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Planar Decompositions of Tableaux and Schur Function Determinants

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In this paper we describe planar decompositions of skew shape tableaux into strips and use the shapes of these strips to generate a determinant. We then prove that each of these determinants is equal to the Schur function for the skew shape. The Jacobi–Trudi identity, the dual Jacobi–Trudi identity, the Giambelli identity and the rim ribbon identity of Lascoux and Pragacz are all special cases of this theorem. A compact Gessel–Viennot lattice path argument provides the proof.

1. Introduction and Background

The literature on symmetric functions contains many determinantal results. Some of the most fundamental are determinantal expressions for the (skew) Schur function; the Jacobi–Trudi determinant [6], its dual [9], the Giambelli determinant [3], and Lascoux and Pragacz's recent rim ribbon determinant [8] are all of this type. In this paper we give a general result that contains these equally as special cases. This general result uses Gessel–Viennot methods [2] and describes a bijection between tableaux and non-intersecting m-tuples of lattice paths, and a bijection between one intersecting m-tuple of lattice paths and a second intersecting m-tuple of lattice paths. The mathematical development requires the language of partitions and symmetric functions, the essentials of which are described below (for a complete account, see Macdonald [9]).

Let λ be a *partition* of k with at most l parts, i.e. $\lambda=(\lambda_1,\ldots,\lambda_l)$ where $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_l$ are non-negative integers and $\lambda_1+\lambda_2+\cdots+\lambda_l=k$ (λ_i is the ith part of λ). The empty partition $\varnothing$ of 0 has no parts. A partition can be represented in the plane by an arrangement of boxes that is left and top justified with λ_i boxes in the ith row. Such an arrangement is called a *Ferrers diagram,* or simply a *diagram,* with *standard shape.* Given two partitions, λ and μ, a Ferrers diagram with *skew shape* λ/μ for $\mu_i\leqslant\lambda_i$, $i\geqslant 1$, has a box in row i column j iff $\mu_i<j\leqslant\lambda_i$. Geometrically, this is the Ferrers diagram of λ with the Ferrers diagram of μ removed from its upper left-hand corner. Note that the standard shape λ is just the skew shape λ/μ with $\mu=\varnothing$. The *content* of a box α in a Ferrers diagram is denoted by $c(\alpha)$ and equals $j-i$ if α lies in column j from the left and row i from the top of the Ferrers diagram (referred to as box (i,j) where convenient). Associated with each skew shape is its *conjugate.* The conjugate of a skew shape λ/μ is defined to be the skew shape λ'/μ' the Ferrers diagram of which is the transpose of the Ferrers diagram of λ/μ. More explicitly, the number of boxes in the ith row of λ'/μ' is the number of boxes in the ith column of λ/μ.

A partition can also be represented in *Frobenius notation*: $\lambda=(\alpha\mid\beta)=(\alpha_1,\ldots,\alpha_m\mid\beta_1,\ldots,\beta_m)$, where m is the number of boxes on the main diagonal of λ, $\alpha_i=\lambda_i-i$ (the number of boxes in the ith row to the right of the diagonal box) and $\beta_i=\lambda_i'-i$ (the number of boxes in the ith column below the diagonal box).

If we insert positive integers into the boxes of a skew shape λ/μ such that the

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entries strictly increase down each column and weakly increase left to right across each row, we say we have a *tableau* of skew shape λ/μ. In a tableau we use $T(\alpha)$ to denote the positive integer in box α of the Ferrers diagram of T.

The (skew) Schur function, $s_{\lambda/\mu}(X)$, in the variables $X=(x_1, x_2, \dots)$, is given by

$$s_{\lambda/\mu}(X)=\sum_T \prod_{\alpha\in\lambda/\mu} x_{T(\alpha)},$$

where the summation is over tableaux T of shape λ/μ and $\alpha\in\lambda/\mu$ means that α ranges over all boxes in the Ferrers diagram of λ/μ. The symmetry of $s_{\lambda/\mu}$ is clear from the following determinantal expressions in terms of the complete symmetric functions, $h_k(X)$, $k\geqslant 0$ and elementary symmetric functions $e_k(X)$, $k\geqslant 0$, where

$$h_k(X)=\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k} x_{i_1}\cdots x_{i_k} \qquad k\geqslant 1,$$

$$e_k(X)=\sum_{1\leqslant i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k}, \qquad k\geqslant 1,$$

and $h_0(X)=e_0(X)=1$.

THEOREM 1.1 (Jacobi–Trudi identity [6]). *Let λ/μ be a skew shape partition, and let $\lambda=(\lambda_1,\dots,\lambda_l)$ and $\mu=(\mu_1,\dots,\mu_l)$ have at most l parts. Then*

$$s_{\lambda/\mu}(X)=\det(h_{\lambda_i-\mu_j-i+j}(X))_{l\times l}. \tag{1}$$

THEOREM 1.2. (dual Jacobi–Trudi identity [9]). *Let λ/μ be a skew shape partition and let $\lambda'=(\lambda_1',\dots,\lambda_l')$ and $\mu'=(\mu_1',\dots,\mu_l')$ have at most l parts. Then*

$$s_{\lambda/\mu}(X)=\det(e_{\lambda_i'-\mu_j'-i+j}(X))_{\lambda_1\times\lambda_1}. \tag{2}$$

Another classical determinantal result for the Schur function of standard shape is stated compactly in terms of Frobenius notation.

THEOREM 1.3. (Giambelli [3]). *Let λ be a partition with m boxes on the main diagonal of the diagram and Frobenius representation $\lambda=(\alpha_1,\dots,\alpha_m\mid\beta_1,\dots,\beta_m)$. Then*

$$s_\lambda(X)=\det(s_{(\alpha_i\mid\beta_j)}(X))_{m\times m}.$$

Now note that $h_k(X)=s_{(k)}(X)$ and $e_k(X)=s_{(1^k)}(X)$, so that Theorems 1.1, 1.2 and 1.3 all express the Schur function as a determinant of Schur functions. This similarity is made much more striking by the main result of this paper, given as Theorem 3.1 in Section 3, which gives a general class of determinantal expressions for the skew Schur function. The class is broad, including as equally direct corollaries Theorems 1.1, 1.2 and 1.3, as well as related results of Lascoux and Pragacz [8] involving 'rim ribbons'. Each member of the class corresponds to a planar geometrical decomposition of the diagram into collections of boxes called 'strips'. This planar decomposition is described in Section 2. In Section 3 we state and prove the main result, and demonstrate that Theorems 1.1, 1.2 and 1.3 above are all immediate corollaries. In Section 4 we discuss the connection between our result and the work of Lascoux and Pragacz [8].

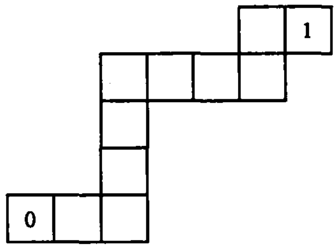

FIGURE 1. An example of a strip.

