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Dyck paths and a bijection for multisets of hook numbers

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Abstract

We give a bijective proof of a result of Regev and Vershik (Electron J. Combin. 4 (1997) R22) on the equality of two multisets of hook numbers of certain skew–Young diagrams. The bijection is given in terms of Dyck paths, a particular type of lattice path. It is extended to also prove a recent, more refined result of Regev (European J. Combin. 21 (2000) 959), which concerns a special class of skew diagrams. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Hook number; Bijection; Dyck path; Projective representation

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1. Introduction

Let n, k be positive integers, and $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a partition with at most k parts, each part at most n, so $n \geqslant \alpha_1 \geqslant \cdots \geqslant \alpha_k \geqslant 0$. The *Young diagram* of α is given by

$$D = \{(i,j) \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \alpha_{k-i+1}\},$$

a collection of unit cells, arranged in rows and columns. Here cell (i,j) appears in row i and column j, rows numbered from bottom to top, and columns numbered from left to right. We regard translates of the diagram in the plane as equivalent, and generally place the bottom-left cell at $(1,1)$. (Note, however that this is not the case for D above when $\alpha_k = 0$.) Also let

$$R = \{(i,j) \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n\},$$

$$T = \{(i,j) \mid 1 \leqslant i \leqslant k, \alpha_1 - \alpha_i + 1 \leqslant j \leqslant n + \alpha_1 - \alpha_i\},$$

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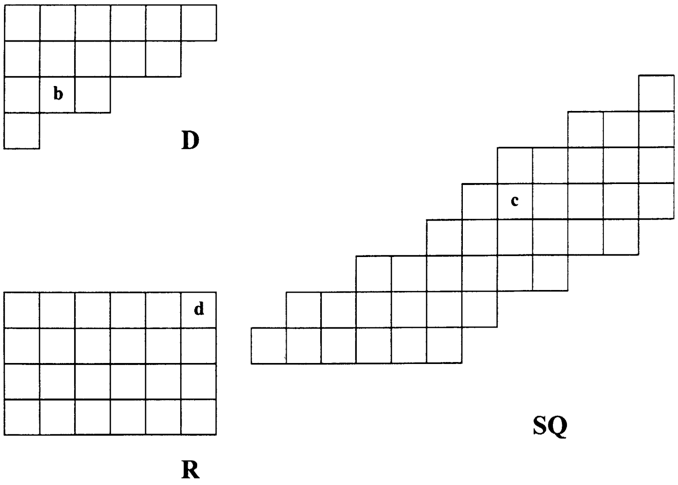

Fig. 1. D, R, SQ for $n=6, k=4, \alpha=(6,5,3,1)$.

$$V=\{(i,j)\mid k+1\leqslant i\leqslant 2k, n+\alpha_1-\alpha_{i-k}+1\leqslant j\leqslant n+\alpha_1\},$$

$$\mathrm{SQ}=T\cup V,$$

so R, T, SQ are *skew diagrams* (in fact, R is also a Young diagram, the $k\times n$ rectangle).

For a skew diagram G, let G^* be the skew diagram obtained by rotating G through 180°. Thus, for example,

$$T^*=\{(i,j)\mid 1\leqslant i\leqslant k, \alpha_{k-i+1}-\alpha_k+1\leqslant j\leqslant n+\alpha_{k-i+1}-\alpha_k\}.$$

Also, let $G^\dagger$ be the collection of cells obtained by reflecting G about a vertical axis.

The *arm length* $a_G(x)$ of a cell x in a skew diagram G is the number of cells of G in the same row of x and to the right of x; the *leg length* $l_G(x)$ of a cell x in a skew diagram G is the number of cells of G in the same column and below. The *coleg* length of a cell x in a skew diagram is the number of cells in the same column and above. The *hook length* $h_G(x)$ is given by $h_G(x)=a_G(x)+l_G(x)+1$. If E is a subset of the cells of G, then $\mathrm{AL}_G(E)$ is the multiset $\{(a_G(x), l_G(x))\mid x\in E\}$, and $H_G(E)$ is the multiset $\{h_G(x)\mid x\in E\}$. When there is no ambiguity, we write $H_G(G)$ as $H(G)$, and $\mathrm{AL}_G(G)$ as $\mathrm{AL}(G)$.

For example, the skew diagrams D, R, SQ are illustrated in Fig. 1 for the case $n=6$, $k=4$, $\alpha=(6,5,3,1)$. For the three cells labelled b, c, d in Fig. 1, we have $a_D(b)=1, l_D(b)=0, a_{\mathrm{SQ}}(c)=4, l_{\mathrm{SQ}}(c)=2$ and $a_R(d)=0, l_R(d)=3$.

