture 12).

For our discussion in the present remark, it is convenient to also consider the framework of a non-commutative probability space $(\mathcal{A}, \varphi)$ – this simply means that $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional normalized such that $\varphi(1_\mathcal{A}) = 1$. For an element $a \in \mathcal{A}$, the linear functional $\mu_a : \mathbb{C}[X] \to \mathbb{C}$ determined by the requirement that

$$\mu_a(1) = 1 \ \text{ and } \ \mu_a(X^n) = \varphi(a^n), \ \ \forall n \in \mathbb{N} \tag{8.12}$$

is called the *distribution of a* with respect to φ. An element $a \in \mathcal{A}$ is said to be a *standard free Poisson element* when its distribution μ_a is the functional ν from Equation (8.10).

Let a be a standard free Poisson element in a non-commutative probability space $(\mathcal{A}, \varphi)$, let us fix a positive integer ℓ, and let us consider the element $a^\ell \in \mathcal{A}$. Let μ denote the distribution of a^ℓ with respect to φ. A basic formula in the combinatorial theory of the R-transform (known as the "formula for free cumulants with products as arguments" – see [NS06], pages 178-181 in Lecture 11) expresses the coefficients of the R-transform of μ ($=$ distribution of a^ℓ) in terms of the coefficients of the R-transform of ν ($=$ distribution of a). This formula says that for every $n \in \mathbb{N}$ we have

$$[z^n]\big(R_\mu(z)\big) = \sum_{\substack{\pi \in NC(\ell n) \ such \\ that \ \pi \vee \lambda_n^{(\ell)} = 1_{\ell n}}} \Big(\prod_{V \in \pi} [z^{|V|}]\big(R_\nu(z)\big) \Big), \tag{8.13}$$

where $\lambda_n^{(\ell)} = \{ \{1, \ldots, \ell\}, \{\ell+1, \ldots, 2\ell\}, \ldots, \{(n-1)\ell+1, \ldots, n\ell\} \in NC(\ell n)$. Since in the case at hand all the coefficients of R_ν are equal to 1, Equation (8.13) amounts to

$$[z^n]\big(R_\mu(z)\big) = \big| \ \{\pi \in NC(\ell n) \mid \pi \vee \lambda_n^{(\ell)} = 1_{\ell n}\} \ \big|. \tag{8.14}$$

Upon considering the $\boxplus$-convolution semigroup $(\mu_t)_{t \in (0,\infty)}$ where $\mu_t = \mu^{\boxplus t}$, we come to the formula

$$[z^n]\big(R_{\mu_t}(z)\big) = t \cdot \big| \ \{\pi \in NC(\ell n) \mid \pi \vee \lambda_n^{(\ell)} = 1_{\ell n}\} \ \big|, \tag{8.15}$$

which holds for every $n \in \mathbb{N}$ and $t \in (0,\infty)$.

A reader who is familiar with the summation formula that gives the moments of a distribution in terms of the coefficients of its R-transform (the so-called "moment$\leftrightarrow$free cumulant" formula, see e.g. [NS06], pages 175-177 in Lecture 11) will now recognize that the first two paragraphs in the proof of Lemma 8.3 are actually performing a transition from free cumulants to moments, which starts from Equation (8.15) in the special case $\ell = 2$ and * arrives to

$$M_{\mu_t}(X^n) = \sum_{\pi \in NC(\ell n)} t^{|\pi \vee \lambda_n^{(\ell)}|}, \ \ \forall n \in \mathbb{N}, t \in (0,\infty). \tag{8.16}$$

Finally, we note that with the specialization $\ell = 2$, Equation (8.16) amounts to the following statement: for every fixed $t \in (0,\infty)$, the generating function $\Psi^{(1)}(z,t)$ is precisely the moment series $M_{\mu_t}(z)$. This is the interpretation in terms of $\boxplus$-convolution for why the series $\Psi^{(1)}(z,t)$ fits in the framework of Proposition 8.1 (or of Remark 8.2).

---

*It is easily seen that the first two paragraphs in the proof of Lemma 8.3 would in fact work for any $\ell \in \mathbb{N}$ instead of $\ell = 2$.

