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Survey Paper

Graph factorization, general triple systems, and cyclic triple systems

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Abstract. In this self-contained exposition, results are developed concerning one-factorizations of complete graphs, and incidence matrices are used to turn these factorization results into embedding theorems on Steiner triple systems. The result is a constructive graphical proof that a Steiner triple system exists for any order congruent to 1 or 3 modulo 6. A pairing construction is then introduced to show that one can also obtain triple systems which are cyclically generated.

1. Introduction

Steiner triple systems are now one of the oldest of combinatorial structures, with applications in many areas from universal algebra to the design of statistical experiments. Basically, a Steiner triple system on v elements is a set of b triples constructed so that each of the v given elements occurs a constant number r of times, and each pair from the v given elements occurs exactly once. It has been known for a long time that such systems exist if and only if v is congruent to 1 or 3, modulo 6.

The aim of this paper is to provide a completely self-contained account of some aspects of graph theory, incidence matrices, and triple systems. However, for additional material, we cite [4] for a fine treatment of graph theory and [3] for a wealth of material on designs; [8] is a general introductory text with much material on graphs, designs, and other aspects of combinatorics; [2] is an exhaustive reference list concerning Steiner triple systems.

We first consider factors of graphs, and obtain an easy conceptual proof of the necessary and sufficient conditions for the existence of triple systems (this may be compared with the omission of the case $6t+1$ in [5] and [8], or with the quite

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lengthy and involved treatment of this case given in [3]). We will also demonstrate the existence of special cyclic triple systems (this was first done in [6], but the methods used there are considerably more complicated than those which we employ; we introduce a simple "pairing" concept, which we find has also been used in [7], but for a different purpose).

We hope that this unified survey will be useful in providing a direct self-contained, and simple account of an extremely important type of combinatorial structure.

2. Complete graphs

A complete graph on n vertices consists of n *vertices* in general position and the $\binom{n}{2}$ joining *edges*. Such a configuration is denoted by K_n; we illustrate K_4 and K_5 in Figure 1.

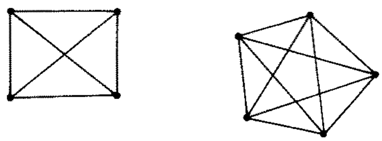

Figure 1
K_4 and K_5

A one-factor of K_{2n} consists of n vertex-disjoint edges. For example, if the vertices of K_6 are named 1,2,3,4,5,6, then the 15 edges may be denoted by (i, j) where $i \neq j$. Here (i, j) is the edge joining vertex i to vertex j. One one-factor would be the set of edges

$$(1, 2),\ (3, 4),\ (5, 6),$$

since all six vertices appear.

A *one-factorization* of K_{2n} consists of $2n-1$ one-factors such that the edges in the one-factors are all distinct. In the case of K_6, the following five one-factors give a complete set of edges, and thus provide a one-factorization:

F_1: (1, 2), (3, 4), (5, 6)

F_2: (1, 3), (2, 5), (4, 6)

F_3: (1, 4), (2, 6), (3, 5)

F_4: (1, 5), (2, 4), (3, 6)

F_5: (1, 6), (2, 3), (4, 5).

A graphic way of looking at this one-factorization is to think of all edges in F_i as being coloured with colour i. Then the one-factorization colours the edges of K_{2n} in $2n-1$ colours in such a way that the $2n-1$ edges leaving any vertex are all coloured distinctly.

The fundamental result on one-factorizations is the following theorem, which can be proved in many different ways. The particular construction we describe was probably first given by König.

THEOREM 2.1. *Every complete graph K_{2n} on $2n$ vertices possesses a one-factorization.*

Proof. Let the vertices be labelled $1, 2, 3, \ldots, 2n$. The one-factors are specified as follows.

$$F_1 : (1, 2n),\ (1+j, 1-j) \quad \text{for} \quad j = 1, 2, \ldots, n-1.$$

$$F_2 : (2, 2n),\ (2+j, 2-j) \quad \text{for} \quad j = 1, 2, \ldots, n-1.$$

and, in general,

$$F_i : (i, 2n),\ (i+j, i-j) \quad \text{for} \quad j = 1, 2, \ldots, n-1.$$

All integers in the edges not involving $2n$ are reduced modulo $2n-1$ to leave integers in the range $1, 2, \ldots, 2n-1$.

The set F_i is a one-factor, since $i+j$ runs over the vertices $i+1$, $i+2, \ldots$, $i+n-1$, and $i-j$ runs over the vertices $i+n$, $i+n+1$, $i+n+2, \ldots, i-1$.

The one-factor F_i is easily drawn; vertex i is joined to vertex $2n$ (see Figure 2). The others are arranged cyclically and joined by a series of parallel chords, as shown.

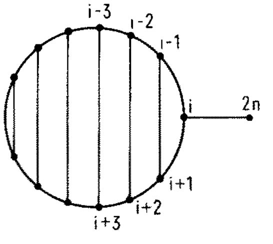

Figure 2

Also the collection of F_i's is a one-factorization. For suppose there were a repeated edge; then

$$(i_1+j_1, i_1-j_1) \equiv (i_2+j_2, i_2-j_2).$$

If $i_1+j_1 \equiv i_2+j_2$, $i_1-j_1 \equiv i_2-j_2 \pmod{2n-1}$, it follows that $i_1=i_2$, $j_1=j_2$. And if

$$i_1+j_1 \equiv i_2-j_2, \qquad i_2+j_2 \equiv i_1-j_1 \pmod{2n-1},$$

we immediately deduce that $i_1=i_2$, $j_1+j_2 \equiv 0$ (a contradiction).

We have thus proved the theorem, and close this section by illustrating it for K_{10}. The one-factorization produced by the theorem is as follows.

F_1: (1, 10), (2, 9), (3, 8), (4, 7), (5, 6)
F_2: (2, 10), (3, 1), (4, 9), (5, 8), (6, 7)
F_3: (3, 10), (4, 2), (5, 1), (6, 9), (7, 8)
F_4: (4, 10), (5, 3), (6, 2), (7, 1), (8, 9)
F_5: (5, 10), (6, 4), (7, 3), (8, 2), (9, 1)
F_6: (6, 10), (7, 5), (8, 4), (9, 3), (1, 2)
F_7: (7, 10), (8, 6), (9, 5), (1, 4), (2, 3)
F_8: (8, 10), (9, 7), (1, 6), (2, 5), (3, 4)
F_9: (9, 10), (1, 8), (2, 7), (3, 6), (4, 5).

3. A special factorization

We shall need the result of Section 2 later; in this section, we develop a rather different factorization.

