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A New Tableau Representation for Supersymmetric Schur Functions

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We give a new tableau definition for supersymmetric skew Schur functions, and obtain a number of properties of these functions as easy corollaries. © 1994 Academic Press, Inc.

1. INTRODUCTION

For partitions λ and μ with $\mu \subseteq \lambda$, the supersymmetric skew Schur function $s_{\lambda/\mu}$ in variables $X_m = (x_1, x_2, \ldots, x_m)$ and $Y_n = (y_1, y_2, \ldots, y_n)$ may be defined by the formula

$$s_{\lambda/\mu}(X_m/Y_n) = \sum_{\mu \subseteq \nu \subseteq \lambda} s_{\nu/\mu}(X_m) s_{\lambda'/\nu'}(Y_n), \qquad (1)$$

where $s_{\lambda/\mu}(X_m)$ denotes the ordinary (symmetric) skew Schur function in variables X_m and λ' denotes the partition conjugate to λ. When $\mu = \emptyset$ these define the characters of certain irreducible representations of the Lie superalgebras $gl(m/n)$ introduced in [13, 14] and studied by several other authors, e.g, [1, 3, 4, 10, 18, 19, 22–24]. The explicit formula (1) appears first in [4], where $s_{\lambda/\mu}(X_m/Y_n)$ is called a "hook Schur function." Several other equivalent combinatorial expressions for $s_{\lambda/\mu}(X_m/Y_n)$ may be found in [8] and [23].

In general, a function $\phi(X_m, Y_n)$ in variables X_m and Y_n is supersymmetric if it is symmetric in X_m and Y_n separately, and satisfies an

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additional cancellation property (described in Section 4, Theorem 4.1). These functions were introduced in [17], where they are called "bisymmetric." Another characterization was conjectured in [20] and proved in [22], where the name "supersymmetric" seems to have been used for the first time. It follows from results in [22] that the supersymmetric Schur functions $s_\lambda(X_m/Y_n)$ form an integral basis for the algebra of all supersymmetric polynomials in X_m and Y_n.

Supersymmetric functions $\phi(X, Y)$ in countably infinite sets of variables X and Y have the following simple characterization: let $f(X \cup Y)$ be a function jointly symmetric in variables $X = (\dots, x_{-1}, x_0, x_1, \dots)$ and $Y = (\dots, y_{-1}, y_0, y_1, \dots)$. Here we use $\mathbb{Z}$ instead of $\mathbb{Z}^+$ as the index set for reasons which will become apparent later. Let ω_Y be the involutory automorphism on $\Lambda(Y)$, the ring of symmetric functions in Y (for a definition of ω and basic notation for partitions and symmetric functions, see, for example, [15]). Then $\omega_Y f$ is symmetric in X and Y separately, and satisfies the cancellation hypothesis of [17] and [22]. Hence $\omega_Y f$ is supersymmetric, and every supersymmetric function arises in this way. From this point of view supersymmetric Schur functions arise from ordinary Schur functions, and may be compactly represented by the formula

$$s_{\lambda/\mu}(X/Y) = \omega_Y s_{\lambda/\mu}(X \cup Y) \tag{2}$$

(where $s_{\lambda/\mu}(X/Y)$ is defined by the right hand side of (1) with X_m and Y_n replaced by X and Y respectively). This was noted in [23], and follows easily from (1), using the well-known expansion

$$s_{\lambda/\mu}(X \cup Y) = \sum_{\mu \subseteq \nu \subseteq \lambda} s_{\nu/\mu}(X) s_{\lambda/\nu}(Y) \tag{3}$$

and the fact that ω acts on skew Schur functions by conjugating the shape.

In this paper we give a simple combinatorial representation of the functions $s_{\lambda/\mu}(X/Y)$ which yields immediate and transparent proofs of many of their important properties. To describe this requires some more notation.

