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Note

Characterizing the number of coloured m-ary partitions modulo m, with and without gaps

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ABSTRACT

In a pair of recent papers, Andrews, Fraenkel and Sellers provide a complete characterization for the number of m-ary partitions modulo m, with and without gaps. In this paper we extend these results to the case of coloured m-ary partitions, with and without gaps. Our method of proof is different, giving explicit expansions for the generating functions modulo m.

1. Introduction

An m-ary partition is an integer partition in which each part is a nonnegative integer power of a fixed integer $m \geq 2$. An m-ary partition *without gaps* is an m-ary partition in which m^j must occur as a part whenever m^{j+1} occurs as a part, for every nonnegative integer j.

Recently, Andrews, Fraenkel and Sellers [2] found an explicit expression that characterizes the number of m-ary partitions of a nonnegative integer n modulo m; remarkably, this expression depended only on the coefficients in the base m representation of n. Subsequently Andrews, Fraenkel and Sellers [3] followed this up with a similar result for the number of m-ary partitions without gaps, of a nonnegative integer n modulo m; again, they were able to obtain a (more complicated) explicit expression, and again this expression depended only on the coefficients in the base m representation of n. See also Edgar [6] and Ekhad and Zeilberger [7] for more on these results.

The study of congruences for integer partition numbers has a long history, starting with the work of Ramanujan (see, e.g., [8]). For the special case of m-ary partitions, a number of authors have studied congruence properties, including Churchhouse [5] for $m = 2$, Rødseth [9] for m a prime, and Andrews [1] for arbitrary positive integers $m \geq 2$. The numbers of m-ary partitions without gaps had been previously considered by Bessenrodt, Olsson and Sellers [4] for $m = 2$.

In this note, we consider m-ary partitions, with and without gaps, in which the parts are *coloured*. To specify the number of colours for parts of each size, we let $\mathbf{k} = (k_0, k_1, \ldots)$ for positive integers $k_0, k_1, \ldots$, and say that an m-ary partition is $\mathbf{k}$-*coloured* when there are k_j colours for the part m^j, for $j \geq 0$. This means that there are k_j different kinds of parts of the same size m^j. Let $b_m^{(\mathbf{k})}(n)$ denote the number of $\mathbf{k}$-coloured m-ary partitions of n, and let $c_m^{(\mathbf{k})}(n)$ denote the number of $\mathbf{k}$-coloured m-ary partitions of n without gaps. For the latter, some part m^j of any colour must occur as a part whenever some part m^{j+1} of any colour (not necessarily the same colour) occurs as a part, for every nonnegative integer j. (In the special case that $k_j = k$ for all $j \geq 0$, where k is a positive integer, we say that the m-ary partitions are k-*coloured*.)

We extend the results of Andrews, Fraenkel and Sellers in [2] and [3] to the case of $\mathbf{k}$-coloured m-ary partitions, where m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$. Our method of proof is different, giving explicit expansions for the generating

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functions modulo m. We then extract the coefficients in these generating functions to determine explicit expressions for the corresponding numbers of partitions modulo m, stated in the following pair of results.

Theorem 1.1. *For $n \geq 0$, suppose that the base m representation of n is given by*

$$n = d_0 + d_1 m + \cdots + d_t m^t, \qquad 0 \leq t.$$

If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have

$$b_m^{(\mathbf{k})}(n) \equiv \binom{k_0 - 1 + d_0}{k_0 - 1} \prod_{j=1}^{t} \binom{k_j + d_j}{k_j} \pmod{m}.$$

Theorem 1.2. *For $n \geq 1$, suppose that n is divisible by m, with base m representation given by*

$$n = d_s m^s + \cdots + d_t m^t, \qquad 1 \leq s \leq t,$$

where $1 \leq d_s \leq m-1$, and $0 \leq d_{s+1}, \ldots, d_t \leq m-1$. If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then for $0 \leq d_0 \leq m-1$ we have

$$c_m^{(\mathbf{k})}(n - d_0) \equiv \binom{k_0 - 1 - d_0}{k_0 - 1} \left(\varepsilon_s + (-1)^{s-1} \left\{ \binom{k_s + d_s - 1}{k_s} - 1 \right\} \sum_{i=s}^{t} \prod_{j=s+1}^{i} \left\{ \binom{k_j + d_j}{k_j} - 1 \right\} \right) \pmod{m},$$

where $\varepsilon_s = 0$ if s is even, and $\varepsilon_s = 1$ if s is odd.

Theorem 1.1 is proved in Section 2, and Theorem 1.2 is proved in Section 3.

2. Coloured *m*-ary partitions

In this section we consider the following generating function for the numbers $b_m^{(\mathbf{k})}(n)$ of $\mathbf{k}$-coloured m-ary partitions:

$$B_m^{(\mathbf{k})}(q) = \sum_{n=0}^{\infty} b_m^{(\mathbf{k})}(n) q^n = \prod_{j=0}^{\infty} \left(1 - q^{m^j}\right)^{-k_j}.$$

The following simple result will be key to the expansion of $B_m^{(\mathbf{k})}(q)$ modulo m.