First, we note that all the edges of K_{2n} fall into n disjoint classes $P_1, P_2, \ldots, P_n$, where edge (i, j) is in P_k if and only if $i-j \equiv k \pmod{2n}$. We call this splitting the *difference partition* of K_{2n}.

EXAMPLE. For K_{12}, the difference partition is as follows.

P_1: (1, 2), (2, 3), (3, 4), ..., (11, 12), (12, 1)
P_2: (1, 3), (2, 4), (3, 5), ..., (11, 1), (12, 2)
P_3: (1, 4), (2, 5), (3, 6), ..., (11, 2), (12, 3)
P_4: (1, 5), (2, 6), (3, 7), ..., (11, 3), (12, 4)
P_5: (1, 6), (2, 7), (3, 8), ..., (11, 4), (12, 5)
P_6: (1, 7), (2, 8), (3, 9), (4, 10), (5, 11), (6, 12).

We now consider the triangles $(1+i, 2+i, 4+i)$ for $i=1,2,\ldots,2n$ (addition modulo $2n$). This gives a set T of $2n$ triangles. We prove

LEMMA 1. *The set T contains exactly those edges from P_1, P_2, P_3.*

Proof. We merely note that the differences between $1+i$ and $2+i$ are ± 1, between $1+i$ and $4+i$ are ± 3, between $2+i$ and $4+i$ are ± 2.

In the rest of this section we consider putting together sets P_{2x} and P_{2x+1} to form one-factors. Here we will assume that $2x+1<n$ (since P_n is special).

LEMMA 2. *Let $(2n, 2x+1)=m$; then the pairs in P_{2x+1} split into two one-factors.*

Proof. The gcd m must be odd; then the pairs of P_{2x+1} split into m cycles of even length $2n/m$.

For example, if $2n=12$, $2x+1=3$, the set P_3 splits into cycles as follows.

$(1,4)$, $(4,7)$, $(7,10)$, $(10,1)$
$(2,5)$, $(5,8)$, $(8,11)$, $(11,2)$
$(3,6)$, $(6,9)$, $(9,12)$, $(12,3)$.

We can use these cycles to give one-factors by taking alternate pairs, namely,

$(1,4)$, $(7,10)$, $(5,8)$, $(11,2)$, $(3,6)$, $(9,12)$,

and

$(4,7)$, $(10,1)$, $(2,5)$, $(8,11)$, $(6,9)$, $(12,3)$.

This procedure is perfectly general. The numbers in the cycles are, in general,

$1, 2x+2, 4x+3, 6x+4, \ldots, -2x$
$2, 2x+3, 4x+4, 6x+5, \ldots, 1-2x$
....
$m, m+2x+1, m+4x+2, \ldots, m-1-2x$

and the one-factors can be written in the general form

$(1, 2x+2)$, $(4x+3, 6x+4), \ldots, (-4x-1, -2x)$,
$(2x+3, 4x+4)$, $(6x+5, 8x+6), \ldots, (1-2x, 2)$,
....
$(m, m+2x+1) \cdots (m-4x-2, m-2x-1)$

and

$$(2x+2, 4x+3) \cdots (-2x, 1)$$
$$(2, 2x+3) \cdots (-4x, 1-2x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(m+2x+1, m+4x+2) \cdots (m-1-2x, m).$$

The behaviour of P_{2x} is slightly more complicated since the gcd $(2x, 2n)$ is bound to be even. Our first result is

LEMMA 3. *If* $(2n, 2x) = d$, *and if* $2n/d$ *is even, then* P_{2x} *splits into two one-factors.*

Proof. The proof is the same as in Lemma 2. The pairs of P_{2x} create d cycles of length $2n/d$, namely,

$$1, 2x+1, 4x+1, \ldots, 1-2x$$
$$2, 2x+2, 4x+2, \ldots, 2-2x$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$d-1, 2x+d-1, 4x+d-1, \ldots, d-1-2x$$
$$d, 2x+d, 4x+d, \ldots, d-2x.$$

Since the length $2n/d$ of each cycle is even, the construction of Lemma 2 applies, and the pairs of P_{2x} split into two one-factors.

We now want to consider the case $2n/d$ odd. The cycle decomposition now has cycles of odd lengths. We write the cycles over again, in pairs, making a unit shift in the second cycle of each pair. Thus the cycles are written as

$$\begin{cases} 1, 2x+1, 4x+1, \ldots, 1-2x \\ 2x+2, 4x+2, 6x+2, \ldots, 2 \end{cases}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\begin{cases} d-1, 2x+d-1, 4x+d-1, d-1-2x \\ d+2x, d+4x, d+6x, \ldots, d. \end{cases}$$

We form two one-factors from these cycle pairs as follows. For the first one-factor, use the two first elements in each cycle pair:

$$\begin{array}{lll} (1, 2x+2) & \text{with} & (2x+1, 4x+1)\cdots(1-4x, 1-2x) \\ & & (4x+2, 6x+2)\cdots(2-2x, 2) \\ \cdots & & \\ (d-1, d+2x) & \text{with} & (2x+d-1, 4x+d-1)\cdots(d-1-4x, d-1-2x) \\ & & (d+4x, d+6x)\cdots(d-2x, d). \end{array}$$

For the second one-factor, use the two last elements in each cycle pair:

$$\begin{array}{lll} (1-2x, 2) & \text{with} & (1, 2x+1)\cdots(1-6x, 1-4x) \\ & & (2x+2, 4x+2)\cdots(2-4x, 2-2x) \\ \cdots & & \\ (d-1-2x, d) & \text{with} & (d-1, 2x+d-1)\cdots(d-1-6x, d-1-4x) \\ & & (d+2x, d+4x)\cdots(d-4x, d-2x). \end{array}$$

There is a problem with these two one-factors. They have repeated a set A of edges from P_{2x+1}, and they have omitted a set B of edges from P_{2x}. The set A is obviously given by the "first-last" edges from the cycle pairs.

$$A\begin{cases}(1, 2x+2)\cdots(d-1, d+2x) \\ (1-2x, 2)\cdots(d-1-2x, d).\end{cases}$$

The set B consists of the "first-last" joins in each cycle.

$$B\begin{cases}(1, 1-2x), (2, 2x+2), \dots \\ \qquad (d-1-2x, d-1), (d+2x, d).\end{cases}$$

We now have the apparatus to prove

THEOREM 3.1. *If $2x+1<2n$, then $P_{2x}\cup P_{2x+1}$ splits into four one-factors.*

Proof. If $(2x, 2n)=d$ and $2n/d$ is even, then Lemmas 2 and 3 provide the answer. Each of P_{2x} and P_{2x+1} provides two one-factors.

