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Raney Paths and a Combinatorial Relationship between Rooted Nonseparable Planar Maps and Two-Stack-Sortable Permutations

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An encoding of the set of two-stack-sortable permutations ($\mathscr{TSS}$) in terms of lattice paths and ordered lists of strings is obtained. These lattice paths are called Raney paths. The encoding yields combinatorial decompositions for two complementary subsets of $\mathscr{TSS}$, which are the analogues of previously known decompositions for the set of nonseparable rooted planar maps ($\mathscr{NS}$). This provides a combinatorial relationship between $\mathscr{TSS}$ and $\mathscr{NS}$, and, hence, a bijection is determined between these sets that is different, simpler, and more refined than the previously known bijection. © 1996 Academic Press, Inc.

1. INTRODUCTION

We begin with preliminary combinatorial definitions in Sections 1.1 and 1.2, followed by a description of the background and outline of the paper in Section 1.3.

1.1. *Nonseparable Rooted Planar Maps*

A *nonseparable* planar map is an embedding of a connected graph with no cut-vertices in the surface of the sphere, partitioning the surface into regions called faces. Multiple edges are allowed. This embedding is represented for convenience in the plane by any stereographic projection of the sphere mapping a point interior to a face to the point at infinity. Thus, for example, the same planar map can be represented with any of its faces as the infinite face. A map is rooted by distinguishing one vertex as the root vertex and one edge incident with the root vertex as the root edge. This is represented in diagrams by drawing an arrow on the root edge pointing away from the root vertex. The face that is counterclockwise from the root edge when moving around the root vertex is the root face.

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Let $\mathcal{NS}$ denote the set of all nonseparable rooted planar maps with at least two edges. There is a single map in $\mathcal{NS}$ with two edges, the rooted digon D, consisting of a pair of edges, either of which is the root edge, between a pair of vertices, either of which is the root vertex. For a map M in $\mathcal{NS}$ let $e(M)$ be the number of edges of $M-1$, let $v(M)$ be the number of vertices of $M-2$, and let $r(M)$ be the degree (number of incident edges) of the root face minus 1.

We partition $\mathcal{NS}$ into two subsets; $\mathcal{NS}^{(1)}$ contains all elements of $\mathcal{NS}$ whose underlying graph has no cut-vertices when the root edge is deleted, except D; $\mathcal{NS}^{(2)}$ contains all other maps in $\mathcal{NS}$ (including D). For $M \in \mathcal{NS}$, we let M' denote the (unrooted) map obtained by deleting the root edge of M.

For example, consider the three elements N_1, N_2, N_3 of $\mathcal{NS}$ given in Fig. 1. Then $e(N_1)=2$, $e(N_2)=4$, $e(N_3)=3$, $v(N_1)=0$, $v(N_2)=v(N_3)=1$, $r(N_1)=r(N_2)=1$, $r(N_3)=2$, and $N_1, N_2 \in \mathcal{NS}^{(1)}$ whereas, $N_3 \in \mathcal{NS}^{(2)}$.

1.2. *Two-Stack-Sortable Permutations*

Let $\mathcal{N}$ denote the set of positive integers and $\mathcal{N}_n = \{1, ..., n\}$. If $\omega = \omega_1 \cdots \omega_m \in \mathcal{N}^*$ is a string on the alphabet $\mathcal{N}$ then the *length* of ω is m, denoted $|\omega| = m$. The empty string ε has length 0. A *descent* in ω is a pair (ω_i, ω_{i+1}) with $\omega_i > \omega_{i+1}$, where $1 \leqslant i \leqslant m-1$. The number of descents in ω is denoted by $\mathrm{desc}(\omega)$. A *rlmax* in ω is an element ω_i such that $\omega_i > \omega_j$ for all $j = i+1, ..., m$, where $1 \leqslant i \leqslant m$. In particular, for $m \geqslant 1$, the terminal element ω_m is always a rlmax element. The number of rlmax in ω is denoted by $\mathrm{rl}(\omega)$. Suppose ω contains distinct entries and the largest of these is n, so that we can write $\omega = \tau_1 n \tau_2$, where τ_1, τ_2 are strings. Then consider the mapping π for strings defined by the property

$$\pi(\omega) = \pi(\tau_1)\, \pi(\tau_2)\, n, \tag{1}$$

with the initial condition $\pi(\varepsilon) = \varepsilon$.

A permutation $\sigma = \sigma_1 \cdots \sigma_n$ on $\mathcal{N}_n$ is called *j-stack-sortable* if $\pi^j(\sigma) = \iota_n$, where ι_n is the identity permutation $1 \cdots n$ on $\mathcal{N}_n$ (when the context is clear,

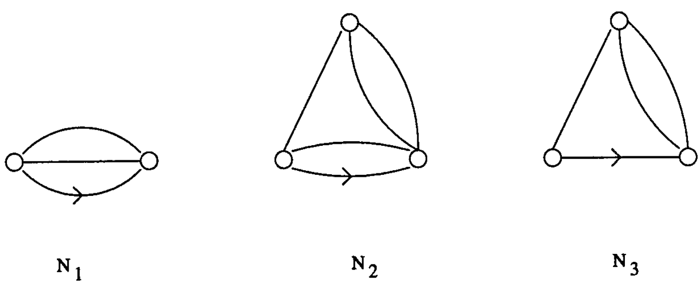

FIG. 1. Three rooted nonseparable planar maps.

we may suppress the subscript and write ι alone; ι_0 is the empty string). In this paper we shall be concerned with the case $j=2$. Let $\mathcal{TSS}_n$ denote the set of two-stack-sortable permutations on $\mathcal{N}_n$, and let $\mathcal{TSS}$ denote the set of all two-stack-sortable permutations for $n \geqslant 1$. For example, if $\rho = 281957364$ then from the defining property of π we find

$$\pi(\rho) = \pi(281)\,\pi(57364)\,9 = \pi(2)\,\pi(1)\,8\pi(5)\,\pi(364)\,79 = \cdots = 218534679,$$

and applying π again we obtain

$$\pi^2(\rho) = \pi(21853467)\,9 = \pi(21)\,\pi(53467)\,89 = \cdots = 123456789,$$

so we conclude that the permutation ρ is in $\mathcal{TSS}$ (and, in particular, in $\mathcal{TSS}_9$). The descents in ρ are (8, 1), (9, 5), (7, 3), (6, 4), so $\operatorname{desc}(\rho)=4$. The rlmax elements in ρ are 9, 7, 6, 4 so $\operatorname{rl}(\rho)=4$. Of course we also have $|\rho|=9$.

Suppose that the rlmax elements of a two-stack-sortable permutation σ are $a_1, ..., a_k$ from left to right as they appear in σ, so that we can write $\sigma = s_1 a_1 \cdots s_k a_k$, where $s_1, ..., s_k$ are strings, and we call this the *rlmax decomposition* for σ. Then we have immediately $n = a_1$ and $a_1 > \ldots > a_k$. We now partition $\mathcal{TSS}$ into two subsets; $\mathcal{TSS}^{(1)}$ contains all elements of $\mathcal{TSS}$ for which $a_k - 1$ appears in s_k; $\mathcal{TSS}^{(2)}$ consists of the remaining elements of $\mathcal{TSS}$, including the permutation 1.

For example, $\mathcal{TSS}^{(1)}$ contains the permutations 312, 6715324, 43215 and $\mathcal{TSS}^{(2)}$ contains the permutations 132, 6713524, 32154.

1.3. *Background and Outline*

In [11], West gave a number of results and conjectures for j-stack-sortable permutations. The name "stack-sortable" was used because when $j=1$ these are precisely the permutations that can be sorted into natural (increasing) order using a single stack. In the latter case the number of such permutations on n elements was shown by Knuth [8] to be the Catalan number $(1/(n+1))\binom{2n}{n}$, $n \geqslant 1$. In the case $j=2$, West conjectured that

$$|\mathcal{TSS}_n| = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}, \qquad n \geqslant 1. \tag{2}$$

Observing that the right side of (2) is also Tutte's formula [10] for the number of maps in $\mathcal{NS}$ with $n+1$ edges, West also pointed out that (2) could be proved by finding an explicit bijection between $\sigma \in \mathcal{TSS}$ and $M \in \mathcal{NS}$ such that $|\sigma| = e(M)$.

Zeilberger [12] proved (2) by examining the combinatorial structure of $\mathcal{TSS}$, using property (1) of π and thus finding a quadratic equation for the generating function with respect to length and a second parameter.

Then he solved the equation indirectly after the second parameter was eliminated, with the assistance of computer algebra. Gessel [6] has simplified this approach and extended it to multipermutations, but neither Zeilberger nor Gessel give any insight into the possible connection with nonseparable maps.