2. OUTSIDE DECOMPOSITIONS OF SKEW STRIPS

In this section we consider a skew diagram geometrically with a special focus on a subset of squares called a strip.

DEFINITION 2.1. A *strip* in a diagram of skew shape is a skew diagram with an edgewise connected set of boxes that contains no 2×2 block of boxes.

We employ an 'active' vocabulary when referring to strips and boxes. For example, a strip 'starts' at a box (called the starting box) if that box is the bottommost and leftmost in the strip, and a strip 'ends' at a box (called the ending box) if that box is the topmost and rightmost in the strip. A strip 'proceeds north' from one box to the one on top of it, and a strip 'proceeds east' from one box to the one to the right of it. A box is 'approached from the left' if either there is a box in the strip immediately to its left or the box is on the left perimeter of the diagram, and a box is 'approached from below' if either there is a box in the strip immediately below it or the box is on the bottom perimeter of the diagram. See Figure 1 for an example of a strip, where the starting box is marked with a 0 and the ending box is marked with a 1.

The strips of Definition 2.1 have a variety of names. Macdonald [9] calls them 'border strips' (in the Russian edition, 'skew hooks'), while Sagan [10] calls them 'rim hooks'. Lascoux and Pragacz [8] call these objects *ribbons,* and use the term *rim* to denote the maximal outer strip of a diagram. See Corollary 3.5 for a determinantal result involving ribbons.

If a strip, θ_i, contains a box from the main diagonal of the diagram, then define θ_i^+ to be the top portion of the strip, i.e. the portion above and to the right of the diagonal box, and define θ_i^- to be the bottom portion of the strip, i.e. the portion below and to the left of the diagonal box.

DEFINITION 2.2. Suppose that $\theta_1, \theta_2, \ldots, \theta_m$ are strips in a skew shape diagram of λ/μ and each strip has a starting box on the left or bottom perimeter of the diagram and an ending box on the right or top perimeter of the diagram. Then if the disjoint union of these strips is the skew shape diagram of λ/μ, we say the totally ordered set $(\theta_1, \theta_2, \ldots, \theta_m)$ is a *(planar) outside decomposition* of λ/μ.

Given a diagram and an outside decomposition of that diagram, then if the diagram is filled with integers to form a tableau, the portion of the tableau that corresponds to a strip in the outside decomposition forms a *tableau of strip shape.* Hence, given an outside decomposition of a shape, a tableau of that shape can be thought of as a union of tableaux of strip shape.

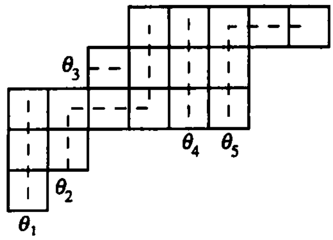

FIGURE 2. An example of an outside decomposition.

Figure 2 gives an example of an outside decomposition into five strips, $\theta_1=1, 1, 1$, $\theta_2=3, 3, 3, 1/2, 2$, $\theta_3=1$, $\theta_4=1, 1, 1$ and $\theta_5=3, 1, 1$. A single diagram may have many outside decompositions, although in general there does not appear to be a systematic method for determining how many outside decompositions are possible for a given diagram. However, the diagram given in Figure 3 has at least 76 outside decompositions, 12 of which are given in Figure 3.

The restrictions of Definition 2.2 force these strips to be 'nested'. Stated another way, this means that boxes that occur on the same top-left-to-bottom-right diagonal (i.e. boxes with the same content) in the diagram are arranged in the same shape, or, to be precise, are approached from the same direction in their respective strips; that is, they are either all approached from below or all approached from the left. For example, in Figure 2, only strips θ_2 and θ_3 have boxes of content 1 (one is a starting box and the other is not) and these are approached from the left, while strips θ_2, θ_4 and θ_5 have boxes of content 3 (one a starting box, one an ending box and one neither) and these are all approached from below. Note that strips in an outside decomposition need not all have boxes of the same content.

In order to state our main result in reasonable generality, it is necessary to consider

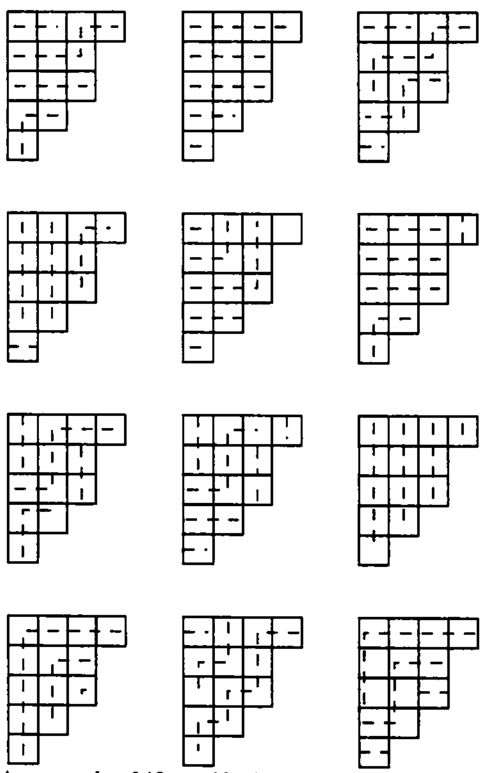

FIGURE 3. An example of 12 outside decompositions of a single diagram.

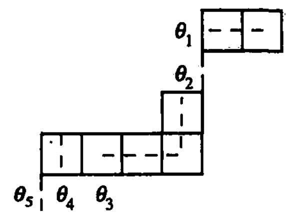

FIGURE 4. An outside decomposition with null strips.

geometrically empty objects in certain contexts called *null strips.* Null strips are edges rather than boxes in the diagram. The content of an edge in a null strip is the content of the box in the diagram containing that edge (as will be seen below, the null strips occur on the perimeter of the diagram, and so there will be a unique box containing them in the diagram). There are four distinct types of null strips:

(1) If $\lambda_i = 0$ we can define a null strip in row i of the diagram. Strictly speaking, the diagram does not have i rows, which is why we use the term 'null strip'. This null strip starts on the left perimeter and the starting box (actually edge) is the left edge of box $(i, 1)$. The ending box is considered to be the same as the starting box.

(2) If $\lambda_i' = 0$ we can define a null strip in column i of the diagram. This null strip ends on the top perimeter and the ending box is the top edge of box $(1, i)$. The starting box is considered to be the same as the ending box.

(3) If $\mu_i = \lambda_i$, if the last box in row i has content c for some $c \in Z$, and if boxes of content $c+1$ are approached from the left, we define a null strip in row i of the diagram. This null strip ends on the right perimeter and the ending box is the right edge of box $(i, c+i)$. The starting box is considered to be the same as the ending box.