Theorem 1.1 below was conjectured by Regev and Vershik [6], and proved by Regev and Zeilberger [7], Janson [2], and Bessenrodt [1] (though only for the case $n=\alpha_1$ in [7]).

Theorem 1.1. *For all* n, k, α,

$$H(\mathrm{SQ}) = H(R) \cup H(D)$$

is a multiset identity.

Regev and Zeilberger note that their proof is not bijective, and ask for a canonical bijection between the multisets. Bessenrodt [1] presents such a bijection, deducing it from a general result about "removable" hooks in Young diagrams. In this paper, we present a different bijection, deducing it from another general result, the main result of the paper. It is convenient to keep arm and leg lengths separately, and thus we prove the following result, which is obviously a generalization of Theorem 1.1.

Theorem 1.2. *For all* n, k, α,

$$\mathrm{AL}(\mathrm{SQ}) = \mathrm{AL}(R) \cup \mathrm{AL}(D)$$

is a multiset identity.

The next result, our main result, is more symmetric and natural looking than Theorem 1.2, but it implies Theorem 1.2. Independently, Theorems 1.2 and 1.3 have also been obtained by Regev [4], and bijective proofs that are different from ours have been given by Krattenthaler [3]. (Note that the bijection that we give for Theorem 1.3 actually yields the same bijection as [3], but it has a different description. The bijection that we give for Theorem 1.2 is quite different, since it is based on a different partitioning, and allows us to apply our proof of Theorem 1.3 directly.)

Theorem 1.3. *For all* n, k, α,

$$\mathrm{AL}(T) = \mathrm{AL}(T^*)$$

is a multiset identity.

We delay the proof of Theorem 1.3 until the next section, and proceed now by giving a bijective proof that it implies Theorem 1.2. The proof involves partitioning the cells of R and T^* into two regions each, and identifying cells in various regions of skew diagrams whose pairs of arm and leg lengths are immediately equal.

Proof that Theorem 1.3 implies Theorem 1.2. Partition the cells of R into two subsets R_1 and R_2, given by

$$R_1 = \{(i,j) \mid 1 \leqslant i \leqslant k, n - \alpha_{k-i+1} + 1 \leqslant j \leqslant n\},$$

$$R_2 = \{(i,j) \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n - \alpha_{k-i+1}\}$$

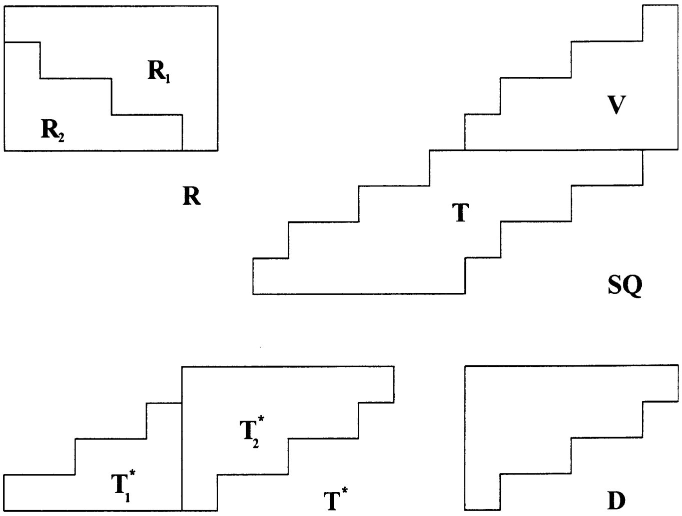

Fig. 2. Skew shapes for $n=6, k=4, \alpha=(6,5,3,1)$.

and the cells of T^* into two subsets T_1^* and T_2^*, given by

$$T_1^* = \{(i,j) \mid 1 \leqslant i \leqslant k, \alpha_{k-i+1} - \alpha_k + 1 \leqslant j \leqslant n - \alpha_k\},$$

$$T_2^* = \{(i,j) \mid 1 \leqslant i \leqslant k, n - \alpha_k + 1 \leqslant j \leqslant n + \alpha_{k-i+1} - \alpha_k\}.$$

The significance of these regions in this proof is that $R_1^\dagger = T_2^* = V^* = D$ and $R_2^\dagger = T_1^*$. These equalities (using appropriate translations) are immediate from the definitions of the regions. See Fig. 2 for an illustration of these regions in the case $n=6, k=4$, $\alpha=(6,5,3,1)$, and to check visually the above equalities in this case.

