The structure of dense triangle-free point sets in a binary projective geometry

We say that $X \subseteq GF(2)^r$ is *triangle-free* if no three points in X sum to 0. The *density* of X is defined as $\frac{|X|}{2^r}$. Bose and Burton [1] showed that triangle-free sets have density at most $\frac{1}{2}$, and that there exist triangle-free sets that attain this bound (such as the set of all vectors with a 1 in the first coordinate). We are interested in the structure of triangle-free sets with a prescribed density. Goeverts and Storme [6] characterised the triangle-free sets with density greater than $\frac{5}{16}$.

This line of research has strong connections with finite geometry and additive combinatorics. The problem is related to Bose's *packing problem*, and has applications in creating codes for data transmission and for the design of efficient experiments [7]. However, our motivation is to find geometric analogues of classical results in graph theory.

The *local density* of a graph $G = (V, E)$ is $\frac{\delta}{|V|}$, where δ is the minimum-degree of G. Evidently, triangle-free graphs have local density at most $\frac{1}{2}$, and this is attained by ballanced complete bipartite graphs. Brandt and Thomassé [2] gave a precise structural characterization of the triangle-free graphs with local density greater than $\frac{1}{3}$. As an easy corollary of their characterization, they show that all such graphs have chromatic number at most 4. Triangle-free graphs with density less than $\frac{1}{3}$ are much more wild. Hajnal (see [4]) showed that for any $\epsilon > 0$ there are triangle-free graphs with local density at least $\frac{1}{3} - \epsilon$ that have arbitrarily large chromatic number.

The natural analogue of chromatic number, in the geometric setting, is *critical number*. Geelen and Nelson [5] proved that, for each $\alpha > \frac{1}{4} + \epsilon$, the triangle-free sets with density greater than $\frac{1}{4} + \epsilon$ have bounded critical number; whereas, there exist triangle-free sets with density greater than $\frac{1}{4} - \epsilon$ that have arbitrarily large critical number. That is to say, below density $\frac{1}{4}$, the class degenerates into chaos.

During my 2014 summer research assistantship under the supervision of Jim Geelen and Peter Nelson, we developed a precise description of all triangle-free sets with density greater than $\frac{33}{128}$ [3]. As a corollary of this description, we showed that if a triangle-free set has density greater than $\frac{33}{128}$, then it has critical number at most 2.

References

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