We now return to the main line of this section. We will show how, by combining Proposition 8.1 and Lemma 8.3, one can evaluate the three averages $\widetilde{d_n}$, d_n and $\widetilde{b_n}$. We begin with $\widetilde{d_n}$ for n even, which, as described above, is the average distance between a non-crossing partition $\rho \in \mathrm{NC}(n)$ and the fixed interval partition $\lambda_n = \{\{1,2\},\ldots,\{n-1,n\}\}$.

Proof of Proposition 1.6(2) . Recall that we need to prove that for n even

$$(i) \quad \widetilde{d_n} = -\frac{3}{2} + \frac{\binom{n}{n/2} 2^{n-1}}{\mathrm{Cat}_n},$$

and that

$$(ii) \quad \lim_{n \to \infty} \left(\widetilde{d_n} - \frac{\sqrt{2}}{2} n \right) = \frac{7\sqrt{2}}{16} - \frac{3}{2}.$$

For part (i), from (8.1) and (8.2) we have

$$\widetilde{d_n} = \frac{n}{2} + \frac{n+1}{2} - \frac{2}{\mathrm{Cat}_n} \left[z^{n/2} \right] \Psi_2^{(1)}(z,1). \tag{8.17}$$

Now by Lemma 8.3 with $i = 1$, we can apply Proposition 8.1 and obtain

$$\Psi_2^{(1)}(z,1) = \Psi^{(1)}(z,1) + z\Psi_1^{(1)}(z,1) - \frac{z\Psi_1^{(1)}(z,1)}{1 + \Psi^{(1)}(z,1)}.$$

But from (8.3) we have

$$\Psi^{(1)}(z,1) = \sum_{n \geq 1} \mathrm{Cat}_{2n} z^n, \qquad z\Psi_1^{(1)}(z,1) = \sum_{n \geq 1} n\mathrm{Cat}_{2n} z^n,$$

and thus from (8.17) we obtain

$$\begin{aligned}
\widetilde{d_n} &= n + \frac{1}{2} - 2 - n + \frac{2}{\mathrm{Cat}_n} \left[z^{n/2} \right] \frac{z\Psi_1^{(1)}(z,1)}{1 + \Psi^{(1)}(z,1)} = -\frac{3}{2} + \frac{2}{\mathrm{Cat}_n} \left[z^{n/2} \right] \frac{z\Psi_1^{(1)}(z,1)}{1 + \Psi^{(1)}(z,1)} \\
&= -\frac{3}{2} + \frac{2}{\mathrm{Cat}_n} \left[z^n \right] \frac{z^2 \Psi_1^{(1)}(z^2,1)}{1 + \Psi^{(1)}(z^2,1)}
\end{aligned}$$

To simplify the final term above, we have the closed form*

$$1 + \Psi^{(1)}(z^2,1) = \sum_{n \geq 0} \mathrm{Cat}_{2n} z^{2n} = \frac{u-v}{4z},$$

where $u = \sqrt{1+4z}$, $v = \sqrt{1-4z}$. Thus, using the chain rule, we obtain

$$z^2 \Psi_1^{(1)}(z^2,1) = \frac{z^2}{2z} \frac{\partial}{\partial z} \Psi^{(1)}(z^2,1) = \frac{z}{2} \left(-\frac{u-v}{4z^2} + \frac{u+v}{2zuv} \right),$$

and, combining these expressions and simplifying via $u^2 - v^2 = 8z$ and $u^2 + v^2 = 2$, we get[†]

$$\begin{aligned}
\frac{z^2 \Psi_1^{(1)}(z^2,1)}{1 + \Psi^{(1)}(z^2,1)} &= -\frac{1}{2} + \frac{z(u+v)}{uv(u-v)} = -\frac{1}{2} + \frac{(u^2-v^2)(u+v)}{8uv(u-v)} \\
&= -\frac{1}{2} + \frac{(u+v)^2}{8uv} = -\frac{1}{2} + \frac{2uv+2}{8uv} = -\frac{1}{4} + \frac{1}{4uv} \\
&= -\frac{1}{4} + \frac{1}{4\sqrt{1-16z^2}} = \sum_{n \geq 1} \binom{2n}{n} 4^{n-1} z^{2n}.
\end{aligned}$$

---

*It is standard that $C(z) := \sum \mathrm{Cat}_n z^n = \frac{1-\sqrt{1-4z}}{2z}$, and here we need $\frac{1}{2}(C(z) + C(-z))$.