Bijective identification of $\mathrm{AL}_{\mathrm{SQ}}(V)$ *and* $\mathrm{AL}_R(R_1)$: Now $V^* = R_1^\dagger$, so the jth columns of V and R, respectively, have the same lengths, for each $j=1,\ldots,\alpha_1$. Furthermore, V appears in SQ with cells added below V to extend all columns of V to length k. Similarly, R_1 appears in R with cells added below R_1 to extend all columns of R_1 to length k. Thus, the arm and leg lengths are equal, for the cells that are i rows from the topmost entry, in the jth column from the left most column, of V in SQ and R_1 in R, respectively. Thus we establish immediately that

$$\mathrm{AL}_{\mathrm{SQ}}(V) = \mathrm{AL}_R(R_1). \tag{1}$$

Bijective identification of $\mathrm{AL}_{T^*}(T_1^*)$ *and* $\mathrm{AL}_R(R_2)$: Now $T_1^* = R_2^\dagger$, so the ith rows of T_1^* and R_2, respectively, have the same lengths, for each $i=1,\ldots,k$ (some of these lengths are zero when $\alpha_1 = n$). Furthermore, T_1 appears in T with cells added to the right of T_1 to extend all rows of T_1 to length n. Similarly, R_2 appears in R with cells added to the right of R_2 to extend all rows of R_2 to length n. Thus, the arm lengths and leg lengths are equal, for the cells that are j columns from the left most entry, in the ith row from the bottom row, of T_1^* in T^* and R_2 in R, respectively. Thus we establish immediately that

$$\mathrm{AL}_{T^*}(T_1^*) = \mathrm{AL}_R(R_2). \tag{2}$$

Bijective identification of $\mathrm{AL}_{T^*}(T_2^*)$ *and* $\mathrm{AL}(D)$: Now $T_2^* = D$, and T_2^* appears in T^* with no cells added to the right nor below, so we establish immediately that

$$\mathrm{AL}_{T^*}(T_2^*) = \mathrm{AL}(D). \tag{3}$$

The result: Suppose Theorem 1.3 is true. Then, applying (1), we obtain

$$\mathrm{AL}_{\mathrm{SQ}}(V) \cup \mathrm{AL}(T) = \mathrm{AL}_R(R_1) \cup \mathrm{AL}(T^*). \tag{4}$$

But $\mathrm{AL}(T) = \mathrm{AL}_{\mathrm{SQ}}(T)$, since T appears in SQ with no cells added to the right nor below. Also, $\mathrm{AL}(T^*) = \mathrm{AL}_{T^*}(T_1^*) \cup \mathrm{AL}_{T^*}(T_2^*)$, since T_1^* and T_2^* partition the cells of T^*. Making these substitutions into (4) gives

$$\begin{aligned}\mathrm{AL}_{\mathrm{SQ}}(V) \cup \mathrm{AL}_{\mathrm{SQ}}(T) &= \mathrm{AL}_R(R_1) \cup \mathrm{AL}_{T^*}(T_1^*) \cup \mathrm{AL}_{T^*}(T_2^*)\\ &= \mathrm{AL}_R(R_1) \cup \mathrm{AL}_R(R_2) \cup \mathrm{AL}(D),\end{aligned}$$

with the second equality from (2) and (3). Now V and T partition the cells of SQ, and R_1 and R_2 partition the cells of R, so the above result becomes $\mathrm{AL}(SQ) = \mathrm{AL}(R) \cup \mathrm{AL}(D)$, and we have established Theorem 1.2. □

How is this proof bijective? To prove Theorem 1.3 bijectively, in the next section we determine an explicit bijection $\phi: T \to T^*$, that preserves arm and leg lengths (this means that for each cell $x \in T$ we have $a_T(x) = a_{T^*}(\phi(x))$ and $l_T(x) = l_{T^*}(\phi(x))$). Similarly, to give a bijective proof of Theorem 1.2, we must determine an explicit bijection $\psi: \mathrm{SQ} \to R \cup D$, that preserves arm and leg lengths.

In terms of ϕ, we now describe such a bijection ψ that is implicit in the above proof. First, note that, to establish (1)–(3) above, we have described three simple bijections, and let us call them $\zeta_1: V \to R_1$, $\zeta_2: T_1^* \to R_2$, and $\zeta_3: T_2^* \to D$.

A bijection ψ that establishes Theorem 1.2. For $x \in \mathrm{SQ}$, we obtain $\psi(x) \in R \cup D$ as follows:

For $x \in V$, let $\psi(x) = \zeta_1(x)$.

