MATH 145 Oral Exam Bank

November 3, 2020

Principles and Structure of Exam

- Total length: 6 questions, each randomly selected from one of four (sometimes three) possible choices (splitting the twelve weeks of term into four equal pieces).
- Each question has up to five possible marks, with half points assigned for answers that a student comes up with after prompting.
- The four chunks of the course are: (1) Introduction to Proofs, Axiomatic Set Theory, and Mathematical Induction; (2) Relations, Functions, Equivalence Relations, Order Relations, Axiom of Choice, Zorn's Lemma, and cardinalities of sets, (3) Introduction to Groups and Rings, (4) Integral Domains, Elementary Number Theory, Polynomials, and Complex Numbers.
- Of the six questions, one will be taken from each of parts (1) and (2), and two each will be taken from parts (3) and (4).
- First approximation of question types:
  - Question 1: Define a key structure or concept in our course.
  - Question 2: Discuss advantages and disadvantages of something in our course.
  - Question 3: Compare and contrast two structures in our course.
  - Question 4: Describe a procedure or algorithm mentioned in our course.
  - Question 5: Discuss how an idea in our course would change if some part of it was modified.
  - Question 6: A ConcepTest-style multiple choice question, in which the way you talk through and justify your final answer will be evaluated.

Question 1: Definition

(1) Define the set of natural numbers, using the rigorous, axiomatic construction given in our course.

(2) Define what it means for a set S to be uncountable, defining all intermediate terms used in giving this definition.

(3) State all the conditions that are part of the definition of a ring.

(4) Given an integral domain R, define the field of fractions of that integral domain.

Question 2: Advantages and Disadvantages

(1) As a proof technique, what are the advantages and disadvantages of proving something by mathematical induction?

(2) What are the advantages and disadvantages of the theory of countable sets?

(3) What do you think the advantages and disadvantages would be to studying only commutative binary operations (for both groups and rings)?

(4) As it relates to what we've discussed in this course, what are the advantages and disadvantages of working with the field of complex numbers?

Question 3: Compare and Contrast

(1) In what ways are proofs by contradiction and contrapositive similar? In what ways are they different?

(2) In what ways are equivalence relations and order relations the same? In what ways are they different?

(3) In what ways are the theory of groups and rings discussed in our course similar? In what ways are they different?

(4) On Assignments 8 and 9, we introduced sets like the Gaussian integers, where the complex number i is replaced with other numbers, like $\sqrt{-2}, \sqrt{-3}, \sqrt{-5}$. In what ways are all these sets of numbers similar to each other? In what ways do they differ?

Question 4: A Procedure or Algorithm

(1) Describe how we would construct the set of functions from a set A to a set B, directly from the axioms of set theory.

(2) Describe how an equivalence relation E on a set A can be used to construct a partition of A, and conversely, how a partition of A can be used to define an equivalence relation on A.

(3) Suppose G is a finite cyclic group with n elements. If you are given an element $g \in G$, can you describe how you would determine whether g is a generator of G? As you work, try to describe how you might do it in as few group computations as possible.

(4) Describe how we determine whether the linear congruence $ax + b \equiv c \pmod{n}$ has solutions in the integers modulo n, and how to locate all those solutions if they exist.

Question 5: What If?

(1) What if we removed the Axiom of Power Set from our axiomatic system? What things about sets would stay the same, and what things would be different?

(2) What if we had allowed rings in our course not to have a multiplicative identity? How would our definitions, examples, and theory of rings have changed?

(3) What if we tried to build a theory of the "integers modulo n" for the Gaussian integers? What things would stay the same, and what things would look different?

Question 6: ConcepTest

(1) Consider the quotient of abelian groups $\mathbb{R}/\mathbb{Q}$. What is the cardinality of this group?

- Finite.
- Countable.
- Uncountable.

(2) Suppose $\phi : R \to S$ is a homomorphism of rings. Which of the following is *not* a possibility?

- $\operatorname{char} R = n$, where $n > 0$, and $\operatorname{char} S = 0$.
- $\operatorname{char} R = 0$ and $\operatorname{char} S = n$, where $n > 0$.
- $\operatorname{char} R = 6$ and $\operatorname{char} S = 2$.

(3) Exactly one of the following three rings has a division algorithm. Which one is it?

- $\mathbb{R}$
- $\mathbb{Z}[\sqrt{-5}]$
- $\mathbb{Z}[x]$ (explain what this is, since we haven't strictly defined this type of polynomial ring).