For $A \subseteq \mathbb{Z}$, let $\mathcal{T}_{\lambda/\mu}(A)$ be the set of skew tableaux of shape λ/μ with entries in A. Thus these consist of elements of A placed in the cells of skew shape λ/μ, weakly increasing along rows and strictly increasing down columns. It is important in what follows that we consider the possibility of tableaux with positive and negative integers: A may be an arbitrary subset of $\mathbb{Z}$. For $T \in \mathcal{T}_{\lambda/\mu}(A)$ and α a cell in λ/μ, let $T(\alpha)$ be the entry of T in cell α, and let $C(\alpha)$ denote the *content* of α; that is, $C(\alpha) = c - r$ if α lies in row r and column c. Let T^C denote the array obtained from T by replacing each $T(\alpha)$ by $T(\alpha) + C(\alpha)$. Clearly the entries in T^C are weakly increasing in columns and strictly increasing in rows, so the transpose of T^C is a tableau of shape λ'/μ'. In fact, the map $T \to (T^C)'$ defines a bijection between $\mathcal{T}_{\lambda/\mu}(\mathbb{Z})$ and $\mathcal{T}_{\lambda'/\mu'}(\mathbb{Z})$, though not between $\mathcal{T}_{\lambda/\mu}(\mathbb{Z}^+)$ and $\mathcal{T}_{\lambda'/\mu'}(\mathbb{Z}^+)$ unless μ is empty. Now consider the combinatorial function

$$G_{\lambda/\mu}(X,Y) = \sum_{T \in \mathcal{T}_{\lambda/\mu}(\mathbb{Z})} \prod_{\alpha \in \lambda/\mu} (x_{T(\alpha)} + y_{T(\alpha)+C(\alpha)}). \tag{4}$$

This is analogous to the familiar combinatorial representation of skew Schur functions given by

$$s_{\lambda/\mu}(X) = \sum_{T \in \mathcal{T}_{\lambda/\mu}(\mathbb{Z})} \prod_{\alpha \in \lambda/\mu} x_{T(\alpha)} \tag{5}$$

Indeed it follows immediately that

$$G_{\lambda/\mu}(X,0) = s_{\lambda/\mu}(X) \tag{6}$$

and using the content-modifying bijection described above,

$$G_{\lambda/\mu}(0,Y) = s_{\lambda'/\mu'}(Y) \tag{7}$$

$$G_{\lambda/\mu}(X,Y) = G_{\lambda'/\mu'}(Y,X). \tag{8}$$

Properties (6), (7), and (8) of $G_{\lambda/\mu}$ are all also true for supersymmetric skew Schur functions, as can be easily seen from (1) and (2). In fact our principal result is the following:

THEOREM 1.1. *For any partitions λ and μ,*

$$G_{\lambda/\mu}(X,Y) = s_{\lambda/\mu}(X/Y)$$

Among other consequences, it follows that $G_{\lambda/\mu}(X,Y)$ is symmetric in X and Y separately, though this is not at all obvious from (4). We give a direct proof of this fact in Section 3.

Much of our present work arose from attempts to understand the symmetry of a closely related family of non-homogeneous polynomials introduced by Biedenharn and Louck [5, 6], and studied further in [9] (where they are called "factorial (skew) Schur functions"), and also [12, 16]. These have the tableau representation

$$t_{\lambda/\mu}(X_m) = \sum_{T \in \mathcal{T}_{\lambda/\mu}([\mathbf{m}])} \prod_{\alpha \in \lambda/\mu} \left(x_{T(\alpha)} - T(\alpha) - C(\alpha) + 1\right) \tag{9}$$

where $[\mathbf{m}] = \{1, 2, \ldots, m\}$. It is shown in [5, 6, 9, 12, 16] that these functions have many properties analogous to the Schur functions, and are themselves symmetric in X_m, although the latter is not an obvious consequence of (9). Unlike Schur functions, it is essential that factorial Schur functions involve a finite number of variables because otherwise unbounded sums arise in (9). Macdonald observed (see [9, 16]) that

$$t_\lambda(X_m) = \frac{\det\big((x_i)_{\lambda_j+m-j}\big)_{m\times m}}{\det\big((x_i)_{m-j}\big)_{m\times m}},$$

where $(x)_k$ denotes the falling factorial $x(x-1)\dots(x-k+1)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$. This is analogous to the classical bi-alternant definition of the Schur functions, and makes the symmetry of $t_{\lambda/\mu}(X_m)$ obvious when $\mu = \varnothing$.