[†]We use here the well-known equality of generating functions $\sum_{n \geq 0} \binom{2n}{2} z^n = \frac{1}{\sqrt{1-4z}}$.

Part (i) of the result follows immediately. For part (ii), the asymptotics of central binomial coefficients and Catalan numbers are well known (see, e.g., [FS09, Pages 381, 383–384]):

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right), \qquad \mathrm{Cat}_n = \frac{4^n}{n\sqrt{\pi n}}\left(1 - \frac{9}{8n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right). \tag{8.18}$$

Combining these, we obtain

$$\frac{\binom{2n}{n}2^{2n-1}}{\mathrm{Cat}_{2n}} = \frac{4^{2n}}{2\sqrt{\pi n}}\left(\frac{4^{2n}}{2n\sqrt{2\pi n}}\right)^{-1}\left(1 + \frac{7}{16n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)$$

$$= \sqrt{2}\,n + \frac{7\sqrt{2}}{16} + \mathcal{O}\left(\frac{1}{n}\right),$$

and part (ii) of the result follows immediately. $\qquad\square$

Now we consider d_n, which, as described above, is the average distance between an interval partition $\pi \in \mathrm{Int}\,(n)$ and a non-crossing partition $\rho \in \mathrm{NC}(n)$. Theorem 1.3 above states that $\lim_{n\to\infty}\left(d_n - \frac{2}{3}n\right) = -\frac{28}{27}$. Before proving it, we first prove the following result:

Proposition 8.5. *For $n \geq 1$,*

$$d_n = 6n + 4 + \frac{(-1)^n}{2^{n-1}\mathrm{Cat}_n} - \frac{3}{2^{n-1}\mathrm{Cat}_n}[y^n]\frac{1}{(1+y)\sqrt{1-8y}}. \tag{8.19}$$

Proof. From (8.1) and (8.2) we have

$$d_n = \tfrac{1}{2}(n+1) + \tfrac{1}{2}(n+1) - \frac{2}{2^{n-1}\mathrm{Cat}_n}[y^n]\Psi_2^{(2)}(1,1). \tag{8.20}$$

Now by Lemma 8.3 with $i = 2$, we can apply Proposition 8.1, and hence obtain

$$\Psi_2^{(2)}(1,1) = \Psi^{(2)}(1,1) + \Psi_1^{(2)}(1,1) - \frac{\Psi_1^{(2)}(1,1)}{1+\Psi^{(2)}(1,1)}.$$

But from (8.3) we have

$$\Psi^{(2)}(1,1) = \sum_{n\geq 1}2^{n-1}\mathrm{Cat}_n y^n, \qquad \Psi_1^{(2)}(1,1) = \sum_{n\geq 1}(n+1)2^{n-2}\mathrm{Cat}_n y^n,$$

and thus from (8.20) we obtain

$$d_n = (n+1) - 2 - (n+1) + \frac{2}{2^{n-1}\mathrm{Cat}_n}[y^n]\frac{\Psi_1^{(2)}(1,1)}{1+\Psi^{(2)}(1,1)}$$

$$= -2 + \frac{2}{2^{n-1}\mathrm{Cat}_n}[y^n]\frac{\Psi_1^{(2)}(1,1)}{1+\Psi^{(2)}(1,1)}.$$

To simplify the final term above, we have the closed forms

$$1 + \Psi^{(2)}(1,1) = \frac{1}{2} + \sum_{n\geq 0}2^{n-1}\mathrm{Cat}_n y^n = \frac{1+4y-\sqrt{1-8y}}{8y},$$

and

$$\Psi_1^{(2)}(1,1) = \sum_{n\geq 1}(n+1)2^{n-2}\mathrm{Cat}_n y^n = \sum_{n\geq 1}2^{n-2}\binom{2n}{n}y^n = \frac{1}{4}\left(\frac{1}{\sqrt{1-8y}} - 1\right).$$

