A SAT + Computer Algebra Attack on Ramsey Problems

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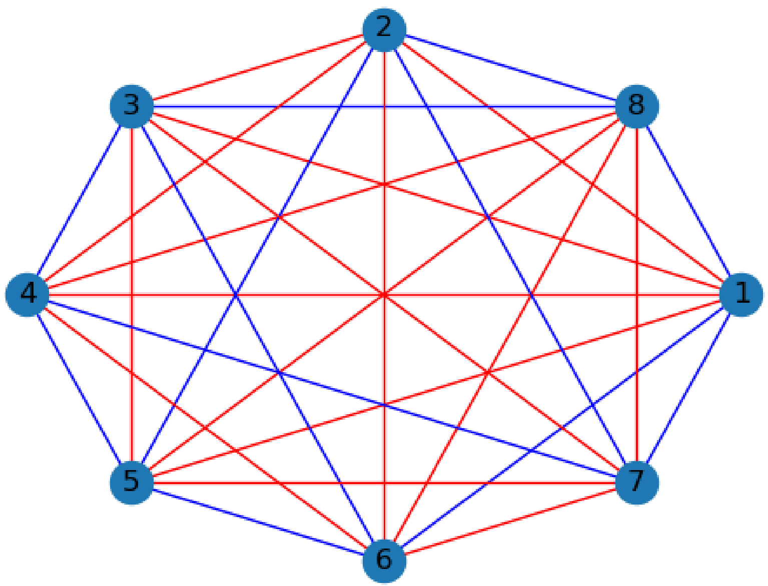
Ramsey Theory – emergence of order

The Ramsey Theorem states that for every $p, q \in \mathbb{Z}$, there exists an $n \in \mathbb{Z}$ such that **every** graph of order n, contains a p-clique or an independent set of size q.

A Ramsey problem is defined as finding the smallest integer n, denoted $R(p, q)$, for some given input p, q.

"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack" – Erdős 1990



A graph on 8 vertices without a blue triangle or red 4-clique, showing *R(3,4) > 8*.

How hard are the problems?

R(p,q)	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		2	3	4	5	6	7	8	9
3			6	9	14	18	23	28	36
4				18	25	36-40	49-58	59-79	73-106
5					43-48	59-85	80-133	101-194	133-282

- Only 9 non-trivial Ramsey numbers are known.
- A graph on n vertices has $\frac{n(n-1)}{2} \in O(n^2)$ edges. There are $\mathrm{O}(2^{n^2})$ possible edge-colourings.
  - For 28 vertices, the search space is huge, there are more than 6×10^{113} possible colourings.

What is a SAT solver?

- A program that takes as input a Boolean formula.
- Determines whether there exists an assignment of variables in the input formula such that the formula evaluates to True.
- Learns through propagation and backtracking.

∨: OR

∧: AND

¬: NEGATION

$$\bigwedge_{K_p \subset K_n} \bigvee_{e \in K_p} \neg e \wedge \bigwedge_{K_q \subset K_n} \bigvee_{e \in K_q} e$$

Example:

$B: (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3) \wedge (x_3 \vee x_4)$

Suppose we choose $x_1 = T$ to propagate and then $x_2 = T$;

1) $(T \vee \neg x_2 \vee x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge (F \vee x_3) \wedge (\neg x_3 \vee x_4)$

2) $(T \vee F \vee x_3) \wedge (F \vee \neg x_3) \wedge (F \vee x_3) \wedge (\neg x_3 \vee x_4)$

Tidying this we see:

3) $(\neg x_3) \wedge (x_3) \wedge (\neg x_3 \vee x_4)$

There is a conflict on x_3 caused by $x_1 \wedge x_2$, thus the solver learns $\neg(x_1 \wedge x_2)$.

What is a CAS?

- Designed to handle mathematical objects symbolically.
- We use it to remove isomorphic graphs from the search space via orderly generation programmatically.