Note that $G_{\lambda/\mu}(X, Y)$ as defined in (4) does not specialize directly to the factorial symmetric functions. In order to permit this, we extend the definition slightly, as follows: for a subset $A \subseteq \mathbb{Z}$, define

$$G^A_{\lambda/\mu}(X,Y) = \sum_{T\in\mathcal{T}_{\lambda/\mu}(A)} \prod_{\alpha\in\lambda/\mu} \left(x_{T(\alpha)} + y_{T(\alpha)+C(\alpha)}\right). \tag{10}$$

Then clearly

$$t_{\lambda/\mu}(X_m) = G^{[\mathbf{m}]}_{\lambda/\mu}(X, Y^*), \tag{11}$$

where here Y^* denotes the result of replacing each y_k by $-k+1$. Thus one may view the polynomials

$$G^{[\mathbf{m}]}_{\lambda/\mu}(X,Y)$$

as two-variable generalizations of factorial Schur functions.[1] These polynomials are symmetric in X_m but not in Y, for finite m, even when $\mu = \varnothing$; the range of y indices is determined by "flag conditions" on the rows of λ/μ.

For similar reasons, the restriction of Theorem 1.1 to sets of variables indexed by positive integers alone must be handled with care, since

$$G^{\mathbb{Z}^+}_{\lambda/\mu}(X,Y) \tag{12}$$

and

$$G_{\lambda/\mu}(X^+, Y^+) \tag{13}$$

[1]As this work developed, we learned that polynomials equivalent to $G^{[\mathbf{m}]}_{\lambda/\mu}(X,Y)$ have been introduced and studied independently by Macdonald ([16, formula (6.16)]), though his point of view is quite different from ours.

are not the same. Here $X^+ = (\ldots, 0, x_1, x_2, \ldots)$ and $Y^+ = (\ldots, 0, y_1, y_2, \ldots)$. For example when $\lambda/\mu = (3,3)/(2,0)$ the tableau

		−1
1	2	4

contributes nothing to (12) since it is not in $\mathscr{T}_{\lambda/\mu}(\mathbb{Z}^+)$, but it contributes

$$y_1 x_1 (x_2 + y_2)(x_4 + y_5)$$

to (13). To obtain a specialization of Theorem 1.1 to countably infinite sets of variables indexed by positive integers one must use (13) rather than (12), and the correct specialization to finite sets of variables (giving the functions defined in (1)) is

$$G_{\lambda/\mu}(X_m^+, Y_n^+),$$

where $X_m^+ = (\ldots, 0, x_1, \ldots, x_m, 0, \ldots)$ and $Y_n^+ = (\ldots, 0, y_1, \ldots, y_n, 0, \ldots)$. Clearly the polynomials $G_{\lambda/\mu}(X_m^+, Y_n^+)$ are symmetric in both X_m and Y_n.

For ordinary skew Schur functions $s_{\lambda/\mu}(X)$ in a single set of variables, the distinction between (12) and (13) is unnecessary. Also, with two sets of variables X and Y, expressions (12) and (13) coincide if $\mu = \varnothing$, i.e., λ/μ is a standard (non-skew) shape. This follows from the fact that cell (1, 1) contains the smallest entry in T and has content zero.

This paper is organized as follows: in Section 2 we obtain a Jacobi–Trudi determinant expansion for $G_{\lambda/\mu}(X, Y)$, which leads via (2) to a proof of Theorem 1.1. In Section 3 we prove that the symmetry of supersymmetric skew Schur functions in X and Y separately follows directly from our tableau representation (4), by a switching argument analogous to the Bender–Knuth proof of symmetry for Schur functions [2]. This symmetry argument is then extended to give a direct bijective proof that (1) follows from (4), thus providing a second proof of Theorem 1.1.

In Section 4 we consider the special case $\mu = \varnothing$, and explore some implications of the "cancellation property" ([17, 22], and see also [18, 19]) which follows immediately from the tableau representation. Finally, we give a simple proof of a theorem due to Berele and Regev [4], which states that

$$s_\lambda(X_m/Y_n) = \left\{ \prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j) \right\} s_\eta(X_m) s_{\beta'}(Y_n) \qquad (14)$$

where $\lambda = (n + \eta_1, n + \eta_2, \ldots, n + \eta_m, \beta_1, \beta_2, \ldots, \beta_k)$, $n \geq \beta_1$, and $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ are partitions.