For $x \in T$,

- if $\phi(x) \in T_1^*$, let $\psi(x) = \zeta_2(\phi(x))$,
- if $\phi(x) \in T_2^*$, let $\psi(x) = \zeta_3(\phi(x))$.

This clearly specifies a bijection ψ of the required type, giving a bijective proof of Theorem 1.2.

2. Dyck paths and the bijection

In this section, we determine a bijection $\phi : T \to T^*$, that preserves arm and leg lengths, as referred to above at the end of Section 1. This provides a bijective proof of Theorem 1.3.

The bijection is described in terms of a particular type of lattice path that will be associated with T and T^*, called a Dyck path. A *Dyck path* of length $2k, k \geqslant 0$, is a sequence (i, y_i), $i = 0,\dots,2k$, of lattice points in the plane, in which $y_0 = y_{2k} = 0$, $y_i \geqslant 0$, for $i = 1,\dots,2k-1$, and $y_i - y_{i-1} = +1$ or -1, for $i = 1,\dots,2k$. Equivalently, a Dyck path is completely specified by its sequence of *steps*; if $y_i - y_{i-1} = +1$ then the ith step is an up step, and if $y_i - y_{i-1} = -1$ then the ith step is a down step. The *height* of the ith step is y_{i-1}, for $i = 1,\dots,2k$. Since $y_{2k} = 0$, then the $2k$ steps consist of k up steps and k down steps. We can visualize a Dyck path as a connected path in the plane by drawing a line segment between the consecutive lattice points in the path.

Let the skew diagrams $T_{[i]}$ and $T_{(i)}$, for $i = 1,\dots,n$, be given by

$$T_{[i]} = \{x \in T \mid a_T(x) = i - 1\},$$

$$T_{(i)} = \{x \in T \mid a_T(x) \leqslant i - 1\}$$

and define $(T^*)_{[i]}$ and $(T^*)_{(i)}$ in the same way. Consider the skew diagram $T_{(i)}$, for each fixed $i = 1,\dots,n$. Label the k cells of $T_{[i]}$ in $T_{(i)}$, successively, $x_1,\dots,x_k$, from bottom to top (there is exactly one cell of $T_{[i]}$ in each of the k rows of $T_{(i)}$). Label the cells of $T_{[0]}$ in $T_{(i)}$, successively, $z_1,\dots,z_k$, from top to bottom (similarly, there is exactly one cell of $T_{[0]}$ in each of the k rows of $T_{(i)}$). In the case $i = 1$, then each cell of $T_{[0]}$ will have two labels, one an x_j and the other z_{k+1-j}, for some $j = 1,\dots,k$.

Now form a permutation σ_i of $x_1,\dots,x_k$, $z_1,\dots,z_k$ as follows: Place the x's and z's from left to right in σ_i in the order that they appear from left to right as labels in the cells of $T_{(i)}$. For labels in the same column of $T_{(i)}$, order them with the x's first, followed by the z's; the x's are ordered as they appear from bottom to top in the same column, and the z's from bottom to top also. For example, in the case $n = 11$, $k = 9$, $\alpha = (11, 11, 9, 8, 8, 6, 3, 1, 0)$, we illustrate $T_{(3)}$ in Fig. 3, with the cells labelled as described above. In this case, the permutation σ_3 is given by

$$\sigma_3 = x_1x_2x_3z_9z_8x_4x_5z_7x_6z_6z_5z_4x_7x_8z_3x_9z_2z_1.$$

Now let ρ_i be the lattice path starting at $(0, 0)$, whose steps are specified by σ_i as follows: the x_j's specify the up steps (labelled x_j), and the z_j's specify the down steps (labelled z_j). For example, the lattice path ρ_3 determined from σ_3 in the example above is illustrated in Fig. 4.

It is a straightforward induction to prove that the height of the up step labelled x_j in ρ_i is equal to the leg length of the cell labelled x_j in $T_{(i)}$, and that the height of the down step labelled z_j in ρ_i is equal to one more than the coleg length of the cell labelled z_j in $T_{(i)}$. But since leg and coleg lengths are always nonnegative, the height of every up step in ρ_i is nonnegative, and the height of every down step in ρ_i is positive, so ρ_i is a Dyck path. For example, the lattice path ρ_3 illustrated in Fig. 4 is clearly a Dyck path.