Dulucq, Gire, and Guibert [3] have given an explicit bijection between $\sigma \in \mathcal{TSS}$ and $M \in \mathcal{NS}$ such that $|\sigma| = e(M)$. In addition, they prove that $\mathrm{rl}(\sigma) = r(M)$ and $\mathrm{desc}(\sigma) = v(M)$ under their bijection, which is given as the composition of a number of bijections. The first of these bijections, given in Dulucq, Gire, and West [4], is between $\mathcal{NS}$ and a set of permutations avoiding a particular class of forbidden subsequences. The remaining eight bijections, given in [3], are between sets of permutations avoiding various classes of forbidden subsequences, the last of which involves the class of forbidden subsequences that was proved by West [11] to characterize the set $\mathcal{TSS}$. In these latter bijections, they show that the *generating trees* for the permutations are isomorphic; a generating tree for a set of permutations is a tree with the permutations as vertices, in which the children of a permutation σ on n elements are all permutations in the set that can be obtained by inserting $n+1$ between elements of σ or at either end of σ.

In this paper we adopt a new approach. We begin in Section 2 by restating Brown's [1] combinatorial decompositions for $\mathcal{NS}^{(1)}$ and $\mathcal{NS}^{(2)}$, with their effect on the map parameters e, r, v. These are given in Lemmas 2.1 and 2.2. Then in Section 3 we considerably refine Zeilberger's [12] combinatorial analysis of $\mathcal{TSS}$ to obtain combinatorial decompositions for $\mathcal{TSS}^{(1)}$ and $\mathcal{TSS}^{(2)}$. These decompositions, given as Lemmas 3.1 and 3.6, are the exact analogues of Brown's decompositions above, with $\mathcal{TSS}^{(i)}$ identified with $\mathcal{NS}^{(i)}$, for $i = 1, 2$, the permutation 1 identified with the map D, and length, rl, desc identified with e, r, v. This immediately establishes a direct combinatorial relationship between $\mathcal{NS}$ and $\mathcal{TSS}$. Moreover, by identifying our combinatorial operations on permutations with Brown's combinatorial operations on maps, we implicitly have a bijection between $M \in \mathcal{NS}$ and $\sigma \in \mathcal{TSS}$ such that $e(M) = |\sigma|$, $r(M) = \mathrm{rl}(\sigma)$, $v(M) = \mathrm{desc}(\sigma)$, and such that $\mathcal{NS}^{(i)}$ is mapped to $\mathcal{TSS}^{(i)}$, for $i = 1, 2$.

In Section 4 we make this bijection explicit by introducing an intermediate combinatorial structure, the *decomposition tree*, which allows the bijection to be carried out in a straightforward manner. We also demonstrate that under this explicit bijection, given as Theorem 4.3, the maps $M \in \mathcal{NS}^{(2)} - \{D\}$ for which M' has m bridges, are mapped to the permutations $\sigma \in \mathcal{TSS}^{(2)} - \{1\}$ that terminate with m descents, for each $m \geqslant 0$ (a more intricate result involving cut-vertices is also described). Thus we have a bijection that is more refined than Dulucq, Gire, and Guibert's.

The decomposition (Lemma 3.1) that we give in Section 3 for $\mathcal{TSS}^{(1)}$ is straightforward. However, for the decomposition (Lemma 3.6) for

$\mathcal{TSS}^{(2)}$, we introduce an encoding, given as Theorem 3.5, which characterizes $\mathcal{TSS}$ by means of lattice paths and ordered lists of strings. The lattice paths, determined by the relative positions of the largest elements in a permutation, are called *Raney paths*, since they were used by Raney [9] in his combinatorial proof of Lagrange's implicit function theorem. Note that the enumerative consequences of our decompositions for permutations are quite direct, since we can use them to enumerate two-stack-sortable permutations precisely as Brown used the analogous decompositions to enumerate nonseparable rooted planar maps. We do not need to apply our bijection and indirectly obtain the permutation enumeration from Brown's map enumeration, as, for example, is required by Dulucq, Gire, and Guibert.

Finally, in Section 5 we give some concluding remarks about related matters, including a comparison of our bijection to Dulucq, Gire, and Guibert's, a second bijection that follows from our methodology, and a direct connection with Lagrange's Theorem.

2. Combinatorial Decompositions for Nonseparable Rooted Planar Maps

Nonseparable rooted planar maps were first enumerated by Tutte [10], with respect to number of edges. Brown [1] and Brown and Tutte [2] then refined this enumeration to include other parameters. See [7] for a comprehensive treatment of planar map enumeration and the quadratic method.

The following pair of decompositions, essentially due to Brown [1], completely describes the set of nonseparable rooted planar maps $\mathcal{NS}$. In each case the element-wise action of the isomorphism is described so that the effect on the combinatorial parameters e, v, r can be made explicit. The image of the first isomorphism, mark($\mathcal{NS}$), is the set whose elements are the elements of $\mathcal{NS}$ with a combinatorial subconfiguration (in this case a nonroot vertex on the root face) marked so as to distinguish it.

Lemma 2.1 (Decomposition for $\mathcal{NS}^{(1)}$).

$$\mathcal{NS}^{(1)} \simeq \text{mark}(\mathcal{NS}) \colon M \mapsto M_1,$$

where M_1 has a nonroot vertex on its root face marked. Moreover, $e(M) = e(M_1) + 1$, $v(M) = v(M_1)$, and $r(M) \leqslant r(M_1)$, and the marked vertex in M_1 is the $r(M)$th encountered after the root vertex when moving around the root face of M_1 in the opposite direction to the root edge.

Proof. Delete the root edge of $M \in \mathcal{NS}^{(1)}$, to obtain M', which is a nonseparable planar map. Root M' at the same vertex as M and at the first edge encountered clockwise around the root vertex from the deleted root edge. This gives a rooted map $M_1 \in \mathcal{NS}$. Mark the nonroot vertex on the root face of M_1 that was incident with the deleted root edge of M. The relationships between the values of the parameters of M and M_1 are easily verified. This construction is reversible, since the marked vertex on the root face of M_1 allows us to uniquely recover M. ∎

As examples of the decomposition in Lemma 2.1 from Fig. 1, the choice $M = N_1$ yields $M_1 = D$ with the nonroot vertex of D marked; the choice $M = N_2$ yields $M_1 = N_3$ with the vertex of N_3 not incident with the root edge as the marked vertex. Note that in the latter case, M_1 has been redrawn with the root face finite.

LEMMA 2.2 (Decomposition for $\mathcal{NS}^{(2)}$). *$\mathcal{NS}^{(2)} - \{D\} \simeq \mathcal{NS} \times (\mathcal{NS}^{(1)} \cup \{D\})$: $M \mapsto (M_1, M_2)$, where $e(M) = e(M_1) + e(M_2)$, $v(M) = v(M_1) + v(M_2) + 1$, and $r(M) = r(M_1) + r(M_2)$.*

Proof. Suppose the root face of $M \in \mathcal{NS}^{(2)} - \{D\}$ has degree k, where $k \geqslant 3$. Let the vertices on the root face be $v_1, ..., v_k$, in the order that they are encountered around the root face beginning with the root vertex and ending at the nonroot vertex incident with the root edge (thus, the root edge is directed from v_1 to v_k). Now M' has at least one cut-vertex, and all such cut-vertices are contained in $\{v_2, ..., v_{k-1}\}$; let v_j be the cut-vertex with the largest subscript among the cut-vertices in this set. Delete the root edge from M and cut the resulting map at v_j into two maps, both containing a copy of v_j. One of these maps contains $v_1, ..., v_j$; add a root edge to this map from v_1 to v_j across the previous root face of M to obtain $M_1 \in \mathcal{NS}$. The other map contains $v_j, ..., v_k$; add a root edge to this map from v_j to v_k across the root face of M to obtain $M_2 \in (\mathcal{NS}^{(1)} \cup \{D\})$. This is clearly reversible and the relationships between the values of the parameters of M, M_1, M_2 are easily verified.

Note that $M_2 = D$ is obtained precisely when edge $\{v_{k-1}, v_k\}$ is a bridge in M', and that, if $j = 2$, then $M_1 = D$ is obtained precisely when edge $\{v_1, v_2\}$ is a bridge in M'. ∎

As an example of the decomposition in Lemma 2.2 from Fig. 1, the choice $M = N_3$ yields $M_1 = D$ and $M_2 = N_1$.

3. RANEY PATHS AND COMBINATORIAL DECOMPOSITIONS FOR TWO-STACK-SORTABLE PERMUTATIONS

In this section we give a pair of decompositions that completely describes the set of two-stack-sortable permutations $\mathcal{TSS}$, by considering the partition into $\mathcal{TSS}^{(1)}$ and $\mathcal{TSS}^{(2)}$ given in Section 1.2.

3.1. *The First Decomposition*

The first isomorphism for two-stack-sortable permutations, analogous to Lemma 2.1 for nonseparable maps, can be given immediately. The image of this isomorphism, mark($\mathcal{TSS}$), is the set containing the permutations in $\mathcal{TSS}$ with one rlmax element marked.