2. A Jacobi–Trudi Determinant For $G_{\lambda/\mu}$

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots u_m)$ be partitions with m parts (possibly some are zero). The Jacobi–Trudi formula for ordinary Schur functions states that

$$s_{\lambda/\mu}(X) = \det\left(h_{\lambda_i - u_j - i + j}(X)\right)_{m \times m}, \tag{15}$$

where $h_n(X)$ is the complete homogenous symmetric function and, by definition, $h_k(X) = 0$ if $k < 0$. In this section we will prove an analogous formula for $G_{\lambda/\mu}(X, Y)$. The argument is a straightforward extension of the method used by Gessel and Viennot [11] to prove (15), and we will merely sketch the details.

Let $P_i = (\mu_i - i + 1, -\infty)$ and $Q_i = (\lambda_i - i + 1, +\infty)$, $i = 1, \ldots, m$. A tableau T in $\mathcal{T}_{\lambda/\mu}(\mathbb{Z})$ can be uniquely represented as an m-tuple of nonintersecting lattice paths with horizontal (increase abscissa by 1) and vertical (increase ordinate by 1) steps where the ith path is from P_i to Q_i and has horizontal steps at ordinates specified by the entries in the ith row of T. This is bijective, and if the weight of a horizontal step from (i, j) to $(i + 1, j)$ is given by $x_j + y_{i+j}$ and the weight of a vertical step is 1, then the generating function for all such m-tuples is precisely $G_{\lambda/\mu}(X, Y)$.

The Gessel–Viennot analysis easily shows that

$$G_{\lambda/\mu}(X, Y) = \det\left(g(P_j, Q_i)\right)_{m \times m}, \tag{16}$$

where $g(P_j, Q_i)$ is the generating function for paths from P_j to Q_i with the given weight. It remains to determine $g(P_j, Q_i)$, or equivalently, to compute $G_\lambda(X, Y)$ when λ consists of a single row. Now clearly

$$g(P_j, Q_i) = S^{\mu_j - j + 1} H_{\lambda_i - u_j - i + j}(X, Y), \tag{17}$$

where

$$H_n(X, Y) = \sum_{d_1 \leq \cdots \leq d_n} \prod_{i=1}^{n} (x_{d_i} + y_{d_i + i - 1}) \tag{18}$$

and S is a shift operator acting on functions $f(X, Y)$ by replacing each y_i by y_{i+1}.

LEMMA 2.1.

$$H_n(X,Y) = \sum_{0 \le k \le n} h_k(X) e_{n-k}(Y) \tag{19}$$

where $h_k(X)$ and $e_{n-k}(Y)$ denote the complete homogeneous and elementary symmetric functions, respectively.

Proof. Terms in the expansion of (18) are monomials of the form

$$x_{a_1} x_{a_2} \cdots x_{a_k} y_{b_1} y_{b_2} \cdots y_{b_{n-k}},$$

where $a_1 \le a_2 \le \cdots \le a_k$ and $b_1 < b_2 < \cdots < b_{n-k}$. To prove (19) it suffices to show that each such monomial appears exactly once, i.e. given such $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_{n-k}\}$ there exists a unique sequence $d_1 \le d_2 \le \cdots \le d_n$ and subset of indices $I \subseteq [\mathbf{n}]$ of size $n-k$ such that $\{d_i | i \notin I\} = \{a_1, \dots, a_k\}$ and $\{d_i + i - 1 | i \in I\} = \{b_1, \dots, b_{n-k}\}$. This may be proved by a "merging" algorithm which constructs both $\{d_i\}$ and I as follows. Initially let $I = \varnothing$ and $i = j = l = 1$. Repeat the following steps until $i = n$.

- If $a_l < b_j - i + 1$, set $d_i = a_k$, $i = i + 1$, and $l = l + 1$.
- If $a_l \ge b_j - i + 1$, set $d_i = b_j - i + 1$, $I = I \cup \{i\}$, $i = i + 1$, and $j = j + 1$.

It is easy to see that this terminates with a sequence $\{d_i\}$ and subset I having the desired properties. ∎

Another way to visualize the merging process in the above argument is as follows: make a list of length n consisting of the b's followed by the a's, with the b's "circled." First replace each b_j by $b_j - j + 1$. Then, as long as there exists a b followed immediately by a smaller a, that is, an adjacent pair (ⓑ, a) with $b > a$, exchange a and b, and subtract one from b. For example, the list

①	⑤	⑨	−1	0	2	2	4

leads first to

①	④	⑦	−1	0	2	2	4

and eventually to

−1	⓪	0	②	2	2	③	4

It is clear that this results in the same sequence $\{d_i\}$ produced by the merging algorithm, and that the set I consists of the positions containing the circled elements. Uniqueness follows from the fact that all of the steps are reversible. A two-dimensional version of this procedure (related to Schützenberger's jeu de Taquin) plays an important role in Section 3 of this paper.

