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Maintaining the spirit of the reflection principle when the boundary has arbitrary integer slope

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Abstract

We provide a direct geometric bijection for the number of lattice paths that never go below the line $y = kx$ for a positive integer k. This solution to the Generalized Ballot Problem is in the spirit of the reflection principle for the Ballot Problem (the case $k = 1$), but it uses rotation instead of reflection. It also gives bijective proofs of the refinements of the Generalized Ballot Problem which consider a fixed number of right-up or up-right corners.

Keywords: Lattice path; Ballot problem; Combinatorial bijection

1. The classical Ballot Problem

A lattice path is a path in the plane consisting of unit up-steps and right-steps, whose ends are points with integer coordinates. The classical *Ballot Problem* was given in [3]:

Theorem 1. *For $n \geqslant m \geqslant 0$, the number of lattice paths from $(0,0)$ to (m,n) that never go below the diagonal $y = x$ is*

$$\frac{n-m+1}{n+1}\binom{m+n}{m}.$$

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The solution to the Ballot Problem in the special case $m = n$ is the well-known Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Perhaps, the best-known solution to the classical Ballot Problem was given by André [1]. André counted the paths that go below the diagonal somewhere (called *bad* paths in this context), motivated by the facts that the solution to the Ballot Problem can be reexpressed in the form

$$\frac{n-m+1}{n+1}\binom{m+n}{m} = \binom{m+n}{m} - \binom{m+n}{m-1}, \tag{1}$$

and that $\binom{m+n}{m}$ is the number of *all* paths from $(0,0)$ to (m,n). André gave a direct geometric bijection between the subset of bad paths and the set $\mathscr{A}$ of *all* paths from $(1,-1)$ to (m,n), and the result then follows immediately, since $|\mathscr{A}| = \binom{m+n}{m-1}$. In the bijection, the initial portion of the path up to the first point that lies on the line $y = x - 1$ is reflected about the line $y = x - 1$, and so André's beautiful method of proof is called the *reflection principle*.

2. The Generalized Ballot Problem

Barbier [2] generalized the classical Ballot Problem by introducing a positive integer parameter k of slope:

Theorem 2. *For $k \geqslant 1$, and $n \geqslant km \geqslant 0$, the number of lattice paths from $(0,0)$ to (m,n) that never go below the line $y = kx$ is*

$$\frac{n-km+1}{n+1}\binom{m+n}{m}.$$

In some sources, a closely related problem is considered:

Theorem 3. *For $k \geqslant 1$, and $n > km \geqslant 0$, the number of lattice paths from $(0,0)$ to (m,n) that never touch the line $y = kx$ after the point $(0,0)$ is*

$$\frac{n-km}{n}\binom{m+n-1}{m}.$$

Theorems 3 and 2 are equivalent, as follows: The paths in Theorem 3 from $(0,0)$ to (m,n) must clearly start with an up-step. If this initial up-step is removed, and the remaining portion of the path is translated vertically down by 1, then we obtain a path in Theorem 2 from $(0,0)$ to $(m,n-1)$. This is reversible (by adding an initial up-step to the paths in Theorem 2), excluding the trivial case that $(m,n) = (0,0)$. We conclude that the number of paths in these two cases are equal, and indeed Theorems 2 and 3 above both give this number as $\frac{n-km}{n}\binom{m+n-1}{m}$. Henceforth, we will consider only Theorem 2, and we will call this the *Generalized Ballot Problem*.

Note that the solution to the Generalized Ballot Problem can be reexpressed in the form

$$\frac{n-km+1}{n+1}\binom{m+n}{m} = \binom{m+n}{m} - k\binom{m+n}{m-1}, \tag{2}$$

which is a straightforward generalization of (1). Now, for $k \geqslant 1$, and $n \geqslant km \geqslant 0$, define $\mathscr{B}$ to be the set of paths from $(0,0)$ to (m,n) that go below the line $y = kx$ somewhere (called *bad* paths in this context). Then, in the spirit of the reflection principle for the classical Ballot Problem, one method of solution to the Generalized Ballot Problem would be to find a direct geometric bijection between $\mathscr{B}$ and k disjoint copies of $\mathscr{A}$, defined in Section 1.