If $(2x, 2n)=d$ and $2n/d$ is odd, we make the following construction. Use Lemma 1 to provide two one-factors from P_{2x+1}, namely F_1 and F_2. Use the cycle construction as described to produce two more one-factors F_3 and F_4 from P_{2x} which repeat the edges A and omit edges B.

Now note that all edges A appear in F_1 (they are just the edges with "lower" element odd). We alter F_1, still keeping it a one-factor by switching edge pairs. Thus we change

$$(1, 2x+2)(1-2x, 2) \quad \text{to} \quad (1, 1-2x)(2, 2+2x)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(d-1, d+2x)(d-1-2x, d) \quad \text{to} \quad (d-1, d-1-2x)(d, d+2x).$$

This result is to give an altered one-factor F_1^*, which now contains the missing edges from B but no longer contains the edges from A. So we have

$$P_{2x} \cup P_{2x+1} = F_1^* + F_2 + F_3 + F_4.$$

We must now consider the special set P_n. It is unlike all other P_i, in that it contains only n rather than $2n$ edges.

LEMMA 4. *If n is even, then P_n is a single one-factor. If n is odd, then $P_{n-1} \cup P_n$ can be split into three one-factors.*

Proof. The case of n even is obvious; the edges are $(1, 1+n)$, $(2, 2+n), \ldots,$ $(n, 2n)$. If n is odd, then the pairs of P_n again form a single one-factor. However, we must pair P_n with P_{n-1}; in this case, $(2n, n-1) = 2$ and consequently $2n/d$ is odd; the switching operation between P_n and P_{n-1} still works and so $P_{n-1} \cup P_n$ gives three one-factors.

Our final result is

THEOREM 3.2. *The graph K_{2n} may be factored into a set of triangles covering P_1, P_2, P_3, and a set of $2n-7$ one-factors covering the other P_i.*

Proof. Lemma 1 handles P_1, P_2, P_3. If n is even, then Theorem 2 handles

$$P_4 \cup P_5, \; P_6 \cup P_7, \ldots, P_{n-2} \cup P_{n-1},$$

to give $4[(n-2)/2-1] = 2n-8$ one-factors. P_n is a single one-factor (Lemma 4). This gives our result.

If n is odd, Theorem 2 applies to

$$P_4 \cup P_5, \; P_6 \cup P_7, \ldots, P_{n-3} \cup P_{n-2},$$

to give $4[(n-3)/2-1] = 2n-10$ one-factors. $P_{n-1} \cup P_n$ gives three one-factors (Lemma 4). Again, the total is $2n-7$ one-factors.

4. An example

We wrote out the difference partition for K_{12} at the beginning of Section 3.

The set T is immediate. Also, P_6 is a one-factor. So we merely need construct four one-factors from $P_4 \cup P_5$.

Lemma 2 gives two one-factors from P_5, namely,

$$F_1:(1,6),\ (11,4),\ (9,2),\ (7,12),\ (5,10),\ (3,8)$$
$$F_2:(6,11),\ (4,9),\ (2,7),\ (12,5),\ (10,3),\ (8,1).$$

P_4 has $(12,4)=3$; so we get cycles of length 3, namely, 1 5 9, 2 6 10, 3 7 11, 4 8 12. Pair the cycles as

1	5	9	3	7	11
6	10	2	8	12	4

and create one-factors

$$F_3:(1,6),\ (5,9),\ (10,2);\ (3,8),\ (7,11),\ (12,4)$$
$$F_4:(9,2),\ (1,5),\ (6,10);\ (11,4),\ (3,7),\ (8,12).$$

Now A is the set $(1,6)$, $(3,8)$, $(9,2)$, $(11,4)$ appearing in F_1. B is the set $(1,9)$, $(6,2)$, $(3,11)$, $(4,8)$; create F_1^* by replacing A by B and we have

$$F_1^* = (1,9),\ (6,2),\ (3,11),\ (4,8),\ (7,12),\ (5,10).$$

Thus $P_4 \cup P_5 = F_1^* + F_2 + F_3 + F_4$.

5. Steiner triple systems

We now apparently switch our attention to a completely different topic. A *Steiner triple system* is a set of b blocks of three elements each (triples), selected from a total set of v elements, with the property that every element is used r times and every pair of elements occurs once. A well-known example has $v=9$, $b=12$, $r=4$, and is illustrated by the blocks 1 3 9, 1 4 2, 3 5 8, 3 4 6, 4 5 7, 5 6 1, 6 7 9, 7 2 3, 9 8 4, 9 2 5, 2 8 6, 8 1 7.

We can immediately prove the classical

THEOREM 5.1. *In a Steiner triple system,*

$$r=\frac{v-1}{2}, \qquad b=\frac{v(v-1)}{6}.$$

Proof. The number of elements in the blocks can be counted as $3b$ or as rv. The number of pairs is counted as $3b$ or as $\binom{v}{2}$. Thus

$$3b=rv=\frac{v(v-1)}{2},$$

and the Theorem follows.

THEOREM 5.2. *For a Steiner triple system to exist, it is necessary that* $v\equiv 1$ *or* $v\equiv 3 \pmod 6$.

Proof. r is an integer; hence v is odd. Thus $v\equiv 1$, 3, or 5 (mod 6). But $v=6t+5$ gives

$$b=\frac{(6t+5)(6t+4)}{6},$$

and this is non-integral. Hence the Theorem.

Our main aim is to show how our knowledge of one-factors of K_{2n} provides an easy proof of the converse of Theorem 5.2.

6. Incidence matrices

A *design* is merely a set of blocks selected from a variety set of v elements. Thus if we have eight elements, the blocks

$\emptyset$

1 2 4

1 3 5 6 7

1 5

2

3 4 7

are a design (not a very interesting one, admittedly). A Steiner triple system is a design where all blocks have length 3 (contain 3 elements), and each pair of elements occurs once in the design.

One easy way of representing a design is by writing down its *incidence matrix*. This is a matrix A of size $v \times b$ which is made up of zeros and ones, with $a_{ij} = 1$ if variety i is in block j, $a_{ij} = 0$ if variety i is not in block j. For example, the design listed at the beginning of this section has incidence matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The empty column corresponds to the null block; the empty row corresponds to the non-occurrence of variety 8.

The triple system on 9 elements listed in Section 5 has a 9×12 incidence matrix, namely,

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Every Steiner triple system has such an incidence matrix with row sums equal to r, column sums equal to 3. We shall use incidence matrices to link up the results on one-factorizations of K_{2n} with Steiner triple systems.