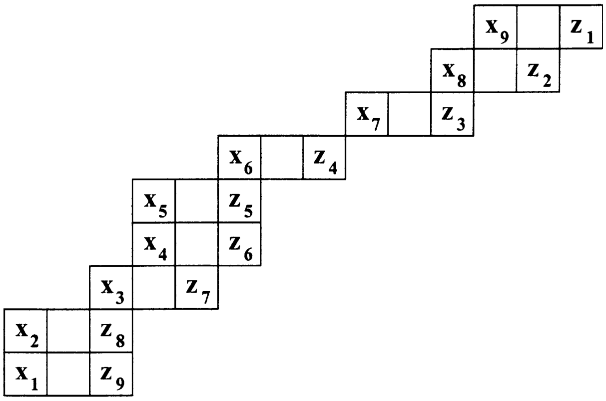

Fig. 3. $T_{(3)}$ for $n=11, k=9, \alpha=(11,11,9,8,8,6,3,1,0)$.

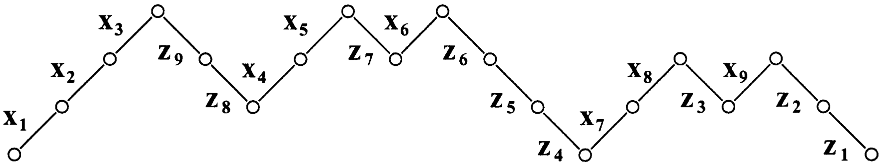

Fig. 4. The Dyck path ρ_3 determined from σ_3.

Now there is a natural bijection between the up steps and down steps in a Dyck path: pair each up step at height j with the first down step at height $j+1$ occurring after that up step (there must be such a down step since the path ends at a vertex with ordinate equal to 0, and down steps decrease the value of the ordinate by exactly 1 for each step). Suppose that the up step labelled x_j is paired with the down step labelled $z_{P_i(j)}$ in this way, for $j=1,\ldots,k$. Then P_i is a bijection on $\{1,\ldots,k\}$, for each fixed i. For example, for the Dyck path illustrated in Fig. 4, we have $P_3(1)=4$, $P_3(2)=8$, $P_3(3)=9$, $P_3(4)=5$, $P_3(5)=7$, $P_3(6)=6$, $P_3(7)=1$, $P_3(8)=3$, and $P_3(9)=2$.

Now rotate $T_{(i)}$, with its cells labelled as above, through 180°, to obtain δ. Now $\delta=(T_{(i)})^*=(T^*)_{(i)}$, and the cells of $T_{[0]}$ in $T_{(i)}$, labelled with z_j's, become the cells of $(T^*)_{[i]}$ in δ. Moreover, the coleg length of a cell labelled z_j in $T_{(i)}$ equals the leg length of the corresponding cell in δ, so

$$l_{T_{(i)}}(x_j)=l_{(T^*)_{(i)}}(z_{P_i(j)}),$$

where, for example, $l_{T_{(i)}}(x_j)$ means the leg length of the cell labelled x_j in $T_{(i)}$. Also,

$$a_{T_{(i)}}(x_j)=i-1=a_{(T^*)_{(i)}}(z_{P_i(j)})$$

since all cells in $T_{[i]}$ and $(T^*)_{[i]}$ have arm length equal to $i-1$, for each fixed i. But $T_{(i)}$ appears in T with no cells added to the right nor below, so $l_{T_{(i)}}(x_j)=l_T(x_j)$ and $a_{T_{(i)}}(x_j)=a_T(x_j)$. Similarly, $l_{(T^*)_{(i)}}(z_{P_i(j)})=l_{T^*}(z_{P_i(j)})$ and $a_{(T^*)_{(i)}}(z_{P_i(j)})=a_{T^*}(z_{P_i(j)})$. Thus, putting these equalities together, we have

$$l_T(x_j)=l_{T^*}(z_{P_i(j)}), \quad a_T(x_j)=a_{T^*}(z_{P_i(j)}). \tag{5}$$

Proof of Theorem 1.3. This follows from Lemma 2.1 immediately, These equations imply that the mapping from the cell labelled x_j in T to the cell labelled $z_{P_i(j)}$ in T^*, for each $i=1,\dots,n$, is arm and leg length preserving, so we have found the bijection ϕ that we require, as stated below. □

A bijection ϕ that establishes Theorem 1.3. For $w\in T$, we obtain $\phi(w)\in T^*$ as follows. Each w is contained in $T_{[i]}$ for some unique $i=1,\dots,n$. If w has label x_j in $T_{(i)}$, then $\phi(w)$ is the cell with label $z_{P_i(j)}$ in $(T^*)_{(i)}$.

This clearly specifies a bijection, that is arm and leg length preserving from (5), giving a bijective proof of Theorem 1.3.