There are various published solutions for the Generalized Ballot Problem (see, e.g., [4], [7, p. 8]; [8], [9, p. 10]; and [10, p. 2]). However, there appears to be no solution which is in the spirit of the reflection principle; in Section 3 of this paper, we describe a direct geometric bijection for bad paths that gives such a solution for the Generalized Ballot Problem. It replaces reflection of a portion of the path by *rotation*, the rigid geometric transformation that would seem most natural when the boundary has slope k, not necessarily equal to 1. (Rotation has of course been featured in other lattice path bijections, see e.g., [6].) In Section 4, we demonstrate that restricting this bijection to paths with a given number of right-up or up-right corners also gives a proof for the corresponding refinements of the Generalized Ballot Problem.

3. A bijection for bad paths

First, we decompose the set $\mathscr{B}$ of bad paths into k disjoint subsets $\mathscr{B}_1, \ldots, \mathscr{B}_k$. For a path in $\mathscr{B}$, find the first right-step whose right end lies below the line $y = kx$ (this clearly happens by the definition of $\mathscr{B}$). Consider the portion of this right-step which is below the diagonal. Its length must be one of the values $\frac{1}{k}, \frac{2}{k}, \ldots, \frac{k}{k}$. For $i = 1, 2, \ldots, k$ define $\mathscr{B}_i$ to be the set of paths in $\mathscr{B}$ which have $\frac{i}{k}$ as this length. Clearly, the $\mathscr{B}_i$ are disjoint, and their union is $\mathscr{B}$, so

$$|\mathscr{B}| = |\mathscr{B}_1| + \cdots + |\mathscr{B}_k|. \tag{3}$$

Next, we will describe a mapping ϕ_i on $\mathscr{B}_i$ for each $i = 1, \ldots, k$. For a path π in $\mathscr{B}_i$, find the first right-step whose right end (a,b) lies below the line $y = kx$. Then the left end of this right-step is $(a-1,b)$. Let p be the portion of the path from $(0,0)$ to $(a-1,b)$. Now rotate p by 180° to interchange the endpoints $(0,0)$ and $(a-1,b)$, and translate the resulting path vertically down by 1, and horizontally right by 1, to obtain a path p' from $(1,-1)$ to $(a,b-1)$. (Equivalently, the steps of p' are the steps of p in reverse left-to-right order.) Then $\phi_i(\pi) = \pi'$, where π' is obtained by using the path p' from $(1,-1)$ to $(a,b-1)$, followed by an *up-step* to (a,b), and then using the portion of π from (a,b) to (m,n). An example of this mapping is illustrated in Fig. 1.

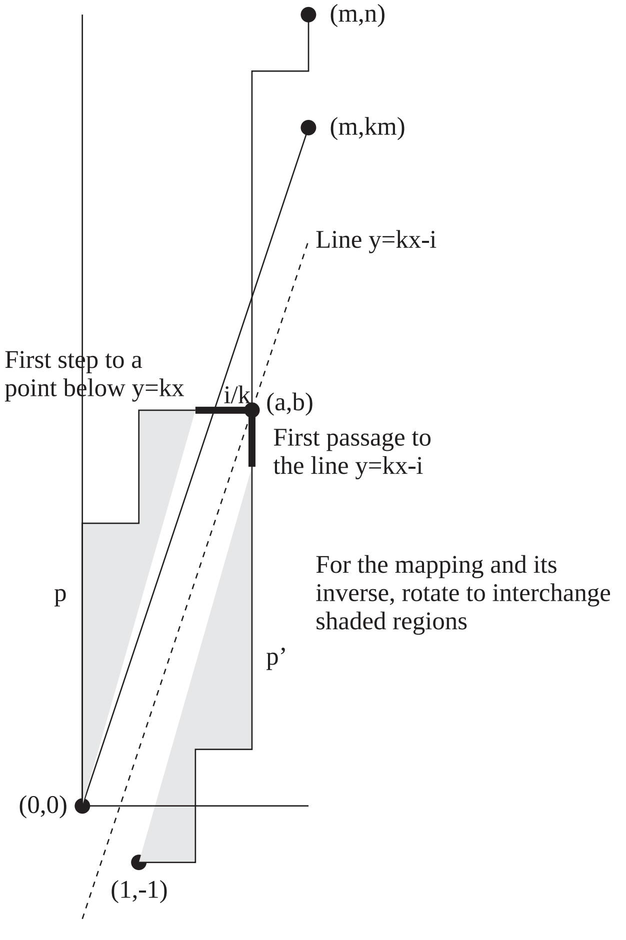

Fig. 1. An example of the mapping ϕ_i.