(4) If $\mu_i' = \lambda_i'$, if the last box in column i has content c for some $c \in Z$, and if boxes of content c are approached from below, we define a null strip in column i of the diagram. This null strip starts on the bottom perimeter and the starting box is the bottom edge of box $(c+i, i)$. The ending box is considered to be the same as the starting box.

We now extend the definition of an outside decomposition to allow null strips of the above four types. For example, Figure 4 gives an example of an outside decomposition with two null strips, one located at $(2, 4)$ and the other located at $(5, 1)$. The strips are $\theta_1 = 2$, $\theta_2 = \text{null}$, $\theta_3 = 3, 3/2$, $\theta_4 = 1$ and $\theta_5 = \text{null}$.

Consider an outside decomposition, $(\theta_1, \theta_2, \ldots, \theta_m)$ of λ/μ, which could include null strips. Our main result involves a determinant the (i, j)th entry of which is defined in terms of a strip, $\theta_i \# \theta_j$, in a non-commutative way from superimposing θ_i and θ_j.

Case I (*θ_i and θ_j have some boxes with the same content*). Superimpose θ_i on θ_j such that the box of content k in θ_i is superimposed on the box of content k in θ_j for all k. This procedure is well-defined since, as noted above, θ_i and θ_j are nested, i.e. the two sets of boxes with the same contents are both arranged in the same shape. Define $\theta_i \# \theta_j$ to be the diagram obtained from this superposition by taking all boxes between the ending box of θ_i and the starting box of θ_j inclusive.

Case II (*θ_i and θ_j do not have any boxes with the same content, but for all integers c between the minimum of the content of ending box of θ_i and the content of ending box of θ_j, and the maximum of the content of starting box of θ_i and the content of starting box of θ_j, there is a box of content c in the diagram*). θ_i and θ_j must be two disconnected pieces. The nesting property forces the starting box of one to be to the right and/or above the ending box of the other. 'Bridge the gap' between θ_i and θ_j by inserting boxes from the ending box of one to the starting box of the other that follow the arrangement dictated by other boxes of the same content in the outside decomposition. Define $\theta_i\#\theta_j$ as in Case I with the following additional conventions. If the ending box of θ_i is edge connected to the starting box of θ_j and occurs before it (that is, below or to the left), then $\theta_i\#\theta_j=\varnothing$. If the ending box of θ_i is not edge connected but occurs before the starting box of θ_j, $\theta_i\#\theta_j$ is undefined.

Case III (θ_i and θ_j do not have any boxes with the same content, and there is some integer c between the minimum of the content of ending box of θ_i and the content of ending box of θ_j, and the maximum of the content of starting box of θ_i and the content of starting box of θ_j, such that there is no box of content c in the diagram). θ_i and θ_j must be two disconnected pieces as in Case II, and the starting box of one must be to the right and/or above the ending box of the other. For contents that have boxes in the diagram, bridge those parts of the gap as in Case II with boxes that follow the same arrangement as dictated by the other boxes. For contents that do not have boxes in the diagram, bridge those parts of the gap in the following manner: for each content c decide from which direction a box of this content should be approached. This choice will be fixed for that particular diagram. Define $\theta_i\#\theta_j$ as in Case I with the additional conventions as given in Case II.

Consider the following examples of the action of '#' using the decompositions in Figures 2 and 4. The strips in Figure 2 are $\theta_1=1,\ 1,\ 1$, $\theta_2=3,\ 3,\ 3,\ 1/2,\ 2$, $\theta_3=1$, $\theta_4=1,\ 1,\ 1$ and $\theta_5=3,\ 1,\ 1$. Some of the strips obtained by superposition are:

$$\begin{aligned}
&\theta_4\#\theta_5=1,1\\
&\theta_5\#\theta_4=3,1,1,1\\
&\theta_1\#\theta_5=\text{undefined}\\
&\theta_5\#\theta_1=5,3,3,3,3,1,1,1/2,2,2,2\\
&\theta_3\#\theta_4=\varnothing\\
&\theta_4\#\theta_2=1,1,1,1
\end{aligned}$$

Suppose we reorder the strips in Figure 4 to be $\theta_1=1$, $\theta_2=3,3/2$, $\theta_3=2$, $\theta_4=\text{null}$ at co-ordinates $(2,4)$ and $\theta_5=\text{null}$ at co-ordinates $(5,1)$. Some of the strips obtained by superposition are:

$$\begin{aligned}
&\theta_4\#\theta_5=\text{undefined}\\
&\theta_5\#\theta_4=3,3,3,1/3,2\\
&\theta_1\#\theta_5=\text{undefined}\\
&\theta_5\#\theta_1=1\\
&\theta_3\#\theta_4=\text{undefined}\\
&\theta_4\#\theta_3=3
\end{aligned}$$

Note that in general $\theta_i\#\theta_i=\theta_i$ for all i.

3. The Main Result

We can now state the main result of this paper. This result provides one determinant for each outside decomposition of a given shape.

THEOREM 3.1. *Let λ/μ be a skew shape partition. Then, for any outside decomposition, $(\theta_1, \theta_2, \ldots, \theta_m)$, of λ/μ,*

$$s_{\lambda/\mu}(X)=\det(s_{\theta_i\#\theta_j}(X))_{m\times m},$$

where $s_{\varnothing}=1$ and $s_{undefined}=0$.

We delay the proof and instead begin with two examples to illustrate the generality of the result. The outside decomposition of Figure 2 leads to the identity:

$$s_{8,6,6,2,1/3,2} = \det\begin{pmatrix} s_{1,1,1} & s_1 & 0 & 0 & 0 \\ s_{3,3,3,1,1,1/2,2} & s_{3,3,3,1/2,2} & s_{1,1,1} & s_{1,1} & s_1 \\ s_{3,1,1,1} & s_{3,1} & s_1 & 1 & 0 \\ s_{3,3,3,3,1,1,1/2,2,2} & s_{3,3,3,3,1/2,2,2} & s_{1,1,1,1} & s_{1,1,1} & s_{1,1} \\ s_{5,3,3,3,3,1,1,1/2,2,2} & s_{5,3,3,3,3,1/2,2,2,2} & s_{3,1,1,1,1} & s_{3,1,1,1} & s_{3,1,1} \end{pmatrix}$$

The outside decomposition of Figure 4 leads to the identity:

$$s_{6,4,4,4} = \det\begin{pmatrix} s_1 & s_{3,3,1/2} & s_{5,3,3,3,1/3,3,2} & s_{3,3,3,1/3,2} & 0 \\ 1 & s_{3,3/2} & s_{5,3,3,3/3,3,2} & s_{3,3,3/3,2} & 0 \\ 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 1 & 0 \\ s_1 & s_{3,3,1/2} & s_{5,3,3,3,1/3,2} & s_{3,3,3,1/3,2} & 1 \end{pmatrix}.$$

Note that there are many outside decompositions of shape 6,4,4,4 and for each such decomposition we obtain a determinantal expression for $s_{6,4,4,4}$ by applying Theorem 3.1. Indeed, the Schur function generated by the diagram given in Figure 3 will have at least 76 determinants equal to it, since the diagram has at least 76 outside decompositions. Additionally, the strips in each of these could be reordered in 5! different ways. The 12 that correspond to the outside decomposition in Figure 3 are given in Figures 5 and 6.

