

The Jordan Curve Theorem is a classical example in mathematics of a result that is at first glance obvious, but was notoriously tricky to prove. The main difficulty lies in the flexibility of the definition of homeomorphism: indeed, there are examples of simple closed curves that are nowhere differentiable and even examples that have positive area (see [2]). Homology theory gives a straight-forward proof (see for example [5]) not only for the theorem in 2-dimensions, but for the general case in n -dimensions.

In 1992, Carsten Thomassen published a brilliant elementary proof of the classical Jordan Curve Theorem using the fact that $K_{3,3}$ cannot be embedded in \mathbb{R}^2 . This method of attack may seem circular, but in fact proving the non-planarity of $K_{3,3}$ relies only on a weak version of the Jordan Curve Theorem for piecewise-linear simple closed curves. An important observation in Thomassen's argument is that it is relatively easy to prove that a planar graph has an embedding in the plane with piecewise-linear edges. Then, if $K_{3,3}$ were planar, there would exist such an embedding, and our weak form of the Jordan Curve Theorem is enough to come to a contradiction. The proof of the Jordan Curve Theorem then follows from a few clever proofs that, assuming the Jordan Curve Theorem were false, lead to constructions of planar embeddings of $K_{3,3}$. Thomassen's method played a fundamental role in inspiring and helping prove what we believe to be a new and interesting generalization of the Jordan Curve Theorem.

Professor Richter, Antoine Vella and I explored the 2-dimensional version of the Jordan Curve Theorem in more abstract topological spaces. We believe our result gives insight about the essential properties required for a Jordan Curve-like theorem to exist, and these are simply properties about how the removal of points disconnects the space. Specifically, our result concerns the Jordan Curve Theorem in what we have called "generalized planes". To define a generalized plane, first we must define a generalized line. A generalized line is a connected topological space in which:

- i) For any three distinct points in the space, there exists one whose removal leaves the other two in different components, and
- ii) The removal of any point disconnects the line.

\mathbb{R} with its usual topology is, as expected, a line, but so are other less familiar spaces, for example \mathbb{Z} with the *Khalimsky* topology: let the basis of the topology be the odd integers and the sets $\{2n-1, 2n, 2n+1\}$, then the resulting space is connected and features alternating open and closed points. A generalized plane is a Cartesian product of two lines that is endowed with the product topology. Immediately we see that \mathbb{R}^2 is a plane, as is \mathbb{Z}^2 where each is given the Khalimsky topology (the resulting space is called the digital plane, and a Jordan Curve Theorem for such a plane has been studied extensively, for example in [4]). Next, it is necessary to define what we mean by a simple arc and a simple closed curve in these spaces. Willard (in [3]) gives a characterization of simple arcs and simple closed curves for more familiar spaces that turns out to be exactly what we want: a simple arc is a connected subspace of a plane that behaves exactly like a line, except that it has two points whose removal does not disconnect the arc, which we call the endpoints. A simple closed curve is a connected subspace of the plane having at least four points so that:

- i) The removal of any one point leaves the space connected, and
- ii) For any four distinct points, there exist two pairs, so that deleting either pair leaves the other two points in different components.

With these definitions in hand, our Jordan Curve Theorem states that, given a (generalized) plane Π , and a locally connected simple closed curve C in Π , then $\Pi \setminus C$ has precisely two components. If Π is T_1 , then each component of $\Pi \setminus C$ has C as its boundary.

The proof relies on bootstrapping from the case where Π is T_1 (in fact, Hausdorff). The proof of this case is essentially that given by Thomassen, with some modifications to accommodate the lack of a metric in our setting. From there, the proof for a general plane Π consists of "lifting" our curve to the T_1 plane Π^* that comes from realizing Π as a natural quotient space of Π^* , and using our T_1 result.

References

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