3. The projective case

A refinement of Theorem 1.2 has been given by Regev [5], in which the partition $\alpha=(\alpha_1,\dots,\alpha_k)$ has a special form. In order to state this result, we require some adaptations of the notation in Section 1. Let $n=k+1$, and α have the form $\alpha=(\lambda_1,\dots,\lambda_m|\lambda_1-1,\dots,\lambda_m-1)$, in *Frobenius notation*, where $k\geqslant\lambda_1>\cdots>\lambda_m>0$, so $\lambda=(\lambda_1,\dots,\lambda_m)$ is a partition with m distinct parts. This means that D, the Young diagram of α, has exactly m cells on the (top-left to bottom-right) *diagonal*, given by the cells $(k+1-j,j)$, for $j=1,\dots,m$, with λ_j cells to the right of the jth of these cells in row $k+1-j$, and λ_j-1 cells below this cell in column j. Let $\mathscr{B}$ consist of all partitions α of this form, for any $m\geqslant 1, k\geqslant 1$ (e.g., R is the Young diagram of a partition in $\mathscr{B}$, with $m=k$ and $\lambda_j=k+1-j$, for $j=1,\dots,k$).

For a Young diagram G, let $p(G)$ consist of the cells of G on or below the diagonal (as described above), and let $q(G)$ consist of the cells of G above the diagonal. For a skew diagram, extend this notation by describing the diagonal: for T, SQ, where $n=k+1$, and $\alpha\in\mathscr{B}$, the diagonal consists of the cells $(k+1-j,\alpha_1+j)$, for $j=1,\dots,k-m$; for T^*, the diagonal consists of the cells $(k+1-j,k+1-\alpha_k+j)$, for $j=1,\dots,m$. For example, the skew diagrams D,R,SQ,T are illustrated in Fig. 5 for the case $k=5$, $m=2$, $\alpha=(5,4,2,1)$, corresponding to $\lambda=(4,2)$. In each of these skew diagrams, there is a thick line extending from top left to bottom right, which partitions the diagram G into the cells of $p(G)$, below and to the left of the line, and the cells of $q(G)$, above and to the right of the line.

The following result has been given by Regev [5], whose proof is not bijective. A bijective proof has been given by Krattenthaler [3]. In the remainder of this paper, we present a different bijective proof, which directly applies the bijection of Section 2, but with some more detailed analysis needed.

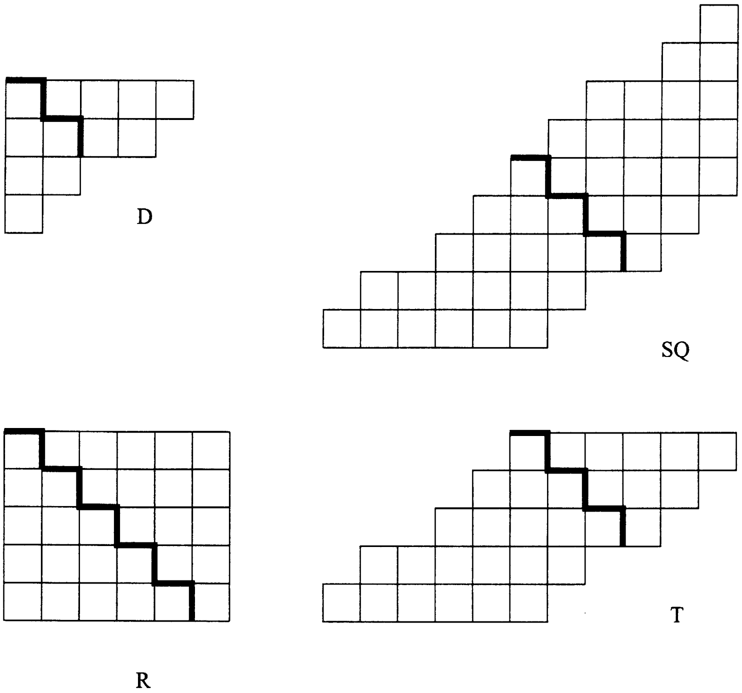

Fig. 5. D, R, SQ, T for $k=5, m=2, \alpha=(5,4,2,1)$.

Theorem 3.1. *For all* k, m *and* $\alpha \in \mathscr{B}$,

$$\mathrm{AL}(p(\mathrm{SQ})) = \mathrm{AL}(p(R)) \cup \mathrm{AL}(q(D))$$

is a multiset identity.