Next we show that the Jacobi–Trudi, dual Jacobi–Trudi and Giambelli identities given in the Introduction follow as corollaries of the main result.

Corollary 3.2 (Jacobi–Trudi [6]). *Let λ/μ be a skew shape partition with at most l parts. Then*

$$s_{\lambda/\mu}(X) \doteq \det(h_{\lambda_i-\mu_j-i+j}(X))_{l\times l}.$$

Proof. Consider the outside decomposition $((\lambda_1)/(\mu_1), (\lambda_2)/(\mu_2), \ldots, (\lambda_l)/(\mu_l))$: the parts in this decomposition are the rows of the diagram. If $\lambda_i = \mu_i$, then θ_i is a null strip of type 3. Then, by Theorem 3.1, the (i, j)th entry in the matrix is $s_{((\lambda_i)/(\mu_i)\#((\lambda_j)/(\mu_j))}$ which equals $s_{(\lambda_i-\mu_j-i+j)}$ since the content of the ending box of $(\lambda_i)/(\mu_i)$ is $\lambda_i - i$ and the content of the starting box of $(\lambda_j)/(\mu_j)$ is $j-\mu_j$, and since the superposition operation amounts to adjoining a number of boxes equal to the content of the ending box of $(\lambda_i)/(\mu_i)$ to a number of boxes equal to the content of the starting box of $(\lambda_j)/(\mu_j)$. In the case that $\lambda_i-\mu_j-i+j$ is negative, the # gives undefined. But $h_k = s_k$ for single part partitions k, with both equal to 1 for $k=0$ and both equal to 0 for $k<0$, and hence the result follows. □

The following corollary is Theorem 1.2 from above.

Corollary 3.3 (dual Jacobi–Trudi [9]). *Let λ/μ be a skew shape partition. Then*

$$s_{\lambda/\mu}(X) = \det(e_{\lambda_i'-\mu_j'-i+j}(X))_{\lambda_1\times\lambda_1}.$$

Proof. Let the ith strip, θ_i, equal the ith column of the diagram of λ/μ, i.e. $((\lambda_1')/(\mu_1'), \ldots, (\lambda_{\lambda_1}')/(\mu_{\lambda_1}'))$. Then the transpose of the matrix that we obtain in Theorem 3.1 is equal to the matrix given in the statement of the corollary above, since $((\lambda_i')/(\mu_i')\#((\lambda_j')/(\mu_j')) = 1^{\lambda_j'-\mu_i'+i-j}$. □

Corollary 3.4 (Giambelli [3]). *Let $\lambda = (\alpha \mid \beta)$ be a partition with m boxes on the main diagonal. Then*

$$s_\lambda(X) = \det(s_{(\alpha_i|\beta_j)}(X))_{m\times m}.$$

$$s_{43321} = \det\begin{pmatrix} s_2 & s_3 & s_4 & s_{51} \\ s_{32/1} & s_{43/2} & s_{54/3} & s_{651/4} \\ s_1 & s_2 & s_3 & s_{41} \\ 0 & 1 & s_1 & s_{21} \end{pmatrix}$$

$$s_{43321} = \det\begin{pmatrix} s_4 & s_5 & s_6 & s_7 & s_8 \\ s_2 & s_3 & s_4 & s_5 & s_6 \\ s_1 & s_2 & s_3 & s_4 & s_5 \\ 0 & 1 & s_1 & s_2 & s_3 \\ 0 & 0 & 0 & 1 & s_1 \end{pmatrix}$$

$$s_{43321} = \det\begin{pmatrix} s_2 & s_{31} & s_{42/1} & s_6 \\ s_{32/1} & s_{431/2} & s_{542/31} & s_{653/42} \\ s_1 & s_{21} & s_{32/1} & s_{43/2} \\ 0 & 0 & 1 & s_1 \end{pmatrix}$$

$$s_{43321} = \det\begin{pmatrix} s_{1111} & s_{111} & s_1 & s_{2222/111} \\ s_{11111} & s_{1111} & s_{11} & s_{22222/1111} \\ s_{211111} & s_{21111} & s_{211} & s_{322222/11111} \\ 1 & 0 & 0 & s_1 \end{pmatrix}$$

$$s_{43321} = \det\begin{pmatrix} s_1 & s_2 & s_3 & 0 & s_4 & s_5 \\ s_{11} & s_{22/1} & s_{33/2} & 0 & s_{44/3} & s_{55/4} \\ s_{111} & s_{222/11} & s_{333/22} & 1 & s_{444/33} & s_{555/44} \\ s_{1111} & s_{2222/111} & s_{3333/222} & s_1 & s_{4444/333} & s_{5555/444} \\ 0 & 1 & s_1 & 0 & s_2 & s_3 \\ 0 & 0 & 0 & 0 & 1 & s_1 \end{pmatrix}$$

$$s_{43321} = \det\begin{pmatrix} s_3 & s_4 & s_5 & 1 & s_{61} \\ s_2 & s_3 & s_4 & 0 & s_{51} \\ s_1 & s_2 & s_3 & 0 & s_{41} \\ s_{33/2} & s_{44/3} & s_{55/4} & s_1 & s_{661/5} \\ 0 & 1 & s_1 & 0 & s_{21} \end{pmatrix}$$

FIGURE 5. Determinants for $\lambda = 4, 3, 3, 2, 1$.

PROOF. Let the ith strip, θ_i, equal the ith hook of the diagram of λ, where $(\alpha_i \mid \beta_i)$ is the Frobenius notation for the ith hook.

The next corollary uses the rim ribbons referred to after Definition 2.1.

COROLLARY 3.5 (rim ribbons (Lascoux and Pragacz [8]). *Let λ be a partition with m boxes on the main diagonal, and let $(\theta_1, \theta_2, \ldots, \theta_m)$ be its decomposition into rim ribbons (maximal outer strips). Then*

$$s_\lambda(X) = \det(s_{\theta_i^- \& \theta_j^+}(X))_{m\times m},$$

where $\theta_i^- \& \theta_j^+$ denotes the partition obtained by replacing the bottom part (that is, the boxes below the main diagonal box) of θ_j by that of θ_i.