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Combining these expressions, we get

$$\frac{\Psi_1^{(2)}(1,1)}{1 + \Psi^{(2)}(1,1)} = \frac{2y\left(\frac{1}{\sqrt{1-8y}} - 1\right)}{1 + 4y - \sqrt{1-8y}}$$

$$= \frac{1}{8(1+y)}\left(\frac{1}{\sqrt{1-8y}} - 1\right)\left(1 + 4y + \sqrt{1-8y}\right)$$

$$= -\frac{y}{2(1+y)} + \frac{3y}{2(1+y)\sqrt{1-8y}}$$

$$= -\frac{y}{2(1+y)} + \frac{3}{2\sqrt{1-8y}} - \frac{3}{2(1+y)\sqrt{1-8y}},$$

where for the second equality we have rationalized the denominator, and for the third and fourth equalities we have routinely simplified. Equation (8.19) follows immediately. $\square$

Proof of Theorem 1.3 . To prove that $\lim_{n\to\infty}\left(d_n - \frac{2}{3}n\right) = -\frac{28}{27}$, we follow the treatment in [FS09, Chapter VI], referred to as *singularity analysis*. First,

$$[y^n]\frac{1}{(1+y)\sqrt{1-8y}} = [y^n]\frac{8}{(8+8y)\sqrt{1-8y}} = 8^n \cdot [y^n]\frac{8}{(8+y)\sqrt{1-y}}.$$

We expand $(8+y)^{-1}$ about $y = 1$, in this case via a geometric series, to obtain

$$\frac{8}{8+y} = \frac{\frac{8}{9}}{1 - \frac{1}{9}(1-y)} = \frac{8}{9}\sum_{k\geq 0}\left(\frac{1}{9}\right)^k(1-y)^k,$$

and hence obtain the expansion

$$\frac{8}{(8+y)\sqrt{1-y}} = \frac{8}{9}\left(\frac{1}{\sqrt{1-y}} + \frac{1}{9}\sqrt{1-y}\right) + \frac{8\cdot(1-y)^{1.5}}{81\cdot(8+y)}.$$

But, also from [FS09, Theorem VI.1], we have

$$[y^n](1-y)^\alpha = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\left(1 + \frac{\alpha(\alpha+1)}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right),$$

where $\Gamma(\alpha)$ is the *Euler Gamma Function* defined as $\Gamma(\alpha) \stackrel{\text{def}}{=} \int_0^\infty e^{-t}t^{\alpha-1}dt$. From this we deduce, using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$, that

$$[y^n]\frac{1}{\sqrt{1-y}} = \frac{1}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)$$

$$[y^n]\sqrt{1-y} = \frac{1}{-2\sqrt{\pi n}}\left(\frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

Finally, since the function $\frac{(1-y)^{1.5}}{8+y}$ is $\mathcal{O}(1-y)^{1.5}$ in a neighborhood of $y = 1$ and satisfies the other assumptions of [FS09, Theorem VI.3], we have, by this theorem, that

$$[y^n]\frac{(1-y)^{1.5}}{8+y} = \mathcal{O}\left(n^{-2.5}\right).$$

We combine all this to obtain

$$
\begin{aligned}
[y^n]\,\frac{1}{(1+y)\sqrt{1-8y}} \;&=\; 8^n\cdot[y^n]\left(\frac{8}{9}\frac{1}{\sqrt{1-y}}+\frac{8}{81}\sqrt{1-y}+\frac{8\cdot(1-y)^{1.5}}{81\cdot(8+y)}\right)\\
&=\; \frac{8}{9}\cdot\frac{8^n}{\sqrt{\pi n}}\left(\left(1-\frac{1}{8n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)+\left(-\frac{1}{18n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)+\mathcal{O}\left(\frac{1}{n^2}\right)\right)\\
&=\; \frac{8}{9}\cdot\frac{8^n}{\sqrt{\pi n}}\left[1-\frac{13}{72n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right].
\end{aligned}
$$