LEMMA 3.1 (Decomposition for $\mathcal{TSS}^{(1)}$). $\mathcal{TSS}^{(1)} \simeq \text{mark}(\mathcal{TSS})$: $\sigma \mapsto \sigma_1$, *where* σ_1 *has a rlmax element marked. Moreover,* $|\sigma| = |\sigma_1| + 1$, $\text{desc}(\sigma) = \text{desc}(\sigma_1)$ *and* $\text{rl}(\sigma) \leqslant \text{rl}(\sigma_1)$, *and the marked rlmax element in* σ_1 *is the* $\text{rl}(\sigma)$*th, when ordered from left to right.*

Proof. Suppose that $\sigma \in \mathcal{TSS}^{(1)}$, with rlmax decomposition $\sigma = s_1 a_1 \cdots s_k a_k$, where $a_k - 1 \in s_k$. Remove a_k from σ and replace j by $j-1$ for each $j = a_k + 1, ..., |\sigma|$ in the resulting string to obtain a permutation σ_1, where $|\sigma_1| = |\sigma| - 1$.

Let $s_k = s_k^{(1)}(a_k - 1)\, s_k^{(2)}$, where $s_k^{(1)}$, $s_k^{(2)}$ are strings. Now all elements of s_k are smaller than a_k, so applying π to σ and σ_1, and using the defining property (1) of π repeatedly for maximum elements down to $a_k - 1$, we obtain

$$\pi(\sigma) = \tau_1 \pi(s_k^{(1)})\, \pi(s_k^{(2)})(a_k - 1)\, a_k \tau_2,$$

$$\pi(\sigma_1) = \tau_1' \pi(s_k^{(1)})\, \pi(s_k^{(2)})(a_k - 1)\, \tau_2',$$

where τ_1, τ_2 are strings (with $\tau_2 = a_{k-1} a_{k-2} \cdots a_1$) and τ_i' is obtained from τ_i for $i = 1, 2$ by replacing j by $j-1$ for each $j = a_k + 1, ..., |\sigma|$. Thus $\pi^2(\sigma) = \iota_{|\sigma|}$ if and only if $\pi^2(\sigma_1) = \iota_{|\sigma_1|}$, so $\sigma_1 \in \mathcal{TSS}$.

Mark element $a_k - 1$ in σ_1. Now $a_k - 1$ is the kth rlmax element in σ_1, since all elements of s_k are smaller than a_k, and the relationships between the parameters of σ and σ_1 are easily verified. This construction is reversible, since σ is uniquely recovered from σ_1 by adding 1 to all elements larger than the marked element of σ_1 and placing an additional element, the marked element plus 1, at the end. ∎

In the isomorphism of Lemma 3.1, for example, $748925136 \in \mathcal{TSS}^{(1)}$ is mapped to $64782\hat{5}13$, where the rlmax element 5 is marked; note that $647825139 \in \mathcal{TSS}^{(1)}$ is mapped to $6478\hat{2}513$ and $758926134 \in \mathcal{TSS}^{(1)}$ is mapped to $6478251\hat{3}$.

3.2. *Raney Paths and a Structural Characterization of Two-Stack-Sortable Permutations*

In order to give the second decomposition for two-stack-sortable permutations, we develop a structural characterization by means of the following proposition. The result follows from considering the positions in the permutation that the largest elements can occupy. This is an extension of Zeilberger's strategy [12] for analyzing the combinatorial structure of $\mathcal{TSS}_n$ (Zeilberger stopped at the largest value of j such that element j appeared to the left of element $j+1$ in the permutation). For notational convenience, if $\tau=(\tau_1, ..., \tau_i)$ is an i-tuple of strings, where $i \geqslant 1$, then we denote $\pi(\pi(\tau_1)\cdots\pi(\tau_i))$ by $\pi^2(\tau)$.

PROPOSITION 3.2. *Consider $\sigma \in \mathcal{TSS}_n$ with rlmax decomposition $\sigma = s_1 a_1 \cdots s_k a_k$.*

(1) *Then for each $m=1, ..., n-a_k+1$, elements $n, n-1, ..., n-m+1$ occur in σ as $a_1, ..., a_\delta$ and $b_{i_1}, ..., b_{i_\gamma}$ for some $\delta+\gamma=m$, $\gamma \geqslant 0$, $\delta \geqslant 1$, where $b_{i_j} \in s_{i_j}$ for $j=1, ..., \gamma$ and $1 \leqslant i_1 < \cdots < i_\gamma \leqslant \delta$, $b_{i_1} > \cdots > b_{i_\gamma}$. Thus, if $s_{i_j} = s_{i_j}^{(1)} b_{i_j} s_{i_j}^{(2)}$, where $s_{i_j}^{(l)} \in \mathcal{N}^*$ for $l=1, 2$ and $j=1, ..., \gamma$, we let*

$$W_j = (s_{i_{j-1}+1}, ..., s_{i_j-1}, s_{i_j}^{(1)}, s_{i_j}^{(2)}) \tag{3}$$

for $j=1, ..., \gamma$, where $i_0=0$, and let

$$\hat{W}_{\gamma+1} = (s_{i_\gamma+1}, ..., s_\delta, s_{\delta+1} a_{\delta+1} \cdots s_k a_k).$$

Then

$$\pi^2(W_1) \cdots \pi^2(W_\gamma)\, \pi^2(\hat{W}_{\gamma+1}) = \iota_{n-m}. \tag{4}$$

(2) *There exists a unique $h \geqslant 0$ such that elements $n, n-1, ..., a_k-h$ occur in σ as $a_1, ..., a_k$ and $b_{i_1}, ..., b_{i_g}$ (so $g=n-a_k+h+1-k$), where $b_{i_j} \in s_{i_j}$ for $j=1, ..., g$, $1 \leqslant i_1 < \cdots < i_g \leqslant k$, $b_{i_1} > \cdots > b_{i_g}$, $s_{i_g+1} = \cdots = s_k = \varepsilon$, and*

$$\pi^2(W_1) \cdots \pi^2(W_g) = \iota_{a_k-h-1},$$

where W_j is defined as in (3) above, for $j=1, ..., g$ (if $g=0$ then $i_0=0$).

Proof. (1) We proceed by induction on m. Since $a_1=n$ the defining property (1) of π gives

$$\pi^2(\sigma) = \pi(\pi(s_1)\,\pi(s_2 a_2 \cdots s_k a_k))\, n = \iota_n,$$

so $\pi^2(s_1, s_2 a_2 \cdots s_k a_k) = \iota_{n-1}$ and the result is true for $m=1$ with $\delta=1$, $\gamma=0$.

Assume the result is true for m, where $1 \leqslant m < n-a_k+1$. Now consider the possible location of element $n-m$. It follows from (4) that $n-m$ must appear in $\hat{W}_{\gamma+1}$ (since $m<n-a_k+1$, we must have $\delta<k$, so $\hat{W}_{\gamma+1}$ is nonempty), as the largest element. There are two cases.

Case 1. Element $n-m$ appears in s_t for $i_\gamma+1 \leqslant t \leqslant \delta$. In this case let $s_t = s_t^{(1)}(n-m)\, s_t^{(2)}$ so

$$\pi^2(\hat{W}_{\gamma+1}) = \pi^2(s_{i_\gamma+1}, ..., s_{t-1}, s_t^{(1)}, s_t^{(2)}) \\ \times \pi^2(s_{t+1}, ..., s_\delta, s_{\delta+1}a_{\delta+1} \cdots s_k a_k)(n-m)$$

and the result is true for $m+1$ in this case with γ increased by 1 (and $i_{\gamma+1} = t$) and δ unchanged.

Case 2. Element $n-m$ appears in $s_{\delta+1}a_{\delta+1} \cdots s_k a_k$. Then $n-m = a_{\delta+1}$, so

$$\pi^2(\hat{W}_{\gamma+1}) = \pi^2(s_{i_\gamma+1}, ..., s_\delta, s_{\delta+1}, s_{\delta+2}a_{\delta+2} \cdots s_k a_k)(n-m),$$

and the result is true for $m+1$ in this case with δ increased by 1 and γ unchanged.

(2) From part (1) when $m = n - a_k + 1$ we have $\delta = k$ and $\gamma = f$, where $f = n - a_k + 1 - k$, and $n, n-1, ..., a_k$ occur as $a_1, ..., a_k$ and $b_{i_1}, ..., b_{i_f}$ with

$$\pi^2(W_1) \cdots \pi^2(W_f)\, \pi^2(\hat{W}_{f+1}) = \iota_{a_k - 1} \tag{5}$$

and $\hat{W}_{f+1} = (s_{i_f+1}, ..., s_k)$. Moreover, $i_f < k$ since the largest element of s_k is not one of $b_{i_1}, ..., b_{i_f}$ (because it is smaller than a_k).

If $a_k - 1 = 0$ then the result is true with $h = 0$ and $g = f$ (and, if $g \neq 0$, all strings in $W_1, ..., W_g$ are empty). Otherwise there are two cases for the occurrence of element $a_k - 1$:

Case 1. Element $a_k - 1$ appears in one of the strings of $\hat{W}_{f+1}$, say in $s_{i_{f+1}}$ for $i_{f+1} > i_f$. Thus in this case we have

$$\pi^2(\hat{W}_{f+1}) = \pi^2(W_{f+1})\, \pi^2(s_{i_{f+1}+1}, ..., s_{i_k}) = \iota_{a_k - 2},$$

where W_{f+1} is defined as in (3).