Clearly, in general π' is contained in $\mathscr{A}$. In the next result we show that ϕ_i is in fact a *bijection* between $\mathscr{B}_i$ and $\mathscr{A}$. This gives immediately that $|\mathscr{B}_i| = \binom{m+n}{m-1}$, independently of i, and from (3) we have our geometric bijection for the Generalized Ballot Problem.

Theorem 4. *For each* $i = 1, \ldots, k$,

$$\phi_i : \mathscr{B}_i \rightarrow \mathscr{A} : \pi \mapsto \pi'$$

is a bijection.

Proof. From the description of ϕ_i given above, the path π' begins at the point $(1, -1)$, which lies on the line $y = kx - k - 1$, and ends at the point (m, n), which lies on or above the line $y = kx$. The point (a, b) lies on the line $y = kx - i$, since $(a - \frac{i}{k}, b)$ lies on the line $y = kx$, so the point $(a, b - 1)$ lies on the line $y = kx - i - 1$. Also, under the rotation and translation that sends p to p', the initial point $(0, 0)$ of p is sent to the terminal point $(a, b - 1)$ of p'. Therefore the line $y = kx$ of slope k through $(0, 0)$ is mapped to the line $y = kx - i - 1$ of slope k through $(a, b - 1)$. But p never goes below the line $y = kx$, so p' never goes above the line $y = kx - i - 1$, and we conclude that (a, b) is the first point of π' on the line $y = kx - i$.

Now, every path in $\mathscr{A}$ must pass through at least one point on the line $y = kx - j$, and the first such point must be immediately preceded by an up-step, for each $j = k, \ldots, 0$, so we conclude that the mapping ϕ_i is uniquely reversible, since the point (a, b) is uniquely determined, given i. $\square$

Note that Theorem 4 does *not* specialize to the reflection principle in the case $k = 1$, although it is close. For the reflection principle, in the notation above, we would obtain p' from p by reflection about the line $y = x$, before translating the resulting path vertically down by 1, and horizontally right by 1. (So a further reflection about a line with slope -1 is needed to agree with our rotation.) Of course, reflecting a lattice path about a line with slope $k \neq 1$ will not give a lattice path, so it is perhaps natural to expect that a geometric bijection for the Generalized Ballot Problem should involve rotation rather than reflection.

4. Restricting the bijection by number of corners

There are a number of refinements of results on counting lattice paths, by considering the number of right-up or up-right corners (see [5] for a comprehensive survey).

4.1. Right-up corners

For right-up corners, the following result is given as Theorem 3.4.2 in [5]:

Theorem 5. *For $k \geqslant 1$, and $n \geqslant km \geqslant 0$, the number of lattice paths from $(0, 0)$ to (m, n) with c right-up corners, that never go below the line $y = kx$ is*

$$\binom{m}{c}\binom{n}{c} - k\binom{m+1}{c+1}\binom{n-1}{c-1}.$$

Note that this is a refinement of the Generalized Ballot Problem, whose solution can be obtained by summing the above result over c. The bijection that we have given in Theorem 4 also gives a bijective proof of Theorem 5, as described below.

Let $\mathscr{C}_c$ be the set of paths from $(0, 0)$ to (m, n) with c right-up corners, for $c = 0, 1, \ldots$. (A *right-up corner* is a point where a right-step meets an immediately subsequent up-step.) Clearly, the number of paths in $\mathscr{C}_c$ is $\binom{m}{c}\binom{n}{c}$, since the right-up corners in such a path occur precisely at the points $(X_1, Y_1), \dots, (X_c, Y_c)$, where $1 \leqslant X_1 < \cdots < X_c \leqslant m$, and $0 \leqslant Y_1 < \cdots < Y_c \leqslant n-1$. The number of bad paths with c right-up corners is obtained bijectively in the following result. In the proof, we apply Theorem 4, and use the same notation.