PROOF. Let the ith strip, θ_i, equal the ith rim ribbon of the diagram of λ. □

Note that our notation for rim ribbons differs from that of Lascoux and Pragacz [8] because we write our tableaux differently. Rim ribbons and Corollary 3.5 will be discussed further in Section 4.

$$s_{43321}=\det\begin{pmatrix} s_{11} & s_{22/1} & s_1 & s_{331/2}\\ s_{311} & s_{422/11} & s_{31} & s_{5331/22}\\ s_{111} & s_{222/11} & s_{11} & s_{3331/22}\\ 1 & s_1 & 0 & s_{21}\end{pmatrix}$$

$$s_{43321}=\det\begin{pmatrix} s_{11} & s_{22/1} & s_1 & 0 & s_{33/2} & s_{44/3}\\ s_{211} & s_{322/11} & s_{21} & 1 & s_{433/22} & s_{544/33}\\ s_{111} & s_{222/11} & s_{11} & 0 & s_{333/22} & s_{444/33}\\ s_{2211/1} & s_{3322/211} & s_{221/1} & s_1 & s_{4333/322} & s_{5444/433}\\ 1 & s_1 & 0 & 0 & s_2 & s_3\\ 0 & 0 & 0 & 0 & 1 & s_1\end{pmatrix}$$

$$s_{43321}=\det\begin{pmatrix} s_{11111} & s_{111} & s_1 & 0\\ s_{111111} & s_{1111} & s_{11} & 0\\ s_{1111111} & s_{11111} & s_{111} & 1\\ s_{11111111} & s_{111111} & s_{1111} & s_1\end{pmatrix}$$

$$s_{43321}=\det\begin{pmatrix} s_{41111} & s_{411} & s_4\\ s_{21111} & s_{211} & s_2\\ s_{111111} & s_{111} & s_1\end{pmatrix}$$

$$s_{43321}=\det\begin{pmatrix} s_1 & s_{21} & s_{321/1}\\ s_{11} & s_{221/1} & s_{3321/21}\\ s_{211} & s_{3221/11} & s_{43321/221}\end{pmatrix}$$

$$s_{43321}=\det\begin{pmatrix} s_{411} & s_{522/11} & s_4 & s_8\\ s_{2111} & s_{332/11} & s_2 & s_{433/22}\\ s_{111} & s_{222/11} & s_1 & s_{333/22}\\ 0 & 1 & 0 & s_1\end{pmatrix}$$

FIGURE 6. Determinants for $\lambda = 4, 3, 3, 2, 1$.

As these corollaries demonstrate, the decomposition and superposition techniques are quite powerful. However, they do have limitations. In particular, Theorem 3.1 is not necessarily true if we relax the condition that the strips end on the perimeter of the diagram. For example, consider the decomposition in Figure 7. The strip that ends in box $(2,3)$ does not end on the perimeter, so this is not an outside decomposition. The associated determinant appears below and it is not equal to $s_{4,4,4}$:

$$s_{4,4,4}\neq\det\begin{pmatrix} s_{3,1,1} & s_{3,1} & s_3\\ s_{2,1,1} & s_{2,1} & s_2\\ s_{2,2,2,1,1/1,1} & s_{2,2,2,1/1} & s_{2,2,2/1,1}\end{pmatrix}.$$

In Definition 2.2, we have imposed the condition that the strips in an outside decomposition are totally ordered. For the main result, Theorem 3.1, it makes no

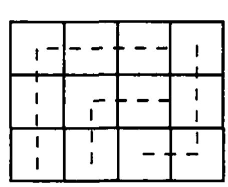

FIGURE 7. A decomposition that is not an outside decomposition.

difference which total order we choose: if we choose a different total order, that merely permutes the rows and columns of the matrix in Theorem 3.1 by the same permutation, resulting in no net change in the determinant. However, for other results, namely the results of Lascoux and Pragacz which we will examine in Section 4, it is very important which total order we choose. For comparison purposes, then, it is certainly to our advantage that Theorem 3.1 is independent of the total order chosen.

Corollaries 3.2–3.5 have each been proved individually by lattice path methods: Corollaries 3.2 and 3.3 by Gessel and Viennot [2], Corollary 3.4 by Stembridge [11] and Corollary 3.5 by Ueno [12]. The motivation for our result was provided by a desire to extend the generality of the planar decomposition of Lascoux and Pragacz [8] featured in the statement of Corollary 3.5 and the lattice path proofs of Stembridge and Ueno.

We now prove the main result. The proof is combinatorial, using a fixed outside decomposition of a given shape. First, it establishes a bijection between the set of all tableaux of a given shape with a given outside decomposition and non-intersecting m-tuples of lattice paths. Each tableau of strip shape in the original tableau is bijectively mapped to a lattice path such that the path has a non-vertical step ending at height i if there is an i in a box of the tableau of strip shape. Second, it invokes the Gessel–Viennot procedure [2], a procedure which defines an involution on intersecting m-tuples of lattice paths. Each intersecting m-tuple is bijectively mapped to another intersecting m-tuple with identical non-vertical steps but permuted starting points. A determinant of symmetric functions give the generating function for all m-tuples of lattice paths, intersecting and non-intersecting.

The procedure given in this paper is different from the Gessel–Viennot procedure since here each path generates a tableau in the shape of a strip, whereas in Gessel–Viennot each path generates a tableau in the shape of a row (since a row is a horizontal strip, the Gessel–Viennot procedure is a special case—see Corollary 3.2).

The lattice paths used here have four types of permissible steps: up-vertical steps that *increase* the y co-ordinate by 1; down-vertical steps that *decrease* the y co-ordinate by 1; right-horizontal (referred to simply as horizontal) steps that increase the x co-ordinate by 1; and down-diagonal (referred to simply as diagonal) steps that increase the x co-ordinate by 1 *and* decrease the y co-ordinate by 1. We specify some additional restrictions: a down-vertical step must not precede an up-vertical step, an up-vertical step must not precede a down-vertical step, a down-vertical step must not precede a horizontal step, an an up-vertical step must not precede a diagonal step. We also require that all steps between lines $x = c$ and $x = c + 1$ for all $c \in Z$ are either all horizontal or all diagonal. The determination of whether these steps are horizontal or diagonal is made by the outside decomposition in the following manner. If boxes of content d are approached from the left, then steps between $x = d$ and $x = d + 1$ must be horizontal; if the boxes of content d are approached from below, then steps between $x = d$ and $x = d + 1$ must be diagonal.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Consider an outside decomposition $(\theta_1, \ldots, \theta_m)$ of λ/μ. We will construct a non-intersecting m-tuple of lattice paths that corresponds to a tableau of shape λ/μ with the outside decomposition $(\theta_1, \ldots, \theta_m)$. See examples in Figures 8, 9 and 10. The ith path begins at P_i and ends at Q_i, $i = 1, \ldots, m$, which are now described. If strip i is not a null strip, fix points $P_i = (t - s, 1)$ if strip i has starting box on left perimeter in box (s, t) of the diagram, or $P_i = (t - s, \infty)$ if strip i has starting box on the bottom perimeter in box (s, t) of the diagram ($P_i = (t - s, 1)$ if both), $i = 1, \ldots, m$. If strip i is not a null strip, fix points $Q_i = (v - u + 1, 1)$ if strip i has ending box on the top perimeter in box (u, v) of the diagram, or $Q_i = (v - u + 1, \infty)$ if