Case 2. Element $a_k - 1$ does not appear in $\hat{W}_{f+1}$. It follows from (5) that all strings in $\hat{W}_{f+1}$ are empty. Thus in this case the result is true with $h = 0$ and $g = f$.

If we were in Case 1, repeat this procedure of locating successively smaller elements $a_k - 1, a_k - 2, ...$ until stopping either at element 0 or in Case 2, giving the required result. ∎

Although the notation of Proposition 3.2 is complicated and multiply indexed, it describes a very simple procedure, in which we consider the largest elements of a permutation in decreasing order until terminating as in part (2) of the result. To illustrate the notation, we give three examples.

EXAMPLE 3.3. (1) *For* $\sigma=\rho_1=3\ 13\ 11\ 2\ 1\ 12\ 6\ 8\ 5\ 7\ 4\ 10\ 9$ *we obtain* $k=4$, $a_1=13$, $a_2=12$, $a_3=10$, $a_4=9$. *Then* $g=2$, $i_1=2$, $i_2=3$, $b_2=11$, $b_3=8$, *so* $W_1=(3, \varepsilon, 21)$ *and* $W_2=(6, 574)$. *Note that, as claimed in the result,* $s_4=\varepsilon$ *and* $\pi^2(W_1)\,\pi^2(W_2)=\iota_7$.

(2) *For* $\sigma=\rho_2=1234$ *we obtain* $k=1$, $a_1=4$, $g=1$, $i_1=1$, $b_1=3$, *so* $W_1=(12, \varepsilon)$.

(3) *For* $\sigma=\rho_3=54321$ *we obtain* $k=5$, $a_1=5$, $a_2=4$, $a_3=3$, $a_4=2$, $a_5=1$, $g=0$. *Moreover, as claimed in the result,* $s_1=\cdots=s_5=\varepsilon$.

Some notation for particular sets of strings and lattice paths is required before we can state the structural consequences of Proposition 3.2.

Let $\mathscr{U}$ be the set of i-tuples of strings in $\mathscr{N}^*$, for $i\geqslant 2$. If $U=(u_1, ..., u_i)$ is in $\mathscr{U}$, then we define

$$|U|=\sum_{j=1}^{i}|u_j|, \qquad \operatorname{desc}(U)=\sum_{j=1}^{i}\operatorname{desc}(u_j).$$

Let $\mathscr{V}$ denote the set of all m-tuples $V=(V_1, ..., V_m)$, where V_j is in $\mathscr{U}$ for $j=1, ..., m$ and $m\geqslant 0$, in which the elements $1, ..., \beta_1+\cdots+\beta_m$ appear exactly once each, where $\beta_j=|V_j|$, and such that

$$\pi^2(V_1)\cdots\pi^2(V_m)=\iota_{\beta_1+\cdots+\beta_m}.$$

In this notation we define

$$\operatorname{type}(V)=(\beta_1-1, ..., \beta_m-1), \qquad |V|=\beta_1+\cdots+\beta_m.$$

We also define $\operatorname{desc}(V)=\operatorname{desc}(V_1)+\cdots+\operatorname{desc}(V_m)$, and $\operatorname{index}(V)$ to be the number of j such that the last string in V_j is nonempty, $j=1, ..., m$.

Next consider lattice paths in the plane beginning at vertex (0, 0), with successive vertices joined by steps that are either up (increasing the x- and y-coordinates by 1) or down by j (increasing the x-coordinate by 1 and decreasing the y-coordinate by j), for $j\geqslant 1$. The *height* of a vertex is its y-coordinate. The set of *Raney paths*, denoted by $\mathscr{R}$, consists of such paths on at least one step in which no vertex has negative height. If a Raney path P has p up steps and q down steps, and the jth of the down steps is down by α_j, for $j=1, ..., q$, then we write $\operatorname{fall}(P)=(\alpha_1, ..., \alpha_q)$ and $\operatorname{rise}(P)=p$. Note that the terminal height of P is $p-\alpha_1-\cdots-\alpha_q$.

We partition $\mathscr{R}$ into two subsets, $\mathscr{R}^{(1)}$ and $\mathscr{R}^{(2)}$; $\mathscr{R}^{(1)}$ consists of all paths in $\mathscr{R}$ that terminate at height 0 (so their last step is down) and whose second last step is up; $\mathscr{R}^{(2)}$ contains the remaining elements of $\mathscr{R}$.

A convenient representation of a path is to give the ordered list of steps, with A representing an up step and B_j representing a step that is down by j; we do not distinguish between the path itself and this list representing it. For example, the path

$$P = AAAB_2AAB_3AAB_1A$$

is a Raney path with $\mathrm{fall}(P) = (2, 3, 1)$ and $\mathrm{rise}(P) = 8$. Since P terminates at height 2, it is contained in $\mathscr{R}^{(2)}$. Other paths in $\mathscr{R}^{(2)}$ are A and $AAB_1AAB_2B_1$. Examples of paths in $\mathscr{R}^{(1)}$ are AB_1, AAB_1AAB_3 and $AAAAB_2B_2AAB_1AAB_3$.

Now we consider these strings and paths simultaneously. Let

$$\mathscr{E} = \{(V, P) \in \mathscr{V} \times \mathscr{R} : \mathrm{type}(V) = \mathrm{fall}(P)\},$$

and partition $\mathscr{E}$ into subsets $\mathscr{E}^{(1)}$ and $\mathscr{E}^{(2)}$, where

$$\mathscr{E}^{(j)} = \{(V, P) \in \mathscr{E} : P \in \mathscr{R}^{(j)}\},$$

for $j = 1, 2$. We now associate an element of $\mathscr{E}$ with $\sigma \in \mathscr{T}\mathscr{S}\mathscr{S}$. In the notation of Proposition 3.2, first define

$$X(\sigma) = (W_1, ..., W_g),$$

so $X(\sigma) \in \mathscr{V}$. Then define $Y(\sigma)$ to be the lattice path with k up steps and g down steps, with the up steps in positions $n+1-a_j$, for $j = 1, ..., k$, and the down steps in positions $n+1-b_{i_j}$ for $j = 1, ..., g$. The jth down step is down by $i_j - i_{j-1}$, where $i_0 = 0$. Note that the terminal height of the path $Y(\sigma)$ is $k - i_g$. Now $Y(\sigma)$ is a Raney path since $a_1 > \cdots > a_k$, $b_{i_1} > \cdots > b_{i_g}$ with $1 \leqslant i_1 < \cdots < i_g \leqslant k$, and $b_{i_j} < a_{i_j}$ for $j = 1, ..., g$ (the last condition follows because $a_1, ..., a_k$ are the rlmax elements of σ). Moreover, by construction we have $\mathrm{type}(X(\sigma)) = \mathrm{fall}(Y(\sigma))$. Thus if we define

$$\phi(\sigma) = (X(\sigma), Y(\sigma)),$$

then $\phi(\sigma) \in \mathscr{E}$.

Example 3.4. *Consider the permutations* ρ_1, ρ_2, ρ_3 *from Example* 3.3. *We obtain*

$$X(\rho_1) = ((3, \varepsilon, 21), (6, 574)), \qquad Y(\rho_1) = AAB_2AAB_1,$$
$$X(\rho_2) = ((12, \varepsilon)), \qquad Y(\rho_2) = AB_1,$$
$$X(\rho_3) = \varepsilon, \qquad Y(\rho_3) = AAAAA.$$

Thus $Y(\rho_1), Y(\rho_3) \in \mathscr{R}^{(2)}$ *so* $\phi(\rho_1), \phi(\rho_3) \in \mathscr{E}^{(2)}$, *and* $Y(\rho_2) \in \mathscr{R}^{(1)}$, *so* $\phi(\rho_2) \in \mathscr{E}^{(1)}$. *Also note that* $\rho_1, \rho_3 \in \mathscr{T}\mathscr{S}\mathscr{S}^{(2)}$, *but* $\rho_2 \in \mathscr{T}\mathscr{S}\mathscr{S}^{(1)}$.

The fact that ϕ acts bijectively on $\mathscr{TSS}$ and, in general, restricts nicely to the subsets $\mathscr{TSS}^{(j)}$, $j=1, 2$, recorded in the following result, is an immediate consequence of Proposition 3.2.

THEOREM 3.5. (Characterization of $\mathscr{TSS}$). $\phi: \mathscr{TSS} \xrightarrow{\sim} \mathscr{E}: \sigma \mapsto (V, P)$, *defined above, is a bijection, in which*

$$|\sigma| = |V| + |\mathrm{fall}(P)| + \mathrm{rise}(P),$$

$$\mathrm{desc}(\sigma) = \mathrm{desc}(V) + \mathrm{index}(V) + \mathrm{rise}(P) - 1,$$

$$\mathrm{rl}(\sigma) = \mathrm{rise}(P).$$

Moreover, ϕ maps $\mathscr{TSS}^{(j)}$ to $\mathscr{E}^{(j)}$ for $j=1, 2$.

Proof. As noted in the preceding discussion, if $\sigma \in \mathscr{TSS}$, then $\phi(\sigma) \in \mathscr{E}$.