Lemma 2.1 shows that $H_n(X,Y)$ is symmetric in Y and hence invariant under the shift operator S. Thus we may write

$$g(P_j, Q_i) = H_{\lambda_i - \mu_i - i + j}(X, Y) \tag{20}$$

and Lemma 2.1 combined with (16) yield the desired Jacobi-Trudi determinant expansion for $G_{\lambda/\mu}$:

COROLLARY 2.2.

$$G_{\lambda/\mu}(X,Y) = \det\left(H_{\lambda_i - \mu_i - i + j}(X,Y)\right)_{m \times m}$$

Now

$$h_n(X \cup Y) = \sum_{0 \le k \le n} h_k(X) h_{n-k}(Y)$$

and since ω_Y exchanges elementary and complete symmetric functions, Lemma 2.1 gives

$$H_n(X,Y) = \omega_Y h_n(X \cup Y)$$

Thus comparison of the two Jacobi-Trudi determinants gives $G_{\lambda/\mu}(X,Y) = \omega_Y s_{\lambda/\mu}(X \cup Y)$, and we have proved Theorem 1.1.

A straightforward extension of this argument gives a Jacobi–Trudi form for the factorial Schur functions (see also [9, 12, 16]). To see this, first note that the proof of Lemma 2.1 also yields the identity

$$\sum_{1 \le d_1 \le \cdots \le d_n \le m} \prod_{i=1}^{n} (x_{d_i} + y_{d_i + i - 1}) = \sum_{0 \le k \le n} h_k(X_m) e_{n-k}(Y_{m+n-1}) \tag{21}$$

Then a modification of the Gessel-Viennot argument for (16) gives

COROLLARY 2.3.

$$G^{[m]}_{\lambda/\mu}(X,Y) = \det\left(S^{\mu_i - j + 1} H_{\lambda_i - \mu_i - i + j}(X_m^+, Y_{m+n-1}^+)\right)_{m \times m}$$

Specializing as in (11) gives the required determinantal expression for factorial Schur functions.

3. Symmetry of $G_{\lambda/\mu}(X,Y)$ via Circled Tableaux

In this section (as well as the next) we consider $G_{\lambda/\mu}(X,Y)$ in light of Theorem 1.1, and show that several important properties of supersymmetric Schur functions can be obtained directly from our tableau definition. First we will be concerned with symmetry. It is convenient to represent the right hand side of formula (4) as a weighted sum over a new class of objects, called *circled tableaux*.[2] These are just ordinary column strict tableaux with some subset of the entries circled. The *weight* of a circled tableau T is defined to be

$$w(T) = w_X(T)w_Y(T),$$

where

$$w_X(T) = \prod_{\alpha \text{ uncircled}} x_{T(\alpha)}$$

$$w_Y(T) = \prod_{\beta \text{ circled}} y_{T(\beta)+C(\beta)}$$

Thus if $\mathcal{T}^{\odot}_{\lambda/\mu}(\mathbb{Z})$ denotes the set of all circled tableaux of shape λ/μ, with entries in $\mathbb{Z}$, we can rewrite (4) as

$$G_{\lambda/\mu}(X,Y) = \sum_{T\in\mathcal{T}^{\odot}_{\lambda/\mu}(\mathbb{Z})} w(T). \tag{22}$$

We now use this representation of $G_{\lambda/\mu}(X,Y)$ to show directly that it is symmetric in X and Y.

Theorem 3.1. *$G_{\lambda/\mu}(X,Y)$ is symmetric in X and Y.*

Proof. By (8) it suffices to show that $G_{\lambda/\mu}(X,Y)$ is invariant under transposition of x_i and x_{i+1}, for all i. We prove this by defining an involution on tableaux $T\in\mathcal{T}^{\odot}_{\lambda/\mu}(\mathbb{Z})$ which exchanges x_i and x_{i+1} in $w_X(T)$ and leaves $w_Y(T)$ invariant.