Corollary 6. *For each $i = 1, \dots, k$, we have*

$$|\mathscr{B}_i \cap \mathscr{C}_c| = \binom{m+1}{c+1}\binom{n-1}{c-1}.$$

Proof. For each $i = 1, \dots, k$, we describe a bijection between $\mathscr{B}_i \cap \mathscr{C}_c$ and the set of X's and Y's satisfying $1 \leqslant X_1 < \cdots < X_{c+1} \leqslant m+1$ and $0 \leqslant Y_1 < \cdots < Y_{c-1} \leqslant n-2$. For $\pi \in \mathscr{B}_i \cap \mathscr{C}_c$, let $\pi' = \phi_i(\pi)$. Given such a set of X's and Y's, let j be the minimum positive integer such that $kX_j - i \leqslant Y_j$ (we use the convention that $Y_c = n-1$, so $kX_c - i \leqslant km - i \leqslant km - 1 \leqslant n-1 = Y_c$, and it is thus always possible to find such a j, at most c). Then it is routine to verify that there is a unique π' for which $(a,b) = (X_j, kX_j - i)$, with the images of right-up corners of π given by $(X_1, Y_1), \dots, (X_{j-1}, Y_{j-1})$ and $(X_{j+1} - 1, kX_j - i), (X_{j+2} - 1, Y_j + 1), \dots, (X_{c+1} - 1, Y_{c-1} + 1)$. (The first $j-1$ of these points are up-right corners in the portion of π' (strictly) before (a,b), and the last $c-j$ of these points are right-up corners in the portion of π' (strictly) after (a,b); the remaining point, $(X_{j+1} - 1, kX_j - i)$, is a right-up corner in π' only if $X_{j+1} - 1 > X_j$, otherwise, it is internal to the vertical segment.) Moreover, π has precisely c right-up corners, given by $(X_j - X_{j-1}, kX_j - i - Y_{j-1} - 1), \dots, (X_j - X_1, kX_j - i - Y_1 - 1)$ and $(X_{j+1} - 1, kX_j - i), (X_{j+2} - 1, Y_j + 1), \dots, (X_{c+1} - 1, Y_{c-1} + 1)$.

The result follows, since the number of such X's and Y's is $\binom{m+1}{c+1}\binom{n-1}{c-1}$. $\square$

The bijective proof of Theorem 5 is now completed, in which the bad paths have been shown to be equally distributed as subsets of the k sets $\mathscr{B}_1, \dots, \mathscr{B}_k$. Note the simple role that the parameter k plays in this proof—the factor k in the subtracted quantity uniquely identifies for which $i = 1, \dots, k$ the bad path first goes below the line $y = kx$ on the line $y = kx - i$. This "purely" bijective proof answers a question raised by Krattenthaler [5, Remark 3.4.2].

4.2. Up-right corners

For up-right corners, the following result is given as Theorem 3.4.3 in [5]:

Theorem 7. *For $k \geqslant 1$, and $n \geqslant km \geqslant 0$, the number of lattice paths from $(0,0)$ to (m,n) with c up-right corners, that never go below the line $y = kx$ is*

$$\binom{m-1}{c-1}\binom{n+1}{c} - k\binom{m}{c}\binom{n}{c-1}.$$

This result is also a refinement of the Generalized Ballot Problem, and the bijection that we have given in Theorem 4 again gives a bijective proof. However, there is a small technical difference from the case of right-up corners, and so we include the details below.