7. An embedding theorem

The set 1 2 4 is a (trivial) STS on three elements; we call it D_1. The set of blocks 1 2 4, 2 3 5, 3 4 6, 4 5 7, 5 6 1, 6 7 2, 7 1 3, is an STS on 7 elements; we call it D_2.

In this example, the blocks of D_1 (there is only one) occur as a subsystem of the blocks of D_2. When this occurs, we say that D_1 is *embedded* in D_2.

From the point of view of incidence matrices, we may permute the blocks of D_2 so that the first columns of the incidence matrix of D_2 (say $I(D_2)$) correspond to those blocks in D_1. Then we may permute rows of $I(D_2)$ so that the first rows of $I(D_2)$ correspond to exactly those varieties appearing in D_1. The incidence matrix of $I(D_2)$ then has the form

$$I(D_2) = \left(\begin{array}{c|c} I(D_1) & B \\ \hline 0 & C \end{array}\right).$$

For example, the designs D_2 and D_1 at the beginning of this section give such an incidence matrix as

$$\begin{array}{r} \text{var} \\ 1 \\ 2 \\ 4 \\ I(D_2) = 3 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \\ \left(\begin{array}{c|cccccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right). \end{array}$$

We now prove a fundamental result.

THEOREM 7.1. *Any Steiner triple system on v elements can be embedded in an STS on $2v+1$ elements.*

Proof. Let the designs be called D_1 and D_2. The number of blocks in D_1 is $v(v-1)/6$, in D_2 is $(2v+1)v/3$.

The r-values for D_1 and D_2 (row sums) are $(v-1)/2$ and v respectively. Now, let us analyze

$$I(D_2)=\left(\begin{array}{c|c} I(D_1) & B \\ \hline 0 & C \end{array}\right).$$

First, there are

$$\frac{(2v+1)v}{3}-\frac{v(v-1)}{6}=\frac{v(v+1)}{2}$$

columns in B and in C. Also, there are $(2v+1)-(v)=v+1$ rows in C.

Now every pair of varieties corresponding to the last $v+1$ rows must occur. Hence there are at least two ones in each column of C.

The number of ones in each row of B is $v-(v-1)/2=(v+1)/2$. Since each of the first v varieties must occur with each of the last $v+1$, the structure of B and C is now constrained: each of the $(v+1)/2$ occurrences of variety i $(i=1,2,\ldots,v)$ in row i of B occurs in a column with exactly two 1's corresponding to varieties between $v+1$ and $2v+1$. These last pairs give a 1-factor of K_{v+1} in a complete graph whose vertices are named from $v+1$ to $2v+1$.

This occurs for each i $(i=1,2,3,\ldots,v)$, and so we have v edge-disjoint one-factors of K_{v+1}, that is, a one-factorization. So we build up B by putting $(v+1)/2$ entries of 1 in each row (all columns disjoint). Then C is filled in by placing a 1-factor of K_{v+1} in those entries corresponding to occurrences of variety i. By Theorem 1, this is always possible.

EXAMPLE. We take D_1 as the design on varieties $1, 2, \ldots, 7$ with blocks $124, 235, 346, \ldots, 713$. Then D_1 has incidence matrix

$$A=\begin{pmatrix} 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 0&1&1&0&0&0&1\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&0&1&1&0\\ 0&0&0&1&0&1&1 \end{pmatrix}.$$

The incidence matrix of D_2 must have 4 more ones in each of the first 7 rows.

Thus

$$I(D_2)=\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right),$$

where B is a 7×28 matrix with form

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1
\end{pmatrix}.$$

We now need a 1-factorization of K_8 based on variety names 8, 9, 10, 11, 12, 13, 14, 15. This can be written down by Theorem 1 as follows.

F_1: (8, 15), (9, 14), (10, 13), (11, 12)
F_2: (9, 15), (8, 10), (11, 14), (12, 13)
F_3: (10, 15), (9, 11), (8, 12), (13, 14)
F_4: (11, 15), (10, 12), (9, 13), (8, 14)
F_5: (12, 15), (11, 13), (10, 14), (8, 9)
F_6: (13, 15), (12, 14), (8, 11), (9, 10)
F_7: (14, 15), (13, 8), (12, 9), (10, 11).

As soon as these one-factors are produced, we can write down C. The first 8 columns of C correspond to F_1 and F_2 and are displayed:

$$C=\left(\begin{array}{cccc|cccc|c|cccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 & 0
\end{array}\right).$$

8. A second embedding theorem

As might be imagined, we now use Theorem 3 to give another embedding theorem.

THEOREM 8.1. *Any STS on v elements can be embedded in an STS on $2v+7$ elements.*

Proof. Our method is similar to that employed in the last theorem. If A is the incidence matrix of D_1, then again we have

$$I(D_2)=\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right).$$

The number of rows in C is now $v+7$. The number of columns in B and C is

$$\frac{(2v+7)(2v+6)}{6}-\frac{v(v-1)}{6}=\frac{(v+7)(v+2)}{2}.$$

The numbers of 1's per row of B is

$$\frac{2v+6}{2}-\frac{v-1}{2}=\frac{v+7}{2}.$$

These ones pair up, as in the last Theorem, with 1-factors on $v+1,\ldots,2v+7$, to produce $\frac{v(v+7)}{2}$ columns of C. But this leaves exactly $v+7$ columns over; these last $v+7$ columns have zeros in B and have three ones per column in C.

We thus see that $I(D_2)$ can be written in the more specific form:

$$I(D_2)=\left(\begin{array}{c|c|c} A & B_1 & 0 \\ \hline 0 & C_1 & C_2 \end{array}\right).$$

B_1 has $(v+7)/2$ ones per row; C_1 is made up of v one-factors of K_{v+7} (on the symbols $v+1,\ldots,2v+7$), and C_2 is made up of $v+7$ triples covering exactly those edges of K_{v+7} not already used. This is exactly the sort of factorization provided by Theorem 3; so, our embedding is established.

EXAMPLE. For D_1, we take the trivial design 1 2 3; then

$$A=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

B_1 is merely the matrix with 3 rows and 15 columns given by

$$B_1=\begin{pmatrix}1&1&1&1&1&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&1&1&1&1&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1&1&1&1&1\end{pmatrix}.$$

To get C_1 and C_2, we use Theorem 3 on K_{10}, using symbols $4, 5, 6, \ldots, 13$. Our result first gives C_2 in the form

$$C_2=\begin{pmatrix}1&0&0&0&0&0&0&1&0&1\\1&1&0&0&0&0&0&0&1&0\\0&1&1&0&0&0&0&0&0&1\\1&0&1&1&0&0&0&0&0&0\\0&1&0&1&1&0&0&0&0&0\\0&0&1&0&1&1&0&0&0&0\\0&0&0&1&0&1&1&0&0&0\\0&0&0&0&1&0&1&1&0&0\\0&0&0&0&0&1&0&1&1&0\\0&0&0&0&0&0&1&0&1&1\end{pmatrix}.$$

The partitions P_4 and P_5 of Lemma 4 are just

P_4: (4, 8), (5, 9), (6, 10), . . . , (13, 7)
P_5: (4, 9), (5, 10), (6, 11), (7, 12), (8, 13).

P_5 is a one-factor itself; P_4 yields cycles

4, 8, 12, 6, 10 and 5, 9, 13, 7, 11.

Our construction then writes these as

4, 8, 12, 6, 10
9, 13, 7, 11, 5

and produces one-factors

$F_1:(4, 9), (8, 12), (6, 10), (13, 7), (11, 5),$

and

$F_2:(10, 5), (4, 8), (12, 6), (9, 13), (7, 11).$

Our switching operation then replaces P_5 by the one factor

$F_3:(4, 10), (5, 9), (6, 11), (7, 12), (8, 13).$

These three one-factors then allow us to write down C_1 in the form

$$C_1 = \left(\begin{array}{ccccc|ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

We have thus constructed the incidence matrix of a design on 13 varieties with 26 blocks.

9. A different embedding

For a different touch, we use another method to prove an extra embedding theorem which we do not really need. However, the result is so simple that it fits in nicely with those of the last two sections.

THEOREM 9.1. *Any STS on v symbols can be embedded in an STS on $3v$ symbols.*

Proof. Write down $v(v-1)/6$ blocks on symbols $1_1, 2_1, \ldots, v_1$. Write down a second such design on symbols $1_2, 2_2, \ldots, v_2$; finally, write down a third design on $1_3, 2_3, \ldots, v_3$. This gives $v(v-1)/2$ blocks.

Now write down a $v \times v$ latin square on the symbols $1_1, 2_1, \ldots, v_1$. Any latin square will do. So we really need not know what a latin square is; we might just write down the array

$$\begin{array}{cccccc} 1_1 & 2_1 & 3_1 & \cdots & v_1 \\ v_1 & 1_1 & 2_1 & \cdots & (v-1)_1 \\ (v-1)_1 & v_1 & 1_1 & \cdots & (v-2)_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2_1 & 3_1 & 4_1 & \cdots & 1_1 \end{array}$$

There are v^2 elements in this array. We use them to create v^2 triples by the algorithm:

(Element)(Row co-ordinate sub 2)(Column co-ordinate sub 3).

We now have $v^2 + v(v-1)/2 = (3v-1)v/2$ blocks; this is the correct number for an STS on $3v$ elements. So we merely need check that each pair occurs exactly once.

Pairs (a_i, b_i) occur exactly once in the three embedded systems. Pairs (a_1, b_2) occur exactly once since each symbol occurs exactly once in each row of the latin square; pairs (a_1, b_3) occur exactly once since each symbol occurs exactly once in each column of the latin square; finally, pairs (a_2, b_3) occur exactly once since row and column co-ordinates identify a unique element of the latin square.

EXAMPLE. Start with the design $1_1\,2_1\,3_1$ and form 2 copies $1_2\,2_2\,3_2$ and $1_3\,2_3\,3_3$. Write down the array

$$\begin{array}{ccc} 1_1 & 2_1 & 3_1 \\ 3_1 & 1_1 & 2_1 \\ 2_1 & 3_1 & 1_1 \end{array}$$

and use it to give 9 more blocks, namely,

$$\begin{array}{ccc} 1_1\,1_2\,1_3 & 2_1\,1_2\,2_3 & 3_1\,1_2\,3_3 \\ 3_1\,2_2\,1_3 & 1_1\,2_2\,2_3 & 2_1\,2_2\,3_3 \\ 2_1\,3_2\,1_3 & 3_1\,3_2\,2_3 & 1_1\,3_2\,3_3. \end{array}$$

A similar procedure immediately takes an STS on 7 symbols and produces the $21+49=70$ blocks of an STS on 21 symbols.

10. The existence theorem for triple systems

We now have a means of constructing Steiner triple systems on $6t+1$ or $6t+3$ symbols (the only possibilities, by Theorem 5). We list the first few cases before proving the general theorem.

v	*Construction*
3	One block 1 2 3
7	$2(3)+1$
9	$3(3)=2(1)+7$
13	$2(3)+7$
15	$2(7)+1$
19	$2(9)+1$
21	$2(7)+7$
25	$2(9)+7$
27	$2(13)+1$
31	$2(15)+1$

Note that Theorem 8, the tripling construction, is not really needed except for a system on 9 elements (and we could write $9=2(1)+7$, the STS on 1 element being the null set).

THEOREM 10.1. *An STS exists for any number of elements v, where $v \equiv 1$ or 3 (mod 6).*

Proof. Certainly we have just checked this up to v-values of 31.

Any larger v can be specified as $12t+1$, $12t+3$, $12t+7$, $12t+9$, and we make the induction hypothesis that the designs exist for smaller values. But

$$12t+1=2(6t-3)+7,$$

$$12t+3=2(6t+1)+1,$$

$$12t+7=2(6t+3)+1,$$

$$12t+9=2(6t+1)+7.$$

These equations establish Theorem 10.1 by induction. Indeed, we have proved the stronger result that there exist designs on $12t+3$ and $12t+9$ symbols containing an embedded design on $6t+1$ symbols; there exists a design on $12t+1$ symbols containing an embedded design on $6t-3$ symbols; there exists a design on $12t+7$ symbols containing an embedded design on $6t+3$ symbols.

11. Pairings

We now move from general triple systems to cyclically generated systems. These special systems will be generated using a *pairing* concept.

Let $P(1, n)$ be a set of n pairs of integers in which each of the integers 1 to $2n$ appears exactly once as an element of a pair and each of the integers 1 to n occurs exactly once as a difference between elements of the same pair. For example, the pairs (1, 2), (5, 7), (3, 6), (4, 8) form a $P(1, 4)$. Similarly, $P(1, n)/j$ is defined to be a set of $n-1$ pairs with each of the integers 1 to $2(n-1)$ appearing exactly once and each of the integers from 1 to n except j occurring as a difference exactly once. Thus, the pairs (3, 4), (6, 8), (1, 5), (2, 7) form a $P(1, 5)/3$.

It is convenient to represent $P(1, n)$ pictorially by placing $2n$ points labelled 1 to $2n$ on a line and drawing an edge between points that are paired together. $P(1, n)/j$ would be similarly represented on $2(n-1)$ points. For example, the $P(1, 4)$ and $P(1, 5)/3$ described in the last paragraph would be represented as in Figure 3. Thus, in a $P(1, n)$, each point must be incident with exactly one edge and the n edges represent each of the differences 1 to n exactly once.

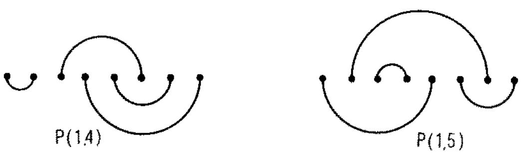

Figure 3

$P(1, n)$ does not exist for all n. If we colour the $2n$ points alternately black and white, there are n of each colour. Pairs with an odd difference contain one point of each colour. In pairs with an even difference, both points are the same colour. Thus, it is necessary that there be an even number of even integers less than or equal to n. Therefore $[n/2]$ must be even, and we have

THEOREM 11.1. *$P(1, n)$ can exist only for $n \equiv 0, 1 \pmod 4$.*

A similar argument determines a necessary condition for the existence of a $P(1, n)/2$.

THEOREM 11.2. *$P(1/n)/2$ can exist only for $n \equiv 2, 3 \pmod 4$.*

In fact, these conditions are also sufficient, as we shall demonstrate.

12. Sufficiency conditions

Consider $n = 4k$. A $P(1, 4)$ was exhibited in Figure 3. $P(1, 8)$ and $P(1, 12)$ are represented in Figure 4.

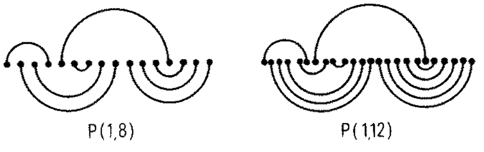

Figure 4

A general solution is suggested by Figure 4, and $P(1, 4k)$ is displayed in Figure 5 for $k \geqslant 2$ (the differences label the edges).

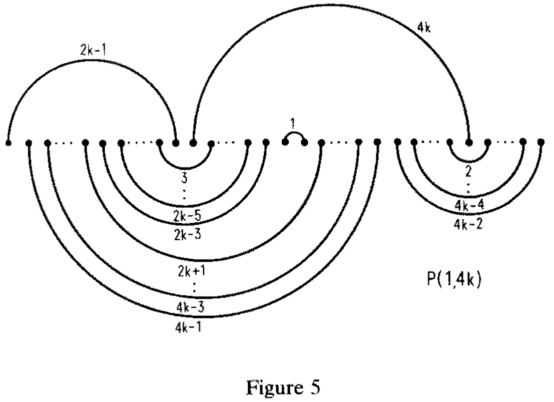

Figure 5

A very simple modification provides a $P(1, 4k+1)$ for $k \geqslant 2$ (see Figure 6).

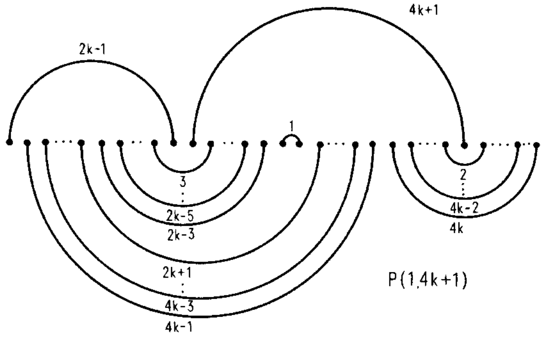

Figure 6

A solution for $P(1, 1)$ and $P(1, 5)$ exists as shown in Figure 7.

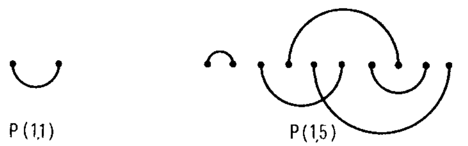

Figure 7

So we have established the following theorem.

THEOREM 12.1. *$P(1, n)$ exists if and only if* $n \equiv 0, 1 \pmod 4$.

Solutions as given by the figures above are as follows.

$P(1,4)$: $(1, 2)$, $(5, 7)$, $(3, 6)$, $(4, 8)$.

$P(1, 4k)$, $k \geqslant 2$: $(3k, 3k+1)$, $(1, 2k)$, $(2k+1, 6k+1)$,

$(2k-i, 2k+1+i)$ for $i = 1, 2, \ldots, k-2$,

$(k+2-i, 3k+1+i)$ for $i = 1, 2, \ldots, k$,

$(6k+1-i, 6k+1+i)$ for $i = 1, 2, \ldots, 2k-1$.

$P(1,1):(1,2)$.

$P(1,5):(1,2),\ (7,9),\ (3,6),\ (4,8),\ (5,10)$.

$$P(1,4k+1),\ k\geq 2:\ (3k,3k+1),\ (1,2k),\ (2k+1,6k+2),$$
$$(2k-i,2k+1+i)\quad\text{for}\quad i=1,2,\ldots,k-2,$$
$$(k+2-i,3k+1+i)\quad\text{for}\quad i=1,2,\ldots,k,$$
$$(6k+2-i,6k+2+i)\quad\text{for}\quad i=1,2,\ldots,2k.$$

For $n=4k+2$, and $k\geq 3$, a solution for $P(1,n)/2$ is represented in Figure 8.

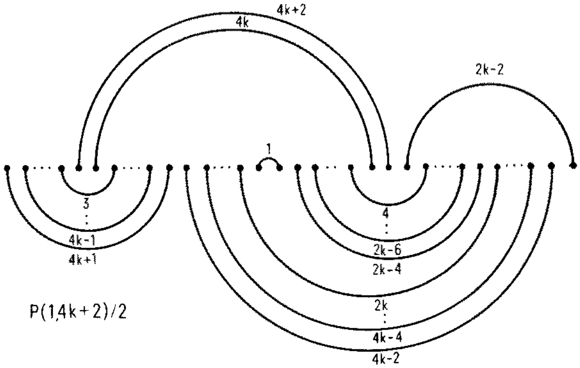

Figure 8

A simple modification yields a $P(1,n)/2$ for $n=4k+3$, $k\geq 2$ (see Figure 9).

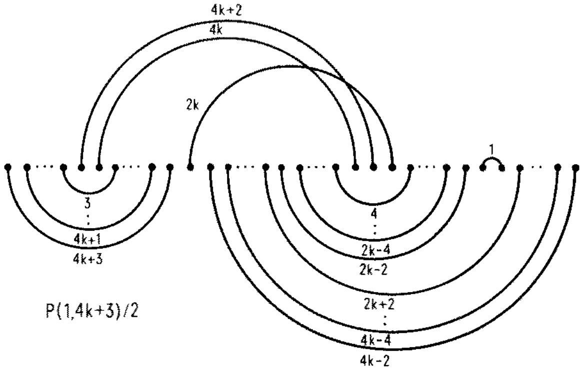

Figure 9

For $n = 2, 3, 6, 7, 10$, we have solutions for $P(1, n)/2$ represented in Figure 10.

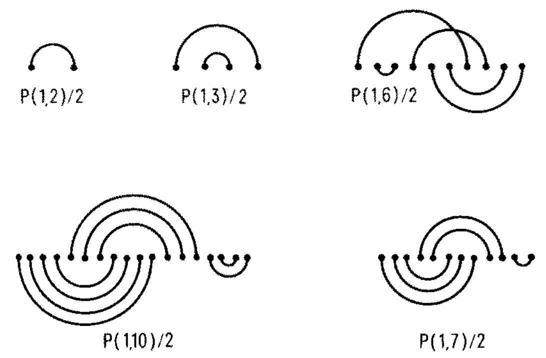

Figure 10

We have thus established the following theorem.

THEOREM 12.2. *$P(1, n)/2$ exists if and only if* $n \equiv 2, 3 \pmod 4$.

The solutions represented by the figures above are as follows.

$P(1, 2)/2 : (1, 2)$.

$P(1, 6)/2 : (2, 3), (6, 9), (4, 8), (5, 10), (1, 7)$.

$P(1, 10)/2 : (1, 11), (2, 10), (3, 9), (4, 8), (5, 14), (6, 13), (7, 12),$
$(15, 18), (16, 17)$.

$P(1, 4k+2)/2,\ k \geq 3 : (2k+2, 6k+2), (2k+1, 6k+3), (6k+4, 8k+2),$
$(5k+3, 5k+4),$
$(2k+1-i, 2k+2+i)$ for $i = 1, 2, \ldots, 2k,$
$(6k+2-i, 6k+4+i)$ for $i = 1, 2, \ldots, k-3,$
$(5k+3-i, 7k+1+i)$ for $i = 1, 2, \ldots, k.$

$P(1, 3)/2 : (2, 3), (1, 4)$.

$P(1, 7)/2 : (1, 8), (2, 7), (3, 6), (4, 10), (5, 9), (11, 12)$.

$$P(1, 4k+3)/2,\ k \geq 2: (2k+3, 6k+3),\ (2k+2, 6k+4),\ (4k+5, 6k+5),$$
$$(7k+4, 7k+5),$$
$$(2k+2-i, 2k+3+i) \quad \text{for} \quad i = 1, 2, \ldots, 2k+1,$$
$$(6k+3-i, 6k+5+i) \quad \text{for} \quad 1, 2, \ldots, k-2,$$
$$(5k+5-i, 7k+5+i) \quad \text{for} \quad i = 1, 2, \ldots, k-1.$$

13. Cyclic Steiner triple systems

As we pointed out earlier, a Steiner triple system on v elements, denoted $S(2, 3, v)$, is a set of triples from v elements in which each pair of distinct elements occur together in a triple precisely once. There are $b = v(v-1)/6$ triples in such a system and each element occurs $r = (v-1)/2$ times; such a system is a balanced incomplete block design with parameters $(v, v(v-1)/6, (v-1)/2, 3, 1)$, and $v \equiv 1, 3 \pmod 6$.

We wish to construct the $t(6t+1)$ triples of an $S(2, 3, 6t+1)$ in the following manner: form t sets of $6t+1$ triples by adding $0, 1, \ldots, 6t \pmod{6t+1}$ to all elements of a set of initial triples (a_i, b_i, c_i), $i = 1, 2, \ldots, t$. Thus all pairs having differences $\pm(a_i - b_i)$, $\pm(b_i - c_i)$, $\pm(c_i - a_i)$, mod $6t+1$, will be represented in the set. Adopt the convention that the differences are in the range $-3t$ to $3t$. Then, to represent each pair of distinct elements exactly once, the set of t initial triples must have each non-zero integer from $-3t$ to $3t$ represented exactly once in the $6t$ differences. For $t = 2$, the pair of triples $(0, 1, 4)$, $(0, 2, 7)$ is such a set of initial triples; we shall say that these produce the difference triples $(1, 3, 4)$ and $(2, 5, 6)$, meaning thereby that they produce the differences ± 1, ± 3, ± 4, and ± 2, ± 5, ± 6. Let $Q(t)$ be the set of difference triples. $Q(t)$ is thus a set of t triples (x_i, y_i, z_i) in which each of the elements $1, 2, \ldots, 3t$ is represented exactly once and either $x_i + y_i = z_i$ or $x_i + y_i + z_i = 6t+1$ ($i = 1$ to t).

The process is reversible since, if $Q(t)$ exists, the set of triples $(0, x_i, x_i + y_i)$ is an appropriate set of initial triples to generate a design $S(2, 3, 6t+1)$ on elements $0, 1, \ldots, 6t$. Thus we have

THEOREM 13.1. *If $Q(t)$ exists, then $S(2, 3, 6t+1)$ exists.*

14. Construction of the sets $Q(t)$

Let us try to find a set $Q(t)$ in the restricted case that $x_i = i$ and $x_i + y_i = z_i$ for all $i = 1, 2, \ldots, t$. Then the set of pairs (y_i, z_i) has each integer from $t+1$ to $3t$ appearing exactly once; since $z_i - y_i = i$, each integer from 1 to t appears exactly once as a difference. Thus, the set of pairs $(y_i - t, z_i - t)$, where $i = 1, 2, \ldots, t$, is a $P(1, t)$. Conversely, given a $P(1, t)$, we can reverse the process by placing the elements of pairs in increasing order, adding t to each, and adjoining the difference to yield a $Q(t)$. Thus, by Theorem 12.1, we have

THEOREM 14.1. *$Q(t)$ exists for $t \equiv 0, 1 \pmod 4$.*

For example, corresponding to the $P(1, 4)$ given by $(1, 2)$, $(5, 7)$, $(3, 6)$, $(4, 8)$, we have the following $Q(4)$:

$(1, 5, 6)$, $(2, 9, 11)$, $(3, 7, 10)$, $(4, 8, 12)$.

Now restrict $Q(t)$ so that $x_i = i$, $x_i + y_i = z_i$, for $i \neq 2$; $(y_2, z_2) = (3t - 1, 3t)$. Then we have the set of $t-1$ pairs (y_i, z_i), for $i = 1, 3, 4, \ldots, t$, with $z_i - y_i = i$; each element from $t+1$ to $3t-2$ appears exactly once. Thus, $(y_i - t, z_i - t)$, for $i = 1, 3, 4, \ldots, t$, is a pairing $P(1, t)/2$. Again, we can reverse the process and Theorem 12.2 gives

THEOREM 14.2. *$Q(t)$ exists for $t \equiv 2, 3 \pmod 4$.*

As an example, if $P(1, 6)/2$ is $(2, 3)$, $(6, 9)$, $(4, 8)$, $(5, 10)$, $(1, 7)$, then the corresponding $Q(6)$ is

$(2, 17, 18)$, $(1, 8, 9)$, $(3, 12, 15)$, $(4, 10, 14)$, $(5, 11, 16)$, $(6, 7, 13)$.

Thus, combining Theorems 13.1, 14.1 and 14.2 we have

THEOREM 14.3. *$S(2, 3, 6t+1)$ exists for all values of t.*

The triples of an $S(2, 3, 6t+1)$ on the elements $0, 1, 2, \ldots, 6t$ are obtained by adding $0, 1, 2, \ldots, 6t \pmod{6t+1}$ to the set of triples given below.

$t = 4: (0, 1, 6), (0, 2, 11), (0, 3, 10), (0, 4, 12)$.

$t = 4k,\ k \geq 2: (0, 1, 7k+1), (0, 2k-1, 6k), (0, 4k, 10k+1),$

$(0, 2i+1, 6k+1+i)$ for $i = 1, 2, \ldots, k-2$,

$(0, 2k-1+2i, 7k+1+i)$ for $i = 1, 2, \ldots, k$,

$(0, 2i, 10k+1+i)$ for $i = 1, 2, \ldots, 2k-1$.

$t=1:(0, 1, 3)$.

$t=5:(0, 1, 7), (0, 2, 14), (0, 3, 11), (0, 4, 13), (0, 5, 15)$.

$t=4k+1,\ k\geq 2:(0, 1, 7k+2), (0, 2k-1, 6k+1), (0, 4k+1, 10k+3)$,

$(0, 2i+1, 6k+2+i)$ for $i=1, 2, \ldots, k-2$,

$(0, 2k-1+2i, 7k+2+i)$ for $i=1, 2, \ldots, k$,

$(0, 2i, 10k+3+i)$ for $i=1, 2, \ldots, 2k$.

$t=2:(0, 1, 4), (0, 2, 7)$.

$t=6:(0, 1, 9), (0, 2, 19), (0, 3, 15), (0, 4, 14), (0, 5, 16)$,

$(0, 6, 13)$.

$t=10:(0, 1, 27), (0, 2, 31), (0, 3, 28), (0, 4, 18), (0, 5, 22)$,

$(0, 6, 19), (0, 7, 23), (0, 8, 20), (0, 9, 24), (0, 10, 21)$.

$t=4k+2,\ k\geq 3:(0, 4k, 10k+4), (0, 4k+2, 10k+5), (0, 2k-2, 12k+4)$

$(0, 1, 9k+6), (0, 2, 12k+7)$,

$(0, 2i+1, 6k+4+i)$ for $i=1, 2, \ldots, 2k$,

$(0, 2i+2, 10k+6+i)$ for $i=1, 2, \ldots, k-3$,

$(0, 2k-2+2i, 11k+3+i)$ for $i=1, 2, \ldots, k$.

$t=3:(0, 1, 6), (0, 2, 10), (0, 3, 7)$.

$t=7:(0, 1, 19), (0, 2, 22), (0, 3, 13), (0, 4, 16), (0, 5, 14)$,

$(0, 6, 17), (0, 7, 15)$.

$t=4k+3,\ k\geq 2:(0, 4k, 10k+6), (0, 4k+2, 10k+7), (0, 2k, 10k+8)$,

$(0, 1, 11k+8), (0, 2, 12k+10)$,

$(0, 2i+1, 6k+6+i)$ for $i=1, 2, \ldots, 2k+1$,

$(0, 2i+2, 10k+8+i)$ for $i=1, 2, \ldots, k-2$,

$(0, 2k+2i, 11k+8+i)$ for $i=1, 2, \ldots, k-1$.

15. Cyclic Steiner systems on $6t+3$ elements

For completeness, we include a well-known construction for $S(2,3,6t+3)$ on elements (i,j), for $i=0,1,2$ and $j=0,1,\ldots 2t$ (Cf. [1], [3] or [5]). The system is obtained by adding $(0,0)$, $(0,1),\ldots,(0,2t)$ to each of the following $3t+1$ triples; addition is mod 3 for the first component and mod $2t+1$ for the second component.

$$[(j,i+1),\ (j,2t-i),\ (j+1,0)] \text{ for } j=0,1,2, \text{ and } i=0,1,\ldots,t-1;$$
$$[(0,0),(1,0),(2,0)].$$

Thus we have

THEOREM 15.1. *$S(2,3,6t+3)$ exists for all values of t.*

A simple cyclic construction, yielding $S(2,3,6t+3)$ on elements $0,1,\ldots,6t+2$, may be obtained for $t\equiv 0 \pmod 3$ (see, for example, [8]).

We have thus given constructive cyclic methods for generating Steiner triple systems on v elements whenever $v\equiv 1,3 \pmod 6$. This provides an elementary proof of

THEOREM 15.2. *$S(2,3,v)$ exists and can be cyclically generated for $v\equiv 1,3$ (mod 6).*

REFERENCES

[1] BOSE, R. C., *On the construction of balanced incomplete block designs.* Annals of Eugenics *9* (1939), 353–399.
[2] DOYEN, J. and ROSA, A., *An extended bibliography and survey of Steiner systems.* Congressus Numerantium *20* (1977), 297–361.
[3] HALL, J., JR., *Combinatorial theory.* Blaisdell Publishing Co., Waltham, Mass., 1967.
[4] HARARY, F., *Graph theory.* Addison-Wesley Publishing Co., Reading, Mass., 1969.
[5] MANN, H. B., *Analysis and design of experiments.* Dover Publications, New York, 1949.
[6] PELTESOHN, R., *Eine Lösung der beiden Heffterschen Differenzenprobleme.* Compositio Math. 6 (1939), 251–257.
[7] ROSA, A., *Poznamka o cyklických Steinerových systémoch trojic.* Mat.-Fyz. Časopis Sloven. Akad. Vied. *16* (1969), 285–290.
[8] STREET, A. P. and WALLIS, W. D., *Combinatorial theory: an introduction.* Chas. Babbage Research Centre Inc., St. Pierre, Manitoba, 1977.

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