In such a T the entries i and $i+1$ appear in a skew subtableau with at most two cells in each column. If entries i and $i+1$ are in the same column (whether circled or not) we say that they are *paired*; all other

[2]One could also imagine calling these objects "supertableaux"; however, the term seems already to appear in the literature (e.g., [10]), with a different meaning.

occurrences of i and $i+1$ are *unpaired*. The involution on $\mathcal{T}^{\odot}_{\lambda/\mu}(\mathbb{Z})$ is defined as follows:

• If i and $i+1$ are paired, and exactly one of them is circled, transfer the circle to the other, leaving i and $i+1$ fixed.

• If i and $i+1$ are paired, and both are circled or both uncircled, do nothing.

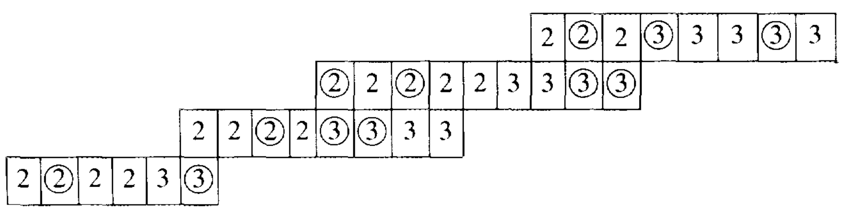
• In each row of T perform the following operations on the remaining unpaired occurrences of i and $i+1$.

(i) Replace each uncircled i by $i+1$ and each uncircled $i+1$ by i.

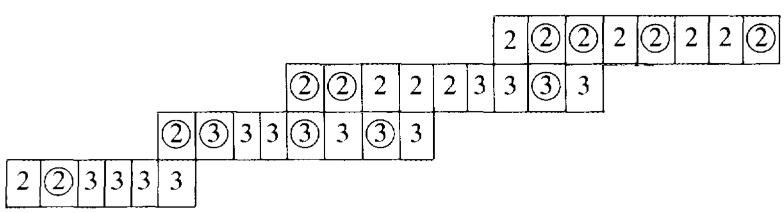
(ii) Sort those (uncircled) entries into increasing order, in place, leaving the other entries fixed.

(iii) If any adjacent pairs $i+1 > i$ remain (circled or not), transpose those pairs, increasing a circled entry by one if it moves to the left and decreasing it by one if it moves to the right. Repeat until no such pairs remain.

For example, suppose that $i = 2$ and the entries 2 and 3 appear in a subtableau as shown:



Then the resulting subtableau is:



Clearly each step is reversible, and the operations define an involution with the stated properties. The symmetry of $G_{\lambda/\mu}(X, Y)$ follows immediately. ∎

It is worth noting that when T contains no circled elements, the involution given in the above proof coincides exactly with the one used by Bender and Knuth [2] to prove symmetry of the ordinary Schur functions.

Next we show how to obtain formula (1) for supersymmetric Schur functions directly from the tableau definition, as an immediate consequence of the symmetry proof. One may regard this as an alternate proof of Theorem 1.1.

THEOREM 3.2. *For any partitions λ and μ with $\mu \subseteq \lambda$,*

$$G_{\lambda/\mu}(X,Y) = \sum_{\mu \subseteq \nu \subseteq \lambda} s_{\nu/\mu}(X) s_{\lambda'/\nu'}(Y)$$

Proof. From (6), (7), and (22) it suffices to construct a bijection between

- circled tableaux $T \in \mathscr{T}^{\odot}_{\lambda/\mu}(\mathbb{Z})$, and
- pairs of circled tableaux $T_1 \in \mathscr{T}^{\odot}_{\nu/\mu}(\mathbb{Z})$, $T_2 \in \mathscr{T}^{\odot}_{\lambda/\nu}(\mathbb{Z})$ for some partition ν, such that every entry of T_1 is uncircled and every entry of T_2 is circled, such that

$$w_X(T) = w_X(T_1)$$
$$w_Y(T) = w_Y(T_2).$$

The bijection is constructed as follows. Start with $T_1 \in \mathscr{T}^{\odot}_{\nu/\mu}(\mathbb{Z})$, with every entry uncircled, and $T_2 \in \mathscr{T}^{\odot}_{\lambda/\nu}(\mathbb{Z})$, with every entry circled. Replace each $k \in T_1$ by $\bar{k} = k - N$ where N is defined as

$$\max\{i \in T_1\} - \min\{j \in T_2^C\} + \max\{C(\alpha) | \alpha \in \lambda/\mu\} + 1$$

(Any canonically chosen $N' > N$ would also suffice.) Let the resulting tableau be denoted by T_1^*. Now every entry of T_1^* is less than every entry of T_2, and hence combining T_1^* and T_2 gives a tableau $T^* \in \mathscr{T}^{\odot}_{\lambda/\mu}(\mathbb{Z})$.