Let $\mathscr{C}^{(c)}$ be the set of paths from $(0,0)$ to (m,n) with c up-right corners, for $c = 0, 1, \ldots$. An *up-right corner* is a point where an up-step meets an immediately subsequent right-step, but in addition, we shall also include a "virtual" up-right corner—the left-most point on an initial right-step. The inclusion of these virtual corners makes no difference in the context of Theorem 7, since any path with an initial right-step must necessarily go below the line $y = kx$, and so will be subtracted as a bad path. In particular, these paths will touch the line $y = kx - k$ after the first right-step, and so will be subtracted as part of the set $\mathscr{B}_k \cap \mathscr{C}^{(c)}$. Clearly, the number of paths in $\mathscr{C}^{(c)}$ is $\binom{m-1}{c-1}\binom{n+1}{c}$, since the up-right corners in such a path occur precisely at the points $(0, Y_1), (X_1, Y_2), \ldots, (X_{c-1}, Y_c)$, where $1 \leqslant X_1 < \cdots < X_{c-1} \leqslant m - 1$, and $0 \leqslant Y_1 < \cdots < Y_c \leqslant n$. (If $Y_1 = 0$, then $(0, Y_1)$ is a virtual up-right corner.) The number of bad paths with c up-right corners is obtained bijectively in the following result, in which we again apply Theorem 4, and use the same notation.

Corollary 8. *For each* $i = 1, \ldots, k$, *we have*

$$|\mathscr{B}_i \cap \mathscr{C}^{(c)}| = \binom{m}{c}\binom{n}{c-1}.$$

Proof. For each $i = 1, \ldots, k$, we describe a bijection between $\mathscr{B}_i \cap \mathscr{C}^{(c)}$ and the set of X's and Y's satisfying $1 \leqslant X_1 < \cdots < X_c \leqslant m$ and $0 \leqslant Y_1 < \cdots < Y_{c-1} \leqslant n-1$. For $\pi \in \mathscr{B}_i \cap \mathscr{C}^{(c)}$, let $\pi' = \phi_i(\pi)$. Given such a set of X's and Y's, let j be the minimum positive integer such that $kX_j - i \leqslant Y_j$ (we use the convention that $Y_c = n - 1$, so $kX_c - i \leqslant km - i \leqslant km - 1 \leqslant n - 1 = Y_c$, and it is thus always possible to find such a j, at most c). Then it is routine to verify that there is a unique π' for which $(a,b) = (X_j, kX_j - i)$, with the images of up-right corners of π given by $(X_1, -1), (X_2, Y_1), \ldots, (X_j, Y_{j-1})$, and $(X_{j+1} - 1, Y_j + 1), \ldots, (X_c - 1, Y_{c-1} + 1)$. (Of the first j of these points, the latter $j - 1$ are right-up corners in the portion of π' (strictly) before (a,b); the first point, $(X_1, -1)$, is a right-up corner of π' only if $X_1 > 1$. The last $c - j$ of these points are up-right corners in the portion of π' after (a,b).) Moreover, π has precisely c up-right corners, given by $(0, kX_j - i - Y_{j-1} - 1), \ldots, (X_j - X_2, kX_j - i - Y_1 - 1), (X_j - X_1, kX_j - i), (X_{j+1} - 1, Y_j + 1), \ldots, (X_c - 1, Y_{c-1} + 1)$.

The result follows, since the number of such X's and Y's is $\binom{m}{c}\binom{n}{c-1}$. $\square$

The bijective proof of Theorem 7 is now completed. Again, the bad paths have been shown to be equally distributed as subsets of the k sets $\mathscr{B}_1, \ldots, \mathscr{B}_k$. Note that in this case we need the additional "virtual" corners to achieve this equidistribution, since otherwise there would be fewer bad paths in $\mathscr{B}_k$ than in the others.

5. Another decomposition of bad paths

There is a second decomposition of the set $\mathcal{B}$ of bad paths into k disjoint subsets, that is induced naturally by the Generalized Ballot Problem and its refinements, as follows. For $i = 1, \ldots, k$, let $\mathcal{B}^{(i)}$ be the set of lattice paths from $(0,0)$ to (m,n) that never go below the line $y = (i-1)x$ but that do go below the line $y = ix$ somewhere (as before, we have $n \geqslant km \geqslant 0$). Clearly, the $\mathcal{B}^{(i)}$ are disjoint, and their union is $\mathcal{B}$, so