In order to prove Theorem 3.1, we first note that

$$\mathrm{AL}(p(\mathrm{SQ})) = \mathrm{AL}(p(T)) \tag{6}$$

so we shall work with T on the left-hand side of the result, instead of SQ. For each $i=1,\ldots,k+1$, let u be the smallest row index among the elements of $T_{[i]}$ above the diagonal of T. Let T^i be the skew diagram obtained from T by shifting rows $u, u+1,\ldots,k$ to the right, where necessary, so that the right most of the $k+1$ cells in each of these rows occurs in column α_1+k+1. (If such a u exists, then T^i is actually the skew diagram T corresponding to the partition $(\alpha_1,\ldots,\alpha_{u-1})$. If no element of $T_{[i]}$ is above the diagonal of T, then we define $T^i = T$.) The diagonals of T^i and T^{i^*} are the same as for T and T^*, respectively, except that we might shift the diagram and diagonal to position the bottom left most cell at $(1,1)$. For example, the skew diagrams T^i, T^{i^*} are illustrated in Fig. 6 for the case $k=5$, $\alpha=(5,4,2,1)$, with $i=3$, for which $u=4$. In each of these skew diagrams, there is again a thick line partitioning the cells

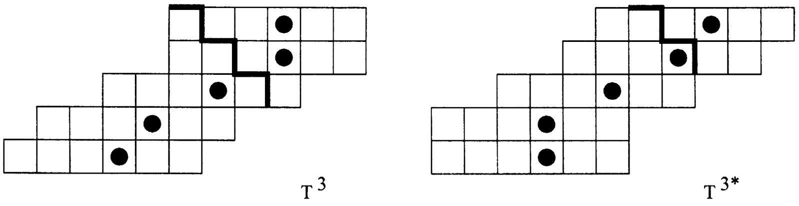

Fig. 6. T^i and T^{i^*} for $k=5, \alpha=(5,4,2,1), i=3$.

into those given by p and q, and there is a dot in every cell with arm length equal to $i-1=2$.

We require the following technical result about the row index u, chosen above for each i.

Proposition 3.2. *Let $\alpha \in \mathscr{B}$, with the diagonal of length m, and with $\alpha_1 \leqslant k+1$. Let u be the smallest row index among the elements of $T_{[i]}$ above the diagonal of T. Then*

1. $u-\alpha_u > i-1$ *and* $u-1-\alpha_{u-1} \leqslant i-1$,
2. $u > m$,
3. $\alpha_{u-i} \geqslant u$ *and* $\alpha_{u-i+1} \leqslant u$,
4. $\alpha_u + i \leqslant \alpha_{u-i}$ *and* $\alpha_{u-1} + i \geqslant \alpha_{u-i+1}$.

Proof. In the row of T with index a, for $a=1,\ldots,k$, the diagonal cell is in column $\alpha_1+k+1-a$, the right most element is in column $\alpha_1+k+1-\alpha_a$, and the unique element of $T_{[i]}$ is therefore in column $\alpha_1+k+1-\alpha_a-(i-1)$. This means that the element of $T_{[i]}$ in row a is above the diagonal of T exactly when $\alpha_1+k+1-\alpha_a-(i-1) > \alpha_1+k+1-a$, or $a-\alpha_a > i-1$. Part 1 of the result follows immediately.

From Part 1, we have $u-\alpha_u > i-1 \geqslant 0$, so $\alpha_u < u$. But, since $\alpha \in \mathscr{B}$, then $\alpha_j \geqslant j$ for $j=1,\ldots,m$, where m is the length of the diagonal of α, giving Part 2 of the result.

Now let $u-\alpha_u = c$ and $u-1-\alpha_{u-1} = d$, where $c > i-1 \geqslant d$, from Part 1. Thus in the Young diagram D of α, the right most cell in row $k+1-u$ is in column $u-c$, and the right most cell in row $k+1-(u-1)$ is in column $u-1-d$. But $\alpha \in \mathscr{B}$, so symmetry of $\mathscr{B}$ implies that the bottom cell in column $u+1$ is in row $k+1-(u-c)$, and the bottom cell in column u is in row $k+1-(u-1-d)$. Thus we have $\alpha_{u-c} \geqslant u+1$, $\alpha_{u+1-c} = \cdots = \alpha_{u-1-d} = u$, $\alpha_{u-d} < u$, and Result 3 follows from $c > i-1 \geqslant d$.

Part 4 follows immediately from Parts 1 and 3. □

Now we are able to give a bijective proof of Theorem 1.3, using the bijective proof of Theorem 1.3.

Proof of Theorem 3.1. Let M_1, M_2, M_3, M_4 be the multisets of leg lengths of the cells with arm lengths equal to $i-1$, in $T^i, (T^i)^*, p(T), q(D)$, respectively. Now, Theorem 1.3 applied to skew diagram T^i gives a bijection between $\mathrm{AL}(T^i)$ and $\mathrm{AL}((T^i)^*)$, which contains a bijection betweem M_1 and M_2.