Now we invoke symmetry and the explicit bijection used in the proof of Theorem 3.1 to successively exchange each $\bar{k} \in T^*$ with k, beginning with the largest, i.e., to restore the original uncircled values. Let the resulting tableau in $\mathscr{T}^{\odot}_{\lambda/\mu}(\mathbb{Z})$ be denoted by T^{**}. The circled entries change in the process, but their content-modified weight does not; hence the map $(T_1, T_2) \to T^* \to T^{**}$ is weight-preserving, and it is clear that every step can be reversed. This completes the proof. ∎

As an example of the bijection given above, suppose that T_1 and T_2 are such that T^* has the form

$\bar{1}$	$\bar{1}$	$\bar{2}$	⓪	①
$\bar{3}$	$\bar{3}$	②	②	
③	④			

Then successively exchanging $\bar{3} \to 3$, $\bar{2} \to 2$, $\bar{1} \to 1$, gives the sequence of circled tableaux

$\bar{1}$	$\bar{1}$	$\bar{2}$	⓪	①
②	③	③	3	
3	④			

$\bar{1}$	$\bar{1}$	①	②	2
②	③	③	3	
3	④			

1	1	①	②	2
②	③	③	3	
3	④			

The final tableau is T^{**}. Note that in the last step no rearrangement of the unbarred entries is necessary.

The bijection described above can be reformulated so that its steps are similar to those of Schützenberger's jeu de Taquin (see [21]). The algorithm as presented above transforms all occurrences of $\bar{k}$'s to k's "in parallel," using modified Bender–Knuth operations. Instead one can perform individual transformations $\bar{k} \to k$ sequentially, applying the following procedure to the rightmost occurrence of the largest $\bar{k}$ until no barred elements remain.

Modified jeu de Taquin

1. Let the rightmost occurrence of the largest $\bar{k}$ be denoted by x (the "active" element).

2. Repeat the following until $x = k$:

(a) Set $x \leftarrow x + 1$.

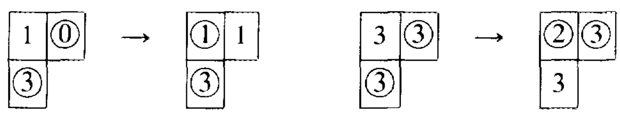
(b) If the resulting tableau is row-weak and column-strict, do nothing further.

(c) Otherwise exchange x with one of its circled neighbors y to the right or z below, chosen so that the 3-cell subtableau involving $\{x, y, z\}$ is row-weak and column-strict *after the following modification*:

• if a circled element moves to the left, its value is increased by one;

• if a circled element moves up, its value is decreased by one. Repeat the last operation until the tableau is row-weak and column-strict.

It is not difficult to see that exactly one choice is possible in the application of rule (c), and each successive iteration results in a tableau which is row-weak and column-strict. The following diagrams illustrate valid moves of this type.



It can be shown easily that sequential application of modified jeu de Taquin moves has the same effect as parallel application of modified Bender–Knuth moves, and hence the same T^{**} results by either process. Note that unlike the standard jeu de Taquin, our modified procedure is deterministic; indeed there are examples showing that different tableaux may result if the individual steps are not applied exactly as stated above.

4. Standard Shapes

In Sections 2 and 3 we have considered infinite sets of variables X and Y indexed by the integers, and proved that $G_{\lambda/\mu}(X, Y)$ is the supersymmetric Schur function in X and Y. In this section we consider standard shapes λ. As noted in the introduction, cell $(1, 1)$ must contain the smallest entry in any circled tableau T. As a consequence, if T is a circled tableau appearing in the expansion of (22), then negative indices appear on x and y in $w(T)$ if and only if negative entries appear in T.

These considerations make the theory somewhat simpler for standard shapes. In this section, we consider two results which hold when $\mu = \varnothing$. The first is the "cancellation property" of [17] and [22], already discussed in Section 1.