Proposition 2.1. *For positive integers m, a with m relatively prime to $(a-1)!$, we have*

$$(1-q)^{-a} \equiv \left(1 - q^m\right)^{-1} \sum_{\ell=0}^{m-1} \binom{a - 1 + \ell}{a - 1} q^{\ell} \pmod{m}.$$

Proof. From the binomial theorem we have

$$(1-q)^{-a} = \sum_{\ell=0}^{\infty} \binom{a-1+\ell}{a-1} q^{\ell}.$$

Now using the falling factorial notation $(a - 1 + \ell)_{a-1} = (a - 1 + \ell)(a - 2 + \ell) \cdots (1 + \ell)$ we have

$$\binom{a-1+\ell}{a-1} = ((a-1)!)^{-1} (a - 1 + \ell)_{a-1}.$$

But

$$(a - 1 + \ell + m)_{a-1} \equiv (a - 1 + \ell)_{a-1} \pmod{m},$$

for any integer ℓ, and $((a-1)!)^{-1}$ exists in $\mathbb{Z}_m$ since m is relatively prime to $(a-1)!$, which gives

$$\binom{a-1+\ell+m}{a-1} \equiv \binom{a-1+\ell}{a-1} \pmod{m}, \tag{1}$$

and the result follows. □

We are now able to give an explicit expansion for $B_m^{(\mathbf{k})}(q)$ modulo m.

Theorem 2.2. *If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have*

$$B_m^{(\mathbf{k})}(q) \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \prod_{j=1}^{\infty} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m}.$$

Proof. Consider the finite product

$$P_i = \prod_{j=0}^{i} \left(1 - q^{m^j}\right)^{-k_j}, \qquad i \geq 0.$$

We prove that

$$P_i \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \left(1 - q^{m^{i+1}}\right)^{-1} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m}, \tag{2}$$

by induction on i. As a base case, the result for $i = 0$ follows immediately from Proposition 2.1 with $a = k_0$. Now assume that (2) holds for some choice of $i \geq 0$, and we obtain

$$\begin{aligned} P_{i+1} &= \prod_{j=0}^{i+1} \left(1 - q^{m^j}\right)^{-k_j} = \left(1 - q^{m^{i+1}}\right)^{-k_{i+1}} P_i \\ &\equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \left(1 - q^{m^{i+1}}\right)^{-k_{i+1}-1} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m} \\ &\equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \left(1 - q^{m^{i+2}}\right)^{-1} \prod_{j=1}^{i+1} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m}, \end{aligned}$$

where the second last equivalence follows from the induction hypothesis, and the last equivalence follows from Proposition 2.1 with $a = k_{i+1} + 1, q = q^{m^{i+1}}$.

This completes the proof of (2) by induction on i, and the result follows immediately since

$$B_m^{(\mathbf{k})}(q) = \lim_{i \to \infty} P_i. \quad \square$$

We are now able to prove Theorem 1.1, which gives an explicit expression for the coefficients modulo m that follows from the above expansion of the generating function $B_m^{(\mathbf{k})}(q)$.

Proof of Theorem 1.1. In the expansion of the series $B_m^{(\mathbf{k})}(q)$ given in Theorem 2.2, the monomial q^n arises uniquely with the specializations $\ell_j = d_j, j = 0, \ldots, t$ and $\ell_j = 0, j > t$. But for the case $\ell_j = 0$ we have $\binom{k_j+\ell_j}{k_j} = \binom{k_j}{k_j} = 1$, and the result follows immediately. $\square$

Example 2.3. As an example of Theorem 1.1, consider the case that $k_j = k, j \geq 0$, where k is a positive integer, and that m is relatively prime to $k!$. Then the number of k-coloured m-ary partitions of n is congruent to

$$\binom{k - 1 + d_0}{k - 1} \prod_{j=1}^{t} \binom{k + d_j}{k} \tag{3}$$

modulo m.

Specializing the expression given in Theorem 1.1 to the case $k_j = 1$ for $j \geq 0$ (or, equivalently, specializing (3) to the case $k = 1$), provides an alternative proof to Andrews, Fraenkel and Sellers' characterization of $m-$ary partitions modulo m, which was given as Theorem 1 of [2].

3. Coloured *m*-ary partitions without gaps

In this section we consider the following generating function for the numbers $c_m^{(\mathbf{k})}(n)$ of **k**-coloured m-ary partitions without gaps:

$$C_m^{(\mathbf{k})}(q) = 1 + \sum_{n=0}^{\infty} c_m^{(\mathbf{k})}(n) q^n = 1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \left(\left(1 - q^{m^j}\right)^{-k_j} - 1 \right).$$

The following result gives an explicit expansion for $C_m^{(\mathbf{k})}(q)$ modulo m. The proof uses Proposition 2.1 in a similar way as for the expansion of $B_m^{(\mathbf{k})}(q)$ modulo m in Theorem 2.2 of the previous section.