Now, the elements of M_1 can be partitioned into two subsets: M_{11}, corresponding to the cells on or below the diagonal of T^i; and M_{12}, corresponding to the cells above the diagonal. Thus the elements of M_{11} correspond to cells in rows $1,\dots,u-1$ of T^i, and the elements of M_{12} correspond to the cells in rows $u,\dots,k$. But T and T^i differ only in rows $u,\dots,k$, so $M_{11}=M_3$. Also, the right most cell of T^i is in column $k+1+\alpha_1-\alpha_j$, for $j=1,\dots,u-1$. Now let s be chosen so that

$$\alpha_s \leqslant i-1 \quad \text{and} \quad \alpha_{s-1} > i-1. \tag{7}$$

Then the bottom element of column $k+1+\alpha_1-(i-1)$ in T^i is in row s, so $M_{12}=\{u-s,\dots,k-s\}$, giving

$$M_1 = M_3 \cup \{u-s,\dots,k-s\}. \tag{8}$$

For example, when $\alpha=(5,4,2,1)$, $i=3$, as in Fig. 6, we obtain $s=3$.

Similarly, the elements of M_2 can be partitioned into three subsets: M_{21}, corresponding to the cells in columns $1,\dots,k+1$ of T^{i^*}; M_{22}, corresponding to the cells to the right of column $k+1$ but on or below the diagonal of T^{i^*}; and M_{23}, corresponding to the cells above the diagonal of T^{i^*}. Now, the right most cell in rows $1,\dots,k+1-u$ of T^{i^*} is in column $k+1$, and the right most cell in row j of T^{i^*} is in column $k+1+\alpha_{k+1-j}$, for $j=k+2-u,\dots,k$. Therefore, from (7), the cells in M_{21} occur in rows $1,\dots,k+1-s$, and the bottom element in each corresponding column is in row 1, so $M_{21}=\{0,\dots,k-s\}$.

Now, let r be the largest row index of the elements of M_{22}. Then, since the diagonal element of row j is in column $k+1+k+1-j$, for $j=k+2-u,\dots,k$, we have

$$k+1-r+i-1 \geqslant \alpha_{k+1-r} \quad \text{and} \quad k+1-(r+1)+i-1 < \alpha_{k-r} \tag{9}$$

and from Proposition 3.2(3), we immediately have $k-r=u-i$, or $r=k-u+i$. For example, in Fig. 6 we have $r=4$, and indeed, as noted previously, $k-u+i=5-4+3=4$. Also, the bottom element of the columns corresponding to the cells of M_{22} all occur in row $k+2-u$, from the second part of Proposition 3.2(4). Thus, $M_{22}=\{(k+2-s)-(k+2-u),\dots,(k-u+i)-(k+2-u)\}=\{u-s,\dots,i-2\}$.

Finally, the leg lengths of the cells of M_{23} are all the same in T^{i^*} as in T^*, from the first part of Proposition 3.2(4). Thus $M_{23}=M_4$, and we have

$$M_2 = M_{21} \cup M_{22} \cup M_{23} = M_4 \cup \{0,\dots,k-s\} \cup \{u-s,\dots,i-2\}.$$

The bijection between M_1 and M_2 then gives, from (8)

$$M_3 \cup \{u-s,\dots,k-s\} = M_4 \cup \{0,\dots,k-s\} \cup \{u-s,\dots,i-2\} \tag{10}$$

and we have

$$M_3 = M_4 \cup \{0,\dots,i-2\}. \tag{11}$$

Now, Theorem 3.1 follows from (6), and the fact that the cells in $p(R)$ with arm length equal to $i-1$ in R have leg lengths $0,\dots,i-2$. □

How is this proof bijective? In the development above, we have claimed that a bijection follows from (11), but we obtained the latter by "cancelling" the contribution of the set $\{u-s,\dots,k-s\}$ on both sides of (10). In general, a bijection that is deduced from such a cancellation would require the involution principle, but we can avoid this principle here by the following observation about the bijective proof of Theorem 1.3 applied to M_1 and M_2: the cells corresponding to elements of M_{12} all appear in a single column, so in the Dyck path associated with M_1, the up steps associated with M_{12} all appear together, with no down steps between them, and these up steps are followed by a terminating sequence of down steps. This means that, under the bijection ϕ, the cells corresponding to elements of M_{12} are mapped to a subset of cells corresponding to elements of M_{21}. But this leads immediately to a bijection for (11), simply by restricting ϕ to the cells corresponding to elements of M_{11}.

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