Theorem 4.1. *For λ any partition,*

$$G_\lambda(X^+, Y^+)\big|_{x_1 \to -y_1} = G_\lambda(X^+, Y^+)\big|_{\substack{x_1 \to 0 \\ y_1 \to 0}} \tag{23}$$

Proof. let $T \in \mathcal{T}_\lambda(\mathbb{Z})$ be a tableau contributing a non-zero term to expansion (4) of $G_\lambda(X^+, Y^+)$. Then $T(1,1) > 0$, by the above remarks. Furthermore, if x_1 or y_1 appears in $\prod_{\alpha \in \lambda}(x_{T(\alpha)} + y_{T(\alpha)+C(\alpha)})$ then $T(1,1) = 1$. Hence all monomials in $G_\lambda(X^+, Y^+)$ involving x_1 or y_1 occur in the expansion of

$$\sum_{\substack{T \in \mathcal{T}_\lambda(\mathbb{Z}) \\ T(1,1)=1}} \prod_{\alpha \in \lambda} (x_{T(\alpha)} + y_{T(\alpha)+C(\alpha)}). \tag{24}$$

This expression is divisible by $(x_1 + y_1)$ and hence equals zero when $x_1 = -y_1$. The theorem follows immediately. ▮

Remarks.

1. Richard Stanley has observed that for standard shapes λ, Theorem 1.1 can be deduced directly from Theorems 3.1 (symmetry) and Theorem 4.1 (cancellation), and from the fact that $G_\lambda(X,Y)$ specializes to $s_\lambda(X)$ and $s_{\lambda'}(Y)$ when $Y=0$ and $X=0$, as in (6), (7). This is a consequence of Stembridge's characterization of supersymmetric functions [22] (see also [23]). Thus we have a third proof of Theorem 1.1, for standard shapes.

2. We have not been able to find a simple direct proof of the more general cancellation property

$$G_\lambda(X^+, Y^+)\big|_{x_i \to -y_j} = G_\lambda(X^+, Y^+)\big|_{\substack{x_i \to 0 \\ y_j \to 0}} \tag{25}$$

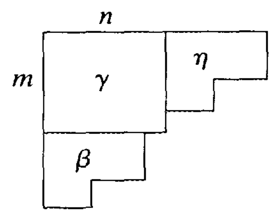
which of course can be deduced from (23) when $\mu = \varnothing$ by symmetry.

Finally we prove Berele and Regev's factorization theorem [4] for supersymmetric Schur functions in finite sets of variables.

THEOREM 4.2. *Let* $\lambda = (n+\eta_1, \ldots, n+\eta_m, \beta_1, \ldots, \beta_k)$, $n \geq \beta_1$, *where* $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ *are partitions. Then*

$$G_\lambda(X_m^+, Y_n^+) = \left\{\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j)\right\} s_\eta(X_m) s_{\beta'}(Y_n) \tag{26}$$

Proof. The diagram of λ consists of an $m \times n$ rectangle γ, with the diagram of partition η to the right and the diagram of partition β below, as illustrated:



Consider a tableau T of shape λ contributing a nonzero term to expansion (4). As in the previous argument, if $T(1,1) < 1$ then $T^C(1,1) < 1$, and if $T(m,n) > m$ then $T^C(m,n) > n$, so for T to provide a non-zero contribution to $G_\lambda(X_m^+, Y_n^+)$, we must have $T(1,1) \geq 1$ and $T(m,n) \leq n$. This forces T to consist entirely of i's in the ith row of γ, and T^C to consist entirely of j's in the jth column of γ. Hence for each such T the contribution of cells in γ to $\prod_{\alpha \in \lambda}(x_{T(\alpha)} + y_{T(\alpha)+C(\alpha)})$ is

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (x_i + y_j)$$

Now the content-modified elements of T in η all exceed n, so there is no y contribution from the elements of T in η. In fact the portion of T in η is an arbitrary column-strict tableau of shape η with elements $1, \ldots, m$ (larger elements would give a zero contribution). Summing over all such T contributes a factor $s_\eta(X_m)$ to (26).

The elements of T in β all exceed m, so there is no x contribution from the elements of T in β. In fact the portion of T^C in β is an arbitrary row-strict tableau of shape β' with elements $1, \ldots, n$ (larger elements would give a zero contribution), contributing $s_{\beta'}(Y_n)$ to (26). Combining the contributions from η, β and γ proves the theorem. ∎

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