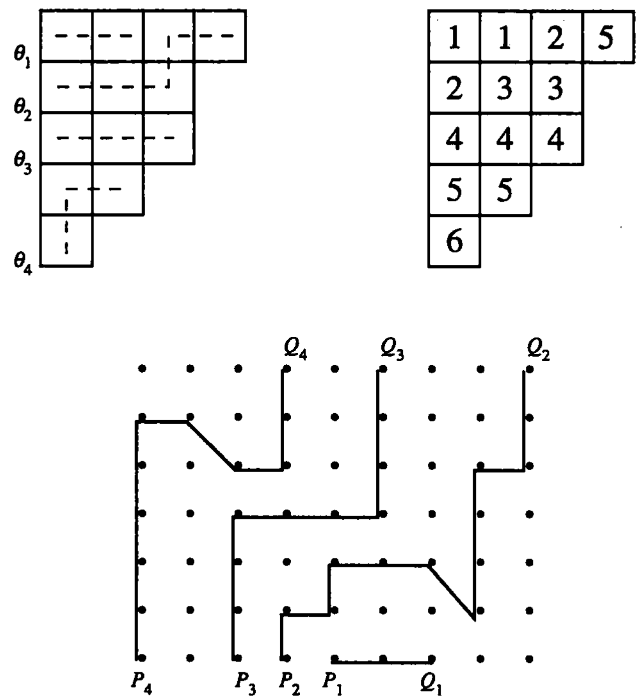

FIGURE 8. A 4-tuple of lattice paths illustrating the tableau–path correspondence.

strip i has ending box on the right perimeter in box (u, v) of the diagram ($Q_i = (v - u + 1, \infty)$ if both), $i = 1, \ldots, m$. If strip i is a null strip with starting box on the bottom perimeter in box (s, t), fix $P_i = (t - s, \infty)$ and $Q_i = (t - s, 1)$. If strip i is a null strip with ending box on the right perimeter in box (s, t), fix $P_i = (t - s + 1, 1)$ and $Q_i = (t - s + 1, \infty)$. If strip i is a null strip with starting box on the left perimeter in box (s, t), fix $P_i = (t - s, 1)$ and $Q_i = (t - s, \infty)$. If strip i is a null strip with ending box on

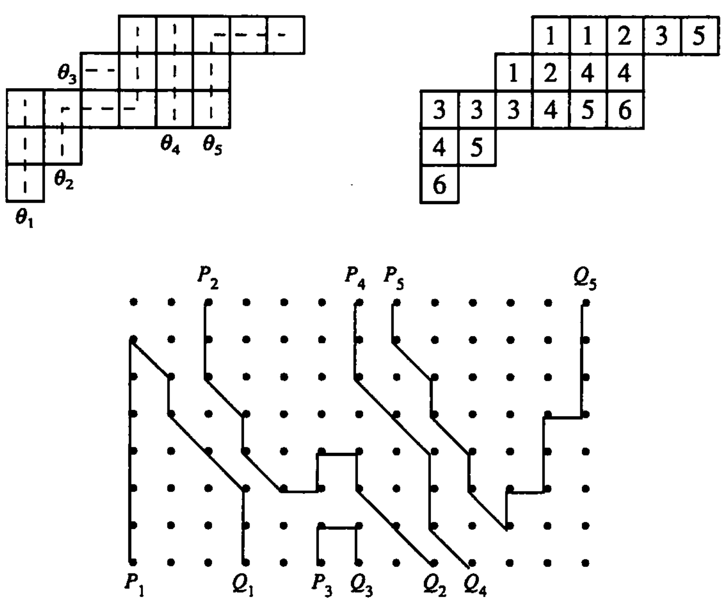

FIGURE 9. A 6-tuple of lattice paths.

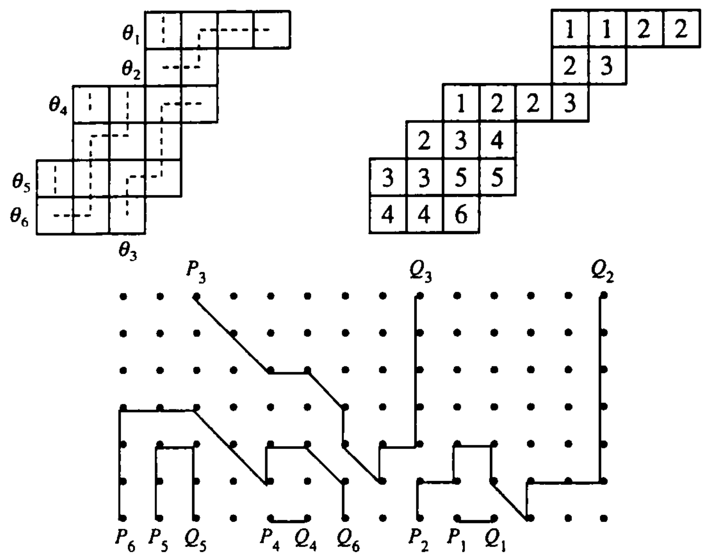

FIGURE 10. A 5-tuple of lattice paths.

the top perimeter in box (s, t), fix $P_i = (t - s + 1, \infty)$ and $Q_i = (t - s + 1, 1)$. For each null strip construct a path consisting only of vertical steps from P_i to Q_i. For each non-null strip construct a path starting at P_i (called the starting point) and ending at Q_i (called the ending point) as follows: if a box containing i and at co-ordinates (a, b) in the diagram is approached from the left in the strip, put a horizontal step from $(b - a, i)$ to $(b - a + 1, i)$; if a box containing i and at co-ordinates (a, b) in the diagram is approached from below in the strip, put a diagonal step from $(b - a, i + 1)$ to $(b - a + 1, i)$. Hence a box of content $b - a$ containing i implies a non-vertical step ending at $(b - a + 1, i)$. So if an m-tuple corresponds to a tableau, every non-vertical step ending at $(b - a + 1, i)$ must correspond to a box of content $b - a$ containing i. Note that no two strips can have the same starting and/or ending points, since that would imply two boxes of the same content on the same section of perimeter. Connect with vertical steps these non-vertical steps just specified. It is routine to verify that there is a unique way of doing this. Consider, for example, strip θ_2 in Figure 8. We show how a path is obtained from it. The strip's starting box is of content -1 and is on the left perimeter of the diagram; hence the path starts at $(-1, 1)$. The integers in the boxes of the strip are 2, 3, 3, 2 and 5, hence the path has non-vertical steps ending at y co-ordinates 2, 3, 3, 2 and 5. The second 2 is the only integer in a box approached from below, hence the step corresponding to it is the only diagonal step. The ending box is of content 3 and ends on both the top and right perimeter. By the convention outlined above, however, we treat it as though the ending box were on the right perimeter and the path ends at $(4, \infty)$.