Theorem 3.1. *If m is relatively prime to $(k_0-1)!$ and to $k_j!$ for $j \geq 1$, then we have*

$$C_m^{(\mathbf{k})}(q) \equiv 1 + \left(\sum_{\ell_0=1}^{m} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0} \right) \sum_{i=0}^{\infty} \left(1-q^{m^{i+1}}\right)^{-1} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j+\ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m}.$$

Proof. Consider the finite product

$$R_i = \prod_{j=0}^{i} \left(\left(1-q^{m^j}\right)^{-k_j} - 1 \right), \qquad i \geq 0.$$

We prove that

$$R_i \equiv \left(\sum_{\ell_0=1}^{m} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0} \right) \left(1-q^{m^{i+1}}\right)^{-1} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j+\ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m}, \tag{4}$$

by induction on i. As a base case, the result for $i = 0$ follows immediately from Proposition 2.1 with $a = k_0$. Now assume that (4) holds for some choice of $i \geq 0$, and we obtain

$$\begin{aligned} R_{i+1} &= \prod_{j=0}^{i+1} \left(\left(1-q^{m^j}\right)^{-k_j} - 1 \right) = \left(\left(1-q^{m^{i+1}}\right)^{-k_{i+1}} - 1 \right) R_i \\ &\equiv \left(\sum_{\ell_0=1}^{m} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0} \right) \left\{ \left(1-q^{m^{i+1}}\right)^{-k_{i+1}-1} - \left(1-q^{m^{i+1}}\right)^{-1} \right\} \\ &\qquad \times \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j+\ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m} \\ &\equiv \left(\sum_{\ell_0=1}^{m} \binom{k_0-1+\ell_0}{k_0-1} q^{\ell_0} \right) \left(1-q^{m^{i+2}}\right)^{-1} \prod_{j=1}^{i+1} \left(\sum_{\ell_j=0}^{m-1} \left\{ \binom{k_j+\ell_j}{k_j} - 1 \right\} q^{\ell_j m^j} \right) \pmod{m}, \end{aligned}$$

where the second last equivalence follows from the induction hypothesis, and the last equivalence follows from Proposition 2.1 with $a = k_{i+1}+1, q = q^{m^{i+1}}$ and $a = 1, q = q^{m^{i+1}}$.

This completes the proof of (4) by induction on i, and the result follows immediately since

$$C_m^{(\mathbf{k})}(q) = 1 + \sum_{i=0}^{\infty} R_i. \quad \square$$

We are now able to prove Theorem 1.2, which gives an explicit expression for the coefficients modulo m that follows from the above expansion of the generating function $C_m^{(\mathbf{k})}(q)$.

Proof of Theorem 1.2. First note that we have

$$n - d_0 = m - d_0 + (m-1)m^1 + \cdots + (m-1)m^{s-1} + (d_s - 1)m^s + d_{s+1}m^{s+1} + \cdots + d_t m^t.$$

Now consider the following specializations: $\ell_0 = m - d_0, \ell_j = m-1, j = 1, \ldots, s-1, \ell_s = d_s - 1, \ell_j = d_j, j = s+1, \ldots, t$, and $\ell_j = 0, j > t$. Then, in the expansion of the series $C_m^{(\mathbf{k})}(q)$ given in Theorem 3.1, the monomial q^n arises once for each $i \geq 0$, in particular with the above specializations truncated to $\ell_0, \ldots, \ell_i$. But with these specializations we have

- for $j = 0$:
$$\binom{k_j - 1 + \ell_j}{k_j - 1} - 1 = \binom{k_0 - 1 + m - d_0}{k_0 - 1} = \binom{k_0 - 1 - d_0}{k_0 - 1}, \qquad \text{from (1)},$$
- for $j = 1, \ldots, s-1$:
$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j - 1}{k_j} - 1 = 0 - 1 = -1,$$

  and
$$\sum_{i=0}^{s-1} \prod_{j=1}^{i} \left\{ \binom{k_j + \ell_j}{k_j} - 1 \right\} = \sum_{i=0}^{s-1} (-1)^i = \varepsilon_s,$$

- for $j = s$:

$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_s + d_s - 1}{k_s} - 1,$$

- for $j = s + 1, \ldots, t$:

$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j + d_j}{k_j} - 1,$$

- for $j > t$:

$$\binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j}{k_j} - 1 = 1 - 1 = 0.$$

The result follows straightforwardly from Theorem 3.1. □

Example 3.2. As an example of Theorem 1.2, consider the case that $k_j = k, j \geq 0$, where k is a positive integer, and that m is relatively prime to $k!$. Then the number of k-coloured m-ary partitions of $n - d_0$ without gaps is congruent to

$$\binom{k - 1 - d_0}{k - 1} \left(\varepsilon_s + (-1)^{s-1} \left\{ \binom{k + d_s - 1}{k} - 1 \right\} \sum_{i=s}^{t} \prod_{j=s+1}^{i} \left\{ \binom{k + d_j}{k} - 1 \right\} \right) \tag{5}$$

modulo m.

Specializing the expression given in Theorem 1.2 to the case $k_j = 1$ for $j \geq 0$ (or, equivalently, specializing (5) to the case $k = 1$), provides an alternative proof to Andrews, Fraenkel and Sellers' characterization of $m-$ary partitions modulo m without gaps, which was given as Theorem 2.1 of [3].

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