$$|\mathcal{B}| = |\mathcal{B}^{(1)}| + \cdots + |\mathcal{B}^{(k)}|.$$

Now note that $\mathcal{B}^{(i)}$ is precisely the symmetric difference of the two sets of paths counted by Theorem 2 when $k = i$ and $i - 1$, respectively, so Theorem 2 gives

$$|\mathcal{B}^{(i)}| = \frac{n-(i-1)m+1}{n+1}\binom{m+n}{m} - \frac{n-im+1}{n+1}\binom{m+n}{m} = \binom{m+n}{m-1},$$

for each $i = 1, \ldots, k$. (Expression (2) makes the above calculation even more transparent.) This means that we would have another proof of Theorem 2 in the spirit of the reflection principle if we could find a direct geometric bijection between $\mathcal{B}^{(i)}$ and $\mathcal{A}$, for each $i = 1, \ldots, k$, but we have been unable to find such a bijection for $k \geqslant 2$.

Moreover, in a similar way Theorems 5 and 7 give

$$|\mathcal{B}^{(i)} \cap \mathcal{C}_c| = \binom{m+1}{c+1}\binom{n-1}{c-1}, \quad |\mathcal{B}^{(i)} \cap \mathcal{C}^{(c)}| = \binom{m}{c}\binom{n}{c-1},$$

for each $i = 1, \ldots, k$, so such a bijection should also allow one to fix the number of right-up or up-right corners.

6. An extension of the Generalized Ballot Problem

Suppose we rescale the x-axis in the Generalized Ballot Problem by a factor of k, thus replacing the unit right-steps $(1,0)$ by k-right-steps $(k,0)$. Then the paths are from $(0,0)$ to (km,n), and are never below the transformed boundary line $y = x$. This rescaling of the problem has a natural extension for the horizontal steps, by allowing j-right-steps for all nonnegative integers j. Now the total number of paths from $(0,0)$ to (M,n) with a_j j-right-steps for $j \geqslant 0$ is given by the multinomial coefficient

$$P(n; a_0, a_1, \ldots) = \frac{(n+m)!}{n! \prod_{j \geqslant 0} a_j!},$$

where $M = \sum_{j \geqslant 0} j a_j$ and $m = \sum_{j \geqslant 0} a_j$. Then, for $n \geqslant M \geqslant 0$, by the cycle lemma (called the method of "penetrating analysis" in [7]), the number of these paths that are never below the line $y = x$ is given by

$$\frac{n - M + 1}{n + 1} P(n; a_0, a_1, \ldots),$$

which can be reexpressed as the difference

$$P(n; a_0, a_1, \ldots) - \sum_{j \geqslant 1} jP(n + 1; a_0, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots),$$

so here the bad paths (those that go below the line $y = x$) are enumerated by the subtracted summation.

The bijection in Section 3 immediately extends to this case, by decomposing the set of bad paths into disjoint subsets $\mathscr{B}_{j,i}$, with $j \geqslant 1$, and $i = 1, \ldots, j$. The set $\mathscr{B}_{j,i}$ consists of the bad paths in which the first right-step whose right end (a, b) lies below the line $y = x$ is a j-right-step, and the portion of this j-right-step below the diagonal has length i. For a path π in $\mathscr{B}_{j,i}$, let p be the portion from $(0, 0)$ to $(a - j, b)$. Now rotate p by 180° to interchange $(0, 0)$ and $(a - j, b)$, and translate by $(j, -1)$, to obtain a path p' from $(j, -1)$ to $(a, b - 1)$. Then p', followed by a vertical step from $(a, b - 1)$ to (a, b), together with the portion of π starting at (a, b), gives the image π' of π. This is uniquely reversible since j is identified by the starting point $(j, -1)$, and i is identified as before. In conclusion, this gives a bijection between $\mathscr{B}_{j,i}$ and the set of all paths from $(j, -1)$ to (M, n) with $n + 1$ unit up-steps, a_t t-right-steps for $t \geqslant 0, t \neq j$, and $a_j - 1$ j-right-steps. The result follows, since there are exactly $P(n + 1; a_0, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots)$ such paths for each $i = 1, \ldots, j$.

Acknowledgments

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