Now we verify that if an m-tuple of lattice paths is intersecting, it does not correspond to a tableau. The essential reason for this is the column strictness and row weakness of the tableau. We give now a more detailed consideration.

Suppose on the contrary that there is some intersecting m-tuple of lattice paths that corresponds to a tableau. Then we will show that we obtain a contradiction by considering the first intersection point from the left. First note that the situations in which intersections occur because there is a diagonal step from (a, b) to $(a + 1, b - 1)$ and a horizontal step from $(a, b - 1)$ to $(a + 1, b - 1)$, or there is a diagonal step from (a, b) to $(a + 1, b - 1)$ and a horizontal step from (a, b) to $(a + 1, b)$, are not permitted because steps between $x = c$ and $x = c + 1$ must all be of the same type. Also note that the column strictness and row weakness requirements in the tableau imply that the tableau is also diagonal strict; that is, that entries increase along top-left-to-bottom-right diagonals, or entries with the same content are strictly increasing. This demonstrates it is not possible for two boxes of the same content to generate two steps at the same position in the plane.

Any intersection must involve either an up-vertical step and a down-vertical step, a horizontal step and an up-vertical step, a horizontal step and a down-vertical step, a diagonal step and an up-vertical step, or a diagonal step and a down-vertical step. Intersections between diagonal steps and horizontal steps will be subsumed by these cases, since the steps in the second path must be preceded by an up-vertical or a down-vertical step (i.e. the restriction that steps between $x = c$ and $x = c + 1$ are of some type and the restriction that boxes of the same content cannot generate two steps at the same position in the plane guarantee this). Consider a number of cases.

Case I (an up-vertical step in path i intersects a down-vertical step in path j and neither path has nonvertical steps before the x co-ordinate of the intersection point). If both of the paths have any non-vertical steps at all, Case I cannot possibly occur since the first non-vertical step in the path that starts at y co-ordinate ∞ must be diagonal, and the first non-vertical step in the path that starts at y co-ordinate 1 must be horizontal, but these first non-vertical steps must both occur between $x = c$ and $x = c + 1$ for some c.

If path i has non-vertical steps and path j has only vertical steps, then there are a number of subcases. Suppose first that the starting box for the strip corresponding to path j is on the bottom perimeter with content c, and the starting box for the strip corresponding to path i is on the left perimeter with content c. Then the starting box for path i must occur above and to the left of the starting box for path j, say in column t. But then $\mu'_t < \mu'_s$, where the starting box for strip j occurs in column s of the tableau. This contradicts the fact that the parts in a partition are weakly decreasing. The other cases are similar.

If the paths have only vertical steps, the paths must correspond to null strips. Then the paths must occupy the same position in the plane, so either 1) the null strip corresponding to path i must be on the right perimeter, the null strip corresponding to path j must be on the bottom perimeter, and both null strips must have the same content, or 2) the null strip corresponding to path i must be on the left perimeter, the null strip corresponding to path j must be on the top perimeter, and both null strips must have the same content. These situations are not possible because boxes of the same content lie on the same top-left-to-bottom-right diagonal.

Case II (a horizontal step at height a in path i intersects an up-vertical step in path j; path j has a step at height d (necessarily horizontal) before the up-vertical steps and a step at height e (necessarily horizontal) after the up-vertical steps). The content of the box containing e is one more than the content of the box containing a, and $e \geqslant a$, so the box containing e is right of and below (or beside) the box containing a by column strictness and row weakness. The content of the box containing d is the same as the content of the box containing a, and $d < a$, so the box containing d is above and to the left of the box containing a by column strictness and row weakness. But the box containing d and the box containing e are in the same strip, yet located on different sides of the box containing a. This provides a contradiction.

Case III (a horizontal step at height a in path i intersects an up-vertical step in path j; path j has a step at height d (necessarily horizontal) before the up-vertical steps and no non-vertical steps after). Since there are no non-vertical steps after, path j ends at y

co-ordinate ∞ and the corresponding strip ends on the right perimeter. But, as in Case II, the box containing d is to the left and above the box containing a, so it is not possible for the strip to end on the right perimeter, and we obtain a contradiction.

Case IV (*a horizontal step at height a in path i intersects an up-vertical step in path j; path j has a step at height e (necessarily horizontal) after the up-vertical steps and no non-vertical steps before*). Since there are no non-vertical steps before, path j starts at y-co-ordinate 1 and the corresponding strip starts on the left perimeter. But, as in Case II, the box containing e is below (or beside) and to the right of the box containing a, so it is not possible for the strip to start on the left perimeter, and we obtain a contradiction.

Case V (*a horizontal step at height a in path i intersects an up-vertical step in path j; path j has no non-vertical steps*). Then the null strip corresponding to path j 1) must be on the right perimeter and must have the same content as the box corresponding to the step at height a, or 2) must be on the left perimeter and must have content one more than the content of the box corresponding to the step at height a. If the null strip is on the right perimeter, its corresponding box must occur below and to the right of the box containing a, say in row t of the tableau. But then $\mu_t > \mu_s$, where the box containing a occurs in row s of the tableau. This contradicts the fact that parts in a partition are weakly decreasing. If the null strip is on the left perimeter, its corresponding box must occur below and left of the box containing a, since such a null strip occurs where the diagram has no rows. But then it is not possible for the content of the null strip box to be larger than the content of the box containing a.

There are four additional cases, similar to Cases II–V, with 'up-vertical' replaced by 'down-vertical', another four with 'horizontal' replaced by 'diagonal', and a final four with 'horizontal' and 'up-vertical' replaced by 'diagonal' and 'down-vertical'. The arguments for these remaining 12 cases are similar to the arguments for Cases II–V. Hence tableaux correspond only to non-intersecting m-tuples of lattice paths.

The construction described above for generating paths given tableaux is reversible, and now we verify that a non-intersecting m-tuple of lattice paths obeying all the path conditions set out above corresponds to a tableau and an outside decomposition where each path in the non-intersecting m-tuple gives rise to a tableau of strip shape. The choice of the starting and ending points and the restrictions on the steps ensure that the m-tuple corresponds to a diagram of the required shape, but we must show that the entries in the tableau obey the column strictness and row weakness rules. We begin by ensuring that a lattice path that starts at P_j and ends at Q_i corresponds to the strips $\theta_i \# \theta_j$. The proof is as follows. Begin with the empty partition. At iteration k, if the kth non-vertical step from the left in the lattice path is horizontal, ending at (i, j), then place a box containing j in the tableau to the right of the previous box; if it is diagonal, ending at (i, j), then place a box containing j in the tableau on top of the previous box. The fact that a down-vertical step does not precede a horizontal step ensures that a horizontal step is at a height higher than or the same as the step just before it. This means the entries in a row of the tableau are weakly increasing. The fact that an up-vertical step does not precede a diagonal step ensures that a diagonal step ends at a height strictly lower than the step just before it. This means the entries in a column of the tableau are strictly increasing. Since the tableau is built by placing boxes always to the right or on top, we know the shape is a strip. Moreover, since the starting and ending points come from θ_j and θ_i, since boxes of the same content correspond to the same type of step, and since the $\#$ operation is based on boxes of the same content, we know that the strip is $\theta_i \# \theta_j$. Consider, for example, Figure 8, in which the path that ends furthest right corresponds to the strip that starts in row 2, column 1, and ends in row 1, column 4.