Combining these results with (8.18) we obtain

$$
\begin{aligned}
\frac{3}{2^{n-1}\mathrm{Cat}_n}\,[y^n]\,\frac{1}{(1+y)\sqrt{1-8y}} \;&=\; \frac{2\cdot3\cdot n\sqrt{\pi n}}{2^n\cdot4^n}\left[1+\frac{9}{8n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right]\frac{8\cdot8^n}{9\sqrt{\pi n}}\left[1-\frac{13}{72n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right]\\
&=\; \frac{16}{3}n\left[1+\frac{17}{18n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right]=\frac{16}{3}n+\frac{136}{27}+\mathcal{O}\left(\frac{1}{n}\right),
\end{aligned}
$$

and the result follows immediately using (8.19). $\qquad\square$

Finally, we consider $\widetilde{b}_n$, which, as described above, is an upper bound on the average distance between two non-crossing partitions $\pi,\rho\in\mathrm{NC}(n)$. Before proving Proposition 1.4 concerning the asymptotics of $\widetilde{b}_n$, we first show:

Proposition 8.6. *For $n\geq1$ we have*

$$
\widetilde{b}_n=-n-1+\frac{2n}{\mathrm{Cat}_n^2}\,[z^n]\log\left(1+\sum_{k\geq1}\mathrm{Cat}_k^2 z^k\right).
$$

Proof. From (8.1) and (8.2) we have

$$
\widetilde{b}_n=\tfrac{1}{2}(n+1)+\tfrac{1}{2}(n+1)-\frac{2}{\mathrm{Cat}_n^2}\,[z^n]\,\Psi_2^{(3)}(z,1). \tag{8.21}
$$

Now by Lemma 8.3 with $i=3$, we can apply Proposition 8.1, and hence obtain

$$
\Psi_2^{(3)}(z,1)=\Psi^{(3)}(z,1)+z\Psi_1^{(3)}(z,1)-\frac{z\Psi_1^{(3)}(z,1)}{1+\Psi^{(3)}(z,1)}.
$$

But from (8.3) we have

$$
\Psi^{(3)}(z,1)=\sum_{n\geq1}\mathrm{Cat}_n^2 z^n,\qquad z\Psi_1^{(3)}(z,1)=\sum_{n\geq1}n\mathrm{Cat}_n^2 z^n,
$$

and thus from (8.21) we obtain

$$
\begin{aligned}
\widetilde{b}_n&=(n+1)-2-2n+\frac{2}{\mathrm{Cat}_n^2}\,[z^n]\,\frac{z\Psi_1^{(3)}(z,1)}{1+\Psi^{(3)}(z,1)}\\
&=-n-1+\frac{2}{\mathrm{Cat}_n^2}\,[z^n]\,\frac{z\Psi_1^{(3)}(z,1)}{1+\Psi^{(3)}(z,1)}\\
&=-n-1+\frac{2}{\mathrm{Cat}_n^2}\,[z^n]\,z\frac{\partial}{\partial z}\log\left(1+\Psi^{(3)}(z,1)\right),
\end{aligned}
$$

and the result follows from the fact that $[z^n]\,z\frac{\partial}{\partial z}f(z)=n[z^n]f(z)$ for any formal power series f. $\qquad\square$

Proof of Proposition 1.4. We need to show that

$$\lim_{n \to \infty} \frac{\widetilde{b_n}}{n} = \frac{3\pi - 8}{8 - 2\pi}.$$

Note that the series

$$\Phi(z) = 1 + \Psi^{(3)}(z, 1)$$

has radius of convergence $\frac{1}{16}$ – this follows, e.g., from the asymptotics of Catalan numbers in (8.18). Then, following the treatment in [FS09, Section VI.10.2] of Hadamard products of series as part of closure properties in singularity analysis, we obtain

$$\lim_{n \to \infty} \frac{1}{\text{Cat}_n^2} \left[z^n \right] \log \Phi(z) = \frac{1}{\Phi\left(\frac{1}{16}\right)}.$$

But from [LZ92, Page 132] we have

$$\Phi\left(\tfrac{1}{16}\right) = \frac{4(4 - \pi)}{\pi},$$

and the result follows immediately. $\qquad\square$

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