To reverse this, consider $(V, P) \in \mathscr{E}$. Then, in the notation of Proposition 3.2, we immediately recover $k = \mathrm{rl}(\sigma) = \mathrm{rise}(P)$ and $g = |\mathrm{fall}(P)|$ and then $n = |\sigma|$, since $n = |V| + k + g$. Also we have $(i_1, ..., i_g) = \mathrm{fall}(P)$, so we can deduce the values of $a_1, ..., a_k$ and $b_{i_1}, ..., b_{i_g}$, and thus reconstruct σ, since the inductive steps of Proposition 3.2 are all reversible. Thus, ϕ is a bijection, and we have already determined $|\sigma|$ and $\mathrm{rl}(\sigma)$ in terms of V and P. To determine $\mathrm{desc}(\sigma)$, note that $a_1, ..., a_{k-1}$ are always followed by smaller elements in σ, giving rise(P)-1 descents in σ. However, a_k and the terminal elements of the $i_1 + \cdots + i_g + g$ strings in V are never followed by smaller elements in σ. Element b_{i_j} is followed by a smaller element in σ exactly when the last string in V_j is nonempty, for $j = 1, ..., g$, giving $\mathrm{index}(V)$ descents in σ. All other descents in σ are descents in the strings of V, giving $\mathrm{desc}(V)$ descents, so $\mathrm{desc}(\sigma) = \mathrm{desc}(V) + \mathrm{index}(V) + \mathrm{rise}(P) - 1$, as required.

Now if $\sigma \in \mathscr{TSS}^{(1)}$ then $a_k - 1$ is in s_k so in part(2) of Proposition 3.2 we have $h=1$, $i_g = k$ and $b_{i_g} = a_k - 1$. Thus the path $Y(\sigma)$ terminates at height $k - i_g = 0$, the last step is down (by $i_g - i_{g-1}$) and the second last step is up (since the rlmax element a_k is the second smallest element in the set $\{n, ..., a_k - h\}$). This means that $Y(\sigma)$ is in $\mathscr{R}^{(1)}$, so $\phi(\sigma) \in \mathscr{E}^{(1)}$.

For $\sigma \in \mathscr{TSS}^{(2)}$, there are two cases. If s_k is empty, then $i_g < k$, so $Y(\sigma)$ terminates at positive height. Otherwise, s_k is not empty but does not contain $a_k - 1$, in which case $i_g = k$, so $Y(\sigma)$ terminates at height 0. However, in this case $b_{i_g} \leqslant a_k - 2$ so the second last step of $Y(\sigma)$ (corresponding to element $a_k - 1$), is down. Thus in both cases we have $Y(\sigma) \in \mathscr{R}^{(2)}$, so $\phi(\sigma) \in \mathscr{E}^{(2)}$. ∎

3.3. *The Second Decomposition*

Now we are able to give the second isomorphism for two-stack-sortable permutations, analogous to Lemma 2.2 for nonseparable maps.

LEMMA 3.6 (Decomposition for $\mathcal{TSS}^{(2)}$). *$\mathcal{TSS}^{(2)} - \{1\} \simeq \mathcal{TSS} \times (\mathcal{TSS}^{(1)} \cup \{1\}): \sigma \mapsto (\sigma_1, \sigma_2)$, where* $|\sigma| = |\sigma_1| + |\sigma_2|$, $\operatorname{desc}(\sigma) = \operatorname{desc}(\sigma_1) + \operatorname{desc}(\sigma_2) + 1$ *and* $\operatorname{rl}(\sigma) = \operatorname{rl}(\sigma_1) + \operatorname{rl}(\sigma_2)$.

Proof. We prove first the path bijection $\psi: \mathcal{R}^{(2)} - \{A\} \simeq \mathcal{R} \times (\mathcal{R}^{(1)} \cup \{A\}): P \mapsto (P_1, P_2)$. Consider $P \in \mathcal{R}^{(2)} - \{A\}$. There are three cases:

Case A. P terminates at positive height, and the last step is an up step. Then $P = R_1 A$, where R_1 is the portion of P before the last up step A. In this case, let $P_1 = R_1 \in \mathcal{R}$ and $P_2 = A$. Note that, by construction, P_1 terminates at height one less that P.

Case B. P terminates at positive height, and the last step is a down step. Suppose the terminal height is j and the last step is down by k. Then $P = R_1 A R_2 A R_3 B_k$, where R_1 is the portion of P before the last up step to height j, R_2 is the portion of P strictly between this up step and the last up step to height $k+j$, and R_3 is the portion of P strictly between this latter up step and the terminal down step B_k. Note that any or all of these R_j, $j = 1, 2, 3$, may be empty (no steps). In this case, let $P_1 = R_1 A R_3$ and $P_2 = R_2 A B_k$. By construction, P_1 terminates at positive height j (the same as P); P_2 terminates at height 0 and the second last step is up, so $P_2 \in \mathcal{R}^{(1)}$.

Case C. P terminates at height 0, but the second last step of P is down. Suppose the last step is down by k. Then $P = R_1 A R_2 B_k$, where R_1 is the portion of P before the last up step to height k, and R_2 is the portion of P strictly between this up step and the terminal down step B_k. Note that R_1 may be empty, but R_2 must contain the second last down step of P, so R_2 cannot be empty. In this case, let $P_1 = R_2$ and $P_2 = R_1 A B_k$. By construction, P_1 terminates at height 0; P_2 terminates at height 0 and the second last step is up, so $P_2 \in \mathcal{R}^{(1)}$.

To reverse this construction, consider an arbitrary $(P_1, P_2) \in \mathcal{R} \times (\mathcal{R}^{(1)} \cup \{A\})$. If $P_2 = A$, then we reverse the construction of Case A above. If $P_2 \in \mathcal{R}^{(1)}$ and P_1 terminates at positive height, then reverse Case B, by writing $P_1 = R_1 A R_3$, where A is the last step up to the terminal height of P_1. If $P_2 \in \mathcal{R}^{(1)}$ and P_1 terminates at height 0, reverse Case C.

Now consider $\sigma \in \mathcal{TSS}^{(2)} - \{1\}$. Then $Y(\sigma) \in \mathcal{R}^{(2)} - \{A\}$ and apply the above construction to obtain

$$\psi(P) = (P_1, P_2),$$

where $P = Y(\sigma)$. Let $X(\sigma) = (W_1, ..., W_g)$ in the notation of Proposition 3.2, so the down steps of $Y(\sigma)$ are indexed $1, ..., g$ from left to right. Suppose that, with respect to this indexing, P_1 has the down steps of P indexed by $\mu \subseteq \{1, ..., g\}$, so P_2 has the remaining down steps, indexed by $\bar{\mu} = \{1, ..., g\} - \mu$. Then let

$$\sigma_1 = \phi^{-1}(\mathrm{Red}(W_i)_{i \in \mu}, P_1),$$
$$\sigma_2 = \phi^{-1}(\mathrm{Red}(W_i)_{i \in \bar{\mu}}, P_2),$$

where Red (for "reduce") means replace the jth largest integer by j, for $j \geqslant 1$. This is clearly well defined and reversible, and the relationships between the values of the parameters of $\sigma, \sigma_1, \sigma_2$ follow immediately from Theorem 3.5. ∎

As illustrations of the decomposition in Lemma 3.6, consider the following examples.

EXAMPLE 3.7. (1) *Let* $\sigma = 492371865 \in \mathscr{TSS}^{(2)}$. *Then* $X(\sigma) = ((4, 23, 1))$ *and* $Y(\sigma) = AAB_2AA$. *Now* $Y(\sigma) \in \mathscr{R}^{(2)}$ *terminates at positive height, and the last step is up, so we use Case* A *to obtain* $\psi(Y(\sigma)) = (AAB_2A, A)$. *Thus we have* $g = 1$ *and* $\mu = \{1\}$, $\bar{\mu} = \{\ \}$, *so*

$$\sigma_1 = \phi^{-1}(\mathrm{Red}((4, 23, 1)), AAB_2A), \qquad \sigma_2 = \phi^{-1}(\mathrm{Red}(\varepsilon), A),$$

and from Theorem 3.5 *we obtain*

$$\sigma_1 = \phi^{-1}((4, 23, 1), AAB_2A) = 48236175, \qquad \sigma_2 = 1.$$

(2) *Let* $\sigma = 2\ 17\ 1\ 18\ 3\ 13\ 16\ 10\ 5\ 4\ 15\ 8\ 6\ 14\ 9\ 7\ 12\ 11 \in \mathscr{TSS}^{(2)}$. *Then* $X(\sigma) = ((2, 1), (3, \varepsilon), (\varepsilon, 54), (86, \varepsilon, 7))$ *and* $Y(\sigma) = AB_1AAAB_1AAB_1B_2$. *Now* $Y(\sigma) \in \mathscr{R}^{(2)}$ *terminates at positive height, and the last step is down, so we use Case* B *to obtain* $\psi(Y(\sigma)) = (AB_1AAB_1, AAB_1AB_2)$. *Thus we have* $g = 4$ *and* $\mu = \{1, 3\}$, $\bar{\mu} = \{2, 4\}$, *so*

$$\sigma_1 = \phi^{-1}(\mathrm{Red}((2, 1), (\varepsilon, 54)), AB_1AAB_1),$$
$$\sigma_2 = \phi^{-1}(\mathrm{Red}((3, \varepsilon), (86, \varepsilon, 7)), AAB_1AB_2),$$

and from Theorem 3.5 *we obtain*

$$\sigma_1 = \phi^{-1}(((2, 1), (\varepsilon, 43)), AB_1AAB_1) = 281954376,$$
$$\sigma_2 = \phi^{-1}((1, \varepsilon), (42, \varepsilon, 3)), AAB_1AB_2) = 179428536.$$

(3) *Let* $\sigma = 2\ 15\ 1\ 13\ 3\ 14\ 10\ 5\ 4\ 12\ 8\ 9\ 6\ 7\ 11 \in \mathscr{TSS}^{(2)}$. *Then* $X(\sigma) = ((2, 1, 3), (\varepsilon, 54), (8, 67))$ *and* $Y(\sigma) = AAB_2AAB_1B_1$. *Now* $Y(\sigma) \in \mathscr{R}^{(2)}$

terminates at height 0, *so we use Case* C *to obtain* $\psi(Y(\sigma))=(AB_1, AAB_2AB_1)$. *Thus we have* $g=3$ *and* $\mu=\{2\}$, $\bar{\mu}=\{1,3\}$, *so*

$$\sigma_1=\phi^{-1}(\mathrm{Red}((\varepsilon, 54)), AB_1),$$

$$\sigma_2=\phi^{-1}(\mathrm{Red}((2,1,3),(8,67)), AAB_2AB_1),$$

and from Theorem 3.5 *we obtain*

$$\sigma_1=\phi^{-1}(((\varepsilon, 21)), AB_1)=3214,$$

$$\sigma_2=\phi^{-1}(((2,1,3),(6,45)), AAB_2AB_1)=2\ 11\ 1\ 9\ 3\ 10\ 6\ 7\ 4\ 5\ 8.$$

Note that Case A of the decomposition in Lemma 3.6 has the following direct description: if $\sigma\in\mathscr{TSS}^{(2)}-\{1\}$ and $Y(\sigma)$ terminates with an up step, then $\sigma_2=1$, and σ_1 is obtained by removing the terminal element a_k from σ and then replacing j by $j-1$ for each $j=a_k+1, ..., |\sigma|$ (just as in Lemma 3.1, although here nothing is marked). As an illustration of this, see part 1 of Example 3.7 above.

Finally we give the following example of reversing the decomposition in Lemma 3.6.

Example 3.8. *Let* $\sigma_1=12\ 2\ 3\ 10\ 1\ 11\ 5\ 6\ 4\ 9\ 8\ 7\in\mathscr{TSS}$ *and* $\sigma_2=6173425\in\mathscr{TSS}^{(1)}$. *Then* $X(\sigma_1)=((\varepsilon, 23, 1), (5,4))$, $Y(\sigma_1)=AAB_2AAAB_1$ *and* $X(\sigma_2)=((\varepsilon, 1), (3,2))$, $Y(\sigma_2)=AB_1AB_1$. *Now* $Y(\sigma_2)\in\mathscr{R}^{(1)}$ *and* $Y(\sigma_1)$ *terminates at positive height, so we reverse Case* B *to obtain*

$$Y(\sigma)=\psi^{-1}(Y(\sigma_1), Y(\sigma_2))=AAB_2AAAB_1AAB_1B_1,$$

and $g=4$, $\mu=\{1,3\}$, *so* $\bar{\mu}=\{2,4\}$. *Thus,*

$$X(\sigma)=\mathrm{Inc}((\varepsilon, 23, 1), (\varepsilon, 1), (5,4), (3,2)),$$

where Inc (*for "increase"*) *is defined as*

$$\mathrm{Inc}(V_1, ..., V_g)=(V_1', ..., V_g'),$$

where V_i' *is obtained from* V_i *by replacing the jth largest integer by* $|V_1|+\cdots+|V_{i-1}|+j$, *for* $i,j\geqslant 1$. *Thus in this case we obtain* $X(\sigma)=((\varepsilon, 23, 1), (\varepsilon, 4), (6,5), (8,7))$, *so finally*

$$\sigma=\phi^{-1}(X(\sigma), Y(\sigma))=19\ 2\ 3\ 17\ 1\ 18\ 13\ 4\ 16\ 6\ 10\ 5\ 15\ 8\ 9\ 7\ 14\ 12\ 11.$$

4. A Bijection between Nonseparable Rooted Planar Maps and Two-Stack-Sortable Permutations

In this section we make explicit the bijection between $\mathcal{NS}$ and $\mathcal{TSS}$ that follows from comparing the decompositions for nonseparable rooted planar maps and two-stack-sortable permutations given in the previous sections.

A *labelled decomposition tree* is a rooted plane tree in which every vertex v has at most two children, with the restriction that if v is itself a right-child of a vertex with two children, then v has at most one child, and a positive integer label $\lambda(v)$ with the following recursively defined restrictions

- if v has no children then $\lambda(v) = 1$,
- if v has one child, v_1, then $1 \leqslant \lambda(v) \leqslant \lambda(v_1)$,
- if v has two children, v_1 and v_2, then $\lambda(v) = \lambda(v_1) + \lambda(v_2)$.

Let $\mathcal{L}$ be the set of all labelled decomposition trees on at least one vertex, let $\mathcal{L}^{(1)}$ be the set of all trees in $\mathcal{L}$ whose root has one child, and let $\mathcal{L}^{(2)} = \mathcal{L} - \mathcal{L}^{(1)}$. An example of a labelled decomposition tree in $\mathcal{L}^{(2)}$ with 27 vertices, and the value of the label beside each vertex, is illustrated in Fig. 2, with the root vertex c_1 (labelled 7) at the top (three other vertices have c_2, c_3, c_4 identifying them for ease of referencing in later examples).

Now associate two statistics $x(v)$ and $y(v)$ with each vertex v of a tree in $\mathcal{L}$ as follows:

- if v has no children then $x(v) = 1$, $y(v) = 0$,
- if v has one child, v_1, then $x(v) = x(v_1) + 1$, $y(v) = y(v_1)$,
- if v has two children, v_1 and v_2, then $x(v) = x(v_1) + x(v_2)$, $y(v) = y(v_1) + y(v_2) + 1$.

For example, in the tree illustrated in Fig. 2, we obtain $x(c_1) = 17$, $x(c_2) = 4$, $x(c_3) = 5$, $x(c_4) = 6$ and $y(c_1) = 10$, $y(c_2) = 1$, $y(c_3) = 2$, $y(c_4) = 3$. Note that the values of these statistics are independent of the labelling λ.

Now for each nonseparable rooted planar map $M \in \mathcal{NS}$ define a unique labelled decomposition tree $\Delta(M)$ recursively as follows:

- if $M = D$ then the root vertex of $\Delta(M)$ has no children,
- if $M \in \mathcal{NS}^{(1)}$ then the root vertex of $\Delta(M)$ has one child, the root vertex of $\Delta(M_1)$, where M_1 is defined in Lemma 2.1;
- if $M \in \mathcal{NS}^{(2)} - \{D\}$ then the root vertex of $\Delta(M)$ has two children; the left-child is the root vertex of $\Delta(M_1)$ and the right-child is the root vertex of $\Delta(M_2)$, where M_1 and M_2 are defined in Lemma 2.2;
- in all cases the root vertex of $\Delta(M)$ has label $r(M)$.

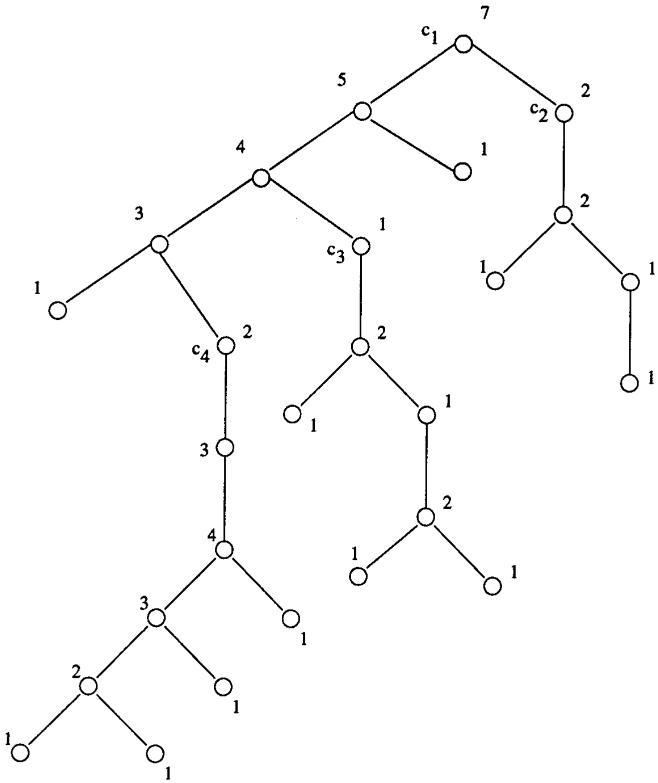

FIG. 2. A labelled decomposition tree.

Thus, to obtain $\Delta(M)$ explicitly, apply Lemmas 2.1 and 2.2 successively to completely decompose M in terms of D. As examples, for the maps $\eta_1, \eta_2, \eta_3, \eta_4$ illustrated in Fig. 3 the decomposition tree $\Delta(\eta_i)$ is the subtree rooted at vertex c_i of the tree in Fig. 2, for $i=1, 2, 3, 4$.

Now $r(D)=1$, $e(D)=1$, and $v(D)=0$. Thus, from the relationships between the parameters r, e, v given in the bijections Lemmas 2.1 and 2.2, we deduce immediately the following result, that Δ is a bijection.

PROPOSITION 4.1. *$\Delta: \mathscr{N}\mathscr{S} \to \mathscr{L}: M \mapsto L$ is a bijection with $r(M)=\lambda(t)$, $e(M)=x(t)$, $v(M)=y(t)$, where t is the root vertex of L. Moreover, Δ maps $\mathscr{N}\mathscr{S}^{(i)}$ to $\mathscr{L}^{(i)}$ for $i=1, 2$.*

Similarly, for each two-stack-sortable permutation $\sigma \in \mathscr{T}\mathscr{S}\mathscr{S}$ define a unique labelled decomposition tree $\Omega(\sigma)$ recursively as follows:

- if $\sigma=1$ then the root vertex of $\Omega(\sigma)$ has no children;

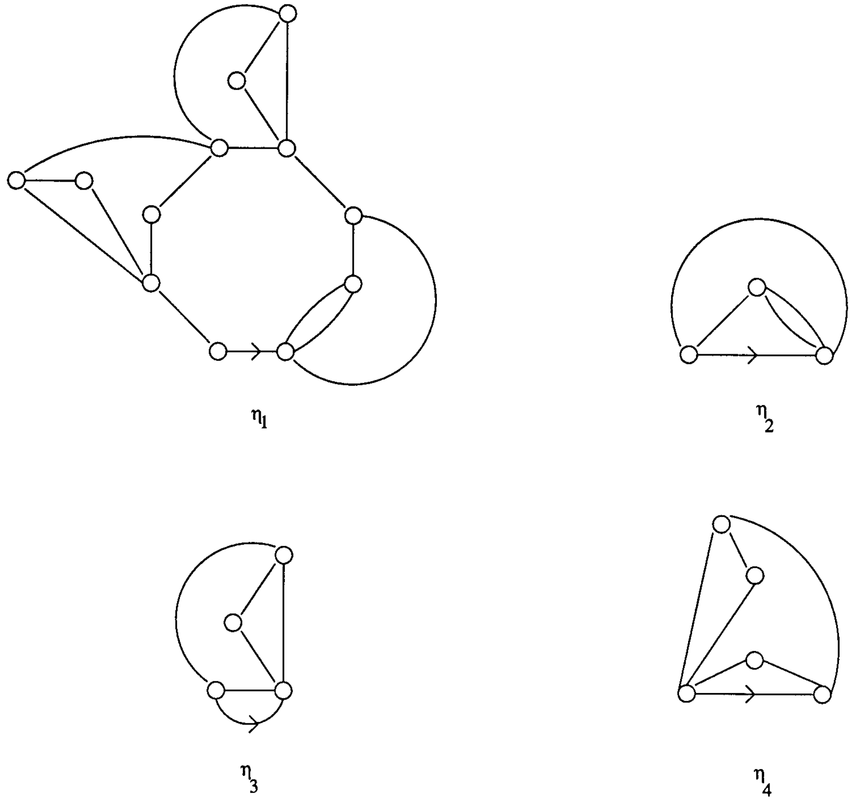

FIG. 3. Four rooted nonseparable planar maps.

- if $\sigma \in \mathscr{T}\mathscr{S}\mathscr{S}^{(1)}$ then the root vertex of $\Omega(\sigma)$ has one child, the root vertex of $\Omega(\sigma_1)$, where σ_1 is defined in Lemma 3.1;
- if $\sigma \in \mathscr{T}\mathscr{S}\mathscr{S}^{(2)} - \{1\}$ then the root vertex of $\Omega(\sigma)$ has two children; the left-child is the root vertex of $\Omega(\sigma_1)$ and the right-child is the root vertex of $\Omega(\sigma_2)$, where σ_1 and σ_2 are defined in Lemma 3.6;
- in all cases the root vertex of $\Omega(\sigma)$ has label $\mathrm{rl}(\sigma)$.

Thus, to obtain $\Omega(\sigma)$ explicitly, apply Lemmas 3.1 and 3.6 successively to completely decompose σ in terms of 1. As examples, for the two-stack-sortable permutations

$$\kappa_1 = 17\ 13\ 2\ 1\ 3\ 16\ 5\ 4\ 12\ 6\ 15\ 7\ 14\ 8\ 11\ 10\ 9, \tag{6}$$

$\kappa_2 = 1423$, $\kappa_3 = 21435$, $\kappa_4 = 642135$, the decomposition tree $\Omega(\kappa_i)$ is the subtree rooted at vertex c_i of the tree in Fig. 2 for $i = 1, 2, 3, 4$.

Now $\mathrm{rl}(1)=1$, $|1|=1$, and $\mathrm{desc}(1)=0$. Thus, from the relationships between the parameters length, rl, desc given in the bijections Lemmas 3.1 and 3.6, we deduce immediately the following result, that Ω is a bijection.

PROPOSITION 4.2. *$\Omega\colon \mathscr{TSS} \rightleftarrows \mathscr{L}\colon \sigma \mapsto L$ is a bijection with $rl(\sigma)=\lambda(t)$, $|\sigma|=x(t)$, $\mathrm{desc}(\sigma)=y(t)$, where t is the root vertex of L. Moreover, Ω maps $\mathscr{TSS}^{(i)}$ to $\mathscr{L}^{(i)}$ for $i=1, 2$.*

We now give our main result, the bijection between $\mathscr{NS}$ and $\mathscr{TSS}$ that follows from comparing Propositions 4.1 and 4.2.

THEOREM 4.3. *$\Omega^{-1}\Delta\colon \mathscr{NS} \rightleftarrows \mathscr{TSS}\colon M \mapsto \sigma$ is a bijection with $r(M)=\mathrm{rl}(\sigma)$, $e(M)=|\sigma|$, $v(M)=\mathrm{desc}(\sigma)$, Moreover, in this bijection*

- *$\mathscr{NS}^{(i)}$ is mapped to $\mathscr{TSS}^{(i)}$ for $i=1, 2$,*
- *the set of all $M \in \mathscr{NS}^{(2)}-\{D\}$ for which M' has m bridges, is mapped to the set of all $\sigma \in \mathscr{TSS}^{(2)}-\{1\}$ for which σ terminates with m descents for each $m \geqslant 0$.*

Proof. The result follows immediately from Propositions 4.1 and 4.2, except for the last part.

For the last part, consider applying Lemma 2.2 to $M \in \mathscr{NS}^{(2)}-\{D\}$, to obtain $M_1 \in \mathscr{NS}$ and $M_2 \in \mathscr{NS}^{(1)} \cup \{D\}$. If $M_1 \in \mathscr{NS}^{(2)}-\{D\}$, apply Lemma 2.2 to M_1 and continue this successively until M has been decomposed into an ordered list of elements of $\mathscr{NS}^{(1)} \cup \{D\}$. Suppose that D occurs m times in this list. Then from the last comment in the proof of Lemma 2.2, M' has m bridges. Moreover, if Lemma 3.6 is applied successively to decompose $\sigma=\Omega^{-1}\Delta(M) \in \mathscr{TSS}^{(2)}-\{1\}$ into an ordered list of elements of $\mathscr{TSS}^{(1)} \cup \{1\}$, then 1 occurs m times in this list (since $\Omega^{-1}\Delta$ has been constructed so that Lemma 3.6 for $\mathscr{TSS}^{(1)}, \mathscr{TSS}^{(2)}$, is isomorphic to Lemma 2.2 for $\mathscr{NS}^{(1)}, \mathscr{NS}^{(2)}$ where 1 is identified with D).

But this means that when the path bijection ψ in the proof of Lemma 3.6 is applied successively to decompose $Y(\sigma)$ into an ordered list of elements of $\mathscr{R}^{(1)} \cup \{A\}$, then A occurs m times in this list. This is equivalent to $Y(\sigma)$ terminating at height m, from the proof that ψ is a bijection, which in turn means that $s_{k-m} \neq \varepsilon$, $s_{k-m+1}=\cdots=s_k=\varepsilon$ in the rlmax decomposition of σ, from part 2 of Proposition 3.2. Thus, $\sigma=s_1 a_1 \cdots s_{k-m} a_{k-m} a_{k-m+1} \cdots a_k$, where $s_{k-m} \neq \varepsilon$, so σ terminates with the m descents (a_{k-m}, a_{k-m+1}), ..., (a_{k-1}, a_k), and the last part of the result follows, since this is reversible. ∎

For example, $\Omega^{-1}\Delta(\eta_i)=\kappa_i$, where η_i is given in Fig. 3 and κ_i is given by (6), for $i=1, 2, 3, 4$. Note that $\eta_1 \in \mathscr{NS}^{(2)}-\{D\}$ and η_1 has two bridges, whereas κ_1 terminates with the two descents $(11, 10)$, $(10, 9)$.

The analysis in the proof of Theorem 4.3 can be extended to prove a refinement involving cut-vertices as well as bridges, although the description of the associated permutations involves the path $Y(\sigma)$, instead of the permutation σ itself, as follows. For each $m \geqslant 0$ and $i_1, ..., i_{m+1} \geqslant 0$, let $\mathcal{TSS}^{(2)}(m; i_1, ..., i_{m+1})$ be the set of all $\sigma \in \mathcal{TSS}^{(2)}$ such that $Y(\sigma)$ terminates at height m, the last i_{m+1} steps are down, and for each $j=1, ..., m$, the last step up to height j in σ is immediately preceded by i_j down steps. For $M \in \mathcal{NS}^{(2)}$ with m bridges in M', index the bridges of M' from $1, ..., m$ as they are encountered when traversing the root face of M, beginning at the root vertex of M and ending at the nonroot vertex incident with the root edge of M. Now remove the m bridges (but not their incident vertices) from M' to obtain an ordered list of maps $M_1, ..., M_{m+1}$, where M_j was incident with bridge $j-1$ and j in M' for $j=2, ..., m$, and M_1 was incident with bridge 1 only, M_{m+1} was incident with bridge m only. Let $\mathcal{NS}^{(2)}(m; i_1, ..., i_{m+1})$ be the set of all $M \in \mathcal{NS}^{(2)}$ such that M' has m bridges, and for each $j=1, ..., m+1$, the map M_j as defined above has $i_j - 1$ cut-vertices if $i_j \geqslant 2$, the map M_j has no cut-vertices and is not the single-vertex map if $i_j = 1$, and the map M_j is the single-vertex map if $i_j = 0$. Then $\Omega^{-1}\Delta$ maps $\mathcal{NS}^{(2)}(m; i_1, ..., i_{m+1})$ to $\mathcal{TSS}^{(2)}(m; i_1, ..., i_{m+1})$ for each $m \geqslant 0$ and $i_1, ..., i_{m+1} \geqslant 0$.

Note that, in terms of this description, $\mathcal{NS}^{(2)}(1; 0, 0) = \{D\}$ and $\mathcal{TSS}^{(2)}(1; 0, 0) = \{1\}$. For a more substantial example, we have $\eta_1 \in \mathcal{NS}^{(2)}(2; 0, 2, 1)$ for η_1 given in Fig. 3. Now, as we have shown previously, $\Omega^{-1}\Delta(\eta_1) = \kappa_1$, where κ_1 is given in (6), and we have $Y(\kappa_1) = AAAAB_2B_1AAAB_2$, so indeed $\kappa_1 \in \mathcal{TSS}^{(2)}(2; 0, 2, 1)$, as required.

5. Concluding Remarks

5.1. *Comparison with the Previous Bijection*

In order to compare our bijection to Dulucq, Gire, and Guibert [3], we applied their bijection between the 91 maps in $\mathcal{NS}$ with six edges and the 91 permutations in $\mathcal{TSS}$ with five elements. We found the bijections to act quite differently for this case, so we conclude that the bijection in this paper is different from theirs. Moreover, their bijection is sufficiently different in this case that it does not map $\mathcal{NS}^{(i)}$ to $\mathcal{TSS}^{(i)}$, for $i=1, 2$, even in a permuted order.

5.2. *Another Bijection*

We have found a second bijection between $\mathcal{NS}$ and $\mathcal{TSS}$ using Raney paths. The details are parallel, so we give only the following sketch.

Let $\mathcal{TSS}^{(3)}$ consist of elements of $\mathcal{TSS}$ in which (i) s_k is not empty and (ii) either $a_k = n$ or $a_k = a_{k-1} - 1$. Then Lemma 3.1 holds with

$\mathscr{TSS}^{(1)}$ replaced by $\mathscr{TSS}^{(3)}$; we use the same construction (remove a_k and reduce everything larger by 1), but reverse it differently, with an appropriately modified convention for the marked rlmax. Let $\mathscr{R}^{(3)}$ be the Raney paths that terminate at height 0 and either the last two up steps are consecutive or the path is AB_1, and let $\mathscr{E}^{(3)}$ consist of pairs (V, P) in $\mathscr{E}$ with $P \in \mathscr{R}^{(3)}$. As before, let $\mathscr{TSS}^{(4)}$, $\mathscr{R}^{(4)}$ and $\mathscr{E}^{(4)}$ be the complements of $\mathscr{TSS}^{(3)}$, $\mathscr{R}^{(3)}$, and $\mathscr{E}^{(3)}$, respectively. Then Theorem 3.5 holds with ϕ mapping $\mathscr{TSS}^{(i)}$ to $\mathscr{E}^{(i)}$, for $i=3, 4$. Finally, Lemma 3.6 holds with $\mathscr{TSS}^{(1)}$, $\mathscr{TSS}^{(2)}$ replaced by $\mathscr{TSS}^{(3)}$, $\mathscr{TSS}^{(4)}$, respectively; in the proof an appropriate replacement for the path bijection ψ is straightforward. Thus we obtain a bijection in which $\mathscr{NS}^{(i)}$ is mapped to $\mathscr{TSS}^{(i+2)}$ for $i=1, 2$.

5.3. *Raney Paths and Lagrange's Theorem*

The term "Raney path" is used because this is the level-free subset of the set of paths used by Raney [9] in his combinatorial proof of Lagrange's implicit function theorem. This is not just a coincidence; in fact the generating function for two-stack-sortable permutations can actually be obtained as a functional composition by this identification, as follows.

Let $\mathscr{G}$ be the set of Raney paths that terminate at height 0. We first consider generating series for sets of Raney paths, in which the weight of a path is a product of the following contribution for each step: up steps contribute t and down steps by j contribute tf_{j+1}, for $j \geqslant 1$, and all these indeterminates commute. For example the path $AAAB_2AAB_3$ thus has weight $t^7 f_3 f_4$. Let G, R, $R^{(i)}$ be the generating series for $\mathscr{G}$, $\mathscr{R}$, $\mathscr{R}^{(i)}$, respectively, for $i=1, 2$, summing this weight over all paths in the sets. We immediately obtain the functional equation

$$G = \sum_{j \geqslant 1} f_{j+1} t^{j+1} (1+G)^{j+1},$$

by noting that the terminal step of a path in $\mathscr{G}$ must be down by j for some unique $j \geqslant 1$. Then identify the j up steps that are the last up steps to height h, for each $h=1, ..., j$. The terminal down step contributes tf_{j+1} and the up steps each contribute t to the weight, and these steps split the path into $j+1$ segments, each of which is either an element of $\mathscr{G}$ or an empty path. Hence the functional equation above follows.

The generating functions for the sets of Raney paths with which we have been concerned in this paper can be expressed in terms of G in a straightforward manner, using combinatorial decompositions including the bijection ψ of Lemma 3.6. Thus we obtain

$$R^{(1)} = \frac{G}{1+G}, \qquad R^{(2)} = \frac{G^2}{1+G} + \frac{t(1+G)^2}{1-t(1+G)}, \qquad 1+R = \frac{1+G}{1-t(1+G)}.$$

But if we let $W = t(1+G)$, then the functional equation for G above becomes $W = tf(W)$, where $f(x) = 1 + \sum_{j \geqslant 2} f_j x^j$. Thus W satisfies Lagrange's functional equation for an arbitrary formal power series f, with constant term 1 and no linear term (the linear term would correspond to levels in the paths).

Now Theorem 3.5 tells us that the generating series for $\mathscr{TSS}^{(i)}$ with respect to length, rlmax, and descents can be obtained by substitution of indeterminates in $R^{(i)}$ for $i = 1, 2$. In particular, f_{j+1} is replaced by the generating function for $j+1$-tuples of strings $(s_1, ..., s_{j+1})$ such that $\pi^2(s_1, ..., s_{j+1}) = \iota$. This is reminiscent of the equation obtained by Zeilberger [12], but we have been unable to obtain new insights by this methodology, nor to identify Zeilberger's second parameter in the map context. Also, Gessel [5] refined Raney's methodology to give a q-analogue of Lagrange's theorem recording the area under the path, but again we do not know whether area under a Raney path can be identified with natural statistics in the context of this paper. These seem worthwhile topics of further research.

5.4. *j-Stack-Sortable Permutations*

For *j*-stack-sortable permutations, a similar result to Proposition 3.2 holds, in which the "biggest" elements will form essentially *j* interleaved decreasing subsequences. It seems that an associated *j*-dimensional lattice path (or, equivalently, an *j*-rowed tableau with "holes") might be used to shed light in this case, and this seems also a worthwhile topic of further research.

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