Now let $T(l, j)$ denote the entry in box (l, j) of the tableau. We claim that $T(l, j) < T(l+1, j)$ and $T(l, j) \leqslant T(l, j+1)$. These inequalities are obvious if the boxes in question are in the same strip. Suppose they are not. Then the first claim follows by the fact that the paths are non-intersecting. To see this, suppose that the step starting at line $x = c$ in path i starts at height t. If this step is horizontal, $T(l, j) = t$, and the step starting at line $x = c - 1$ but in path $i + 1$ must end at height $t + 1$ or higher to avoid intersection, implying $T(l+1, j) \geqslant t + 1$. If this step is diagonal, then the box $(l+1, j)$ must be in the same strip as (l, j), and so column strictness is guaranteed by the conditions internal to a path. The second claim follows again by the fact that the paths are non-intersecting. To see this, suppose that the step starting at line $x = c$ in path i starts at height t. If this step is horizontal, $T(l, j) = t$, and the step starting at line $x = c + 1$ but in path $i + 1$ must start at height $t + 1$ or higher, implying $T(l, j+1) \geqslant t$. If the step is diagonal, $T(l, j) = t - 1$, and the step starting at line $x = c + 1$ in path $i + 1$ must start at height t or higher, implying $T(l, j+1) \geqslant t - 1$.

For each horizontal or diagonal step, we choose a weight of x_j where the step ends at (i, j). For each up-vertical or down-vertical step, regardless of position, we choose a weight of 1. Since there is a one-to-one correspondence between lattice paths and tableaux which have the shape of a strip, the generating function for these lattice paths is the Schur function for the shape of a strip.

The proof now follows by the well-known Gessel–Viennot lattice path procedure as described in Gessel and Viennot [2], Goulden and Jackson [4] or Sagan [11]. This procedure was originally created to prove the Jacobi–Trudi identity, and it defines a sign-reversing, weight-preserving involution on intersecting m-tuples of lattice paths, thus demonstrating that their contribution to the determinantal sum is 0. To obtain the full generality we require, we invoke the broader result of Stembridge ([11], Theorem 1.2). To do so we must verify that any m-tuple in which P_i is joined to Q_j for some $1 \geqslant i,\ j \leqslant m,\ i \neq j$, necessarily contains an intersection; however, this is routine. Note additionally that although Stembridge does not impose conditions on which steps may follow each other (as we do before this proof), his theorem is still applicable, since the step restrictions actually serve to define the underlying digraph which Stembridge's lattice path uses. □

4. Lascoux and Pragacz's Result

Lascoux and Pragacz's papers, [7, 8], in which they manipulate minors algebraically, give a number of expressions for Schur functions as determinants of Schur functions. The notation that is used to describe the entries is different from the notation used here; in particular, the entries are not, in general, symmetrically described from a single underlying decomposition, and the skew shapes are considered with the underlying standard shape as a subdeterminant. In this section we make explicit the connection between Theorem 3.1 and Lascoux and Pragacz's results [7, 8].

Lascoux and Pragacz's argument involves two operations, $\downarrow$ and $\rhd$. These operations, $\downarrow$ and $\rhd$, are defined on strips, and a prescribed sequence of $\downarrow$ and $\rhd$ on an appropriate sequence of strips is equivalent to the # operation. $S \downarrow T$ is the diagram obtained by 'gluing' the lower left-hand corner box of diagram T on top of the upper right-hand corner box of diagram S. $S \rhd T$ is the diagram obtained by 'gluing' the lower left-hand corner box of diagram I to the right of the upper right-hand corner box of diagram S. Lascoux and Pragacz's results come from the Giambelli matrix (either the standard shape or their skew shape Giambelli matrix [7]). They rewrite this matrix by partitioning the hooks into blocks of boxes, each block of size $\alpha_i - \alpha_{i+1}$ or $\beta_i - \beta_{i+1}$ for some $1 \leqslant i \leqslant m$, and inserting $\triangleright$ between the $\alpha_i - \alpha_{i+1}$ blocks and $\downarrow$ between the $\beta_i - \beta_{i+1}$ blocks. Lascoux and Pragacz then obtain different matrices by interchanging $\downarrow$ and $\triangleright$, with the restriction that the symbol between blocks of size $\alpha_i - \alpha_{i+1}$ (resp. blocks of size $\beta_i - \beta_{i+1}$) must be constant down a column (resp. row) of the matrix. The determinants of these matrices are the same as the Giambelli determinant up to sign.

Those matrices in which $\alpha_i - \alpha_{i+1}$ and $\alpha_{i-1} - \alpha_i$ must have the same symbol between them, and $\beta_i - \beta_{i+1}$ and $\beta_{i-1} - \beta_i$ must have the same symbol between them for $1 \leqslant i \leqslant m$, are precisely those that correspond to outside decompositions and are therefore included in Theorem 3.1. Additionally, these outside decompositions necessarily have the property that each strip must contain a box of content zero. Those matrices in which it is not the case that $\alpha_i - \alpha_{i+1}$ and $\alpha_{i-1} - \alpha_i$ must have the same symbol between them, and $\beta_i - \beta_{i+1}$ and $\beta_{i-1} - \beta_i$ must have the same symbol between them for $1 \leqslant i \leqslant m$, are precisely those that do *not* correspond to outside decompositions, and therefore are not included in Theorem 3.1. This lack of correspondence is due to the fact that the symbol choices preclude 'nesting', and thus will not together fit inside a diagram. For example, for $\lambda = 4, 3, 3$ there is a choice of $\downarrow$ and $\triangleright$ for which Lascoux and Pragacz's matrix is

$$s_{4,3,3} = \begin{pmatrix} s_1 & s_{1,1} & s_{3,2/1} \\ s_2 & s_{2,2/1} & s_{4,3/2} \\ s_3 & s_{3,3/2} & s_{5,4/3} \end{pmatrix}.$$

Finally, note that although Lascoux and Pragacz have determinants that do not come from Theorem 3.1, there is a large class of determinants that Theorem 3.1 can deal with—including the Jacobi–Trudi determinant—that do not come from Lascoux and Pragacz's work, since their paper considered only outside decompositions in which each strip contains a box from the main diagonal of the diagram.

For a complete account of the transformation from Lascoux and Pragacz's matrices to matrices of the form of Theorem 3.1, see Hamel [5].

The techniques used in this paper can be extended to handle shifted tableaux and Schur Q-functions as well, and this work will be the subject of a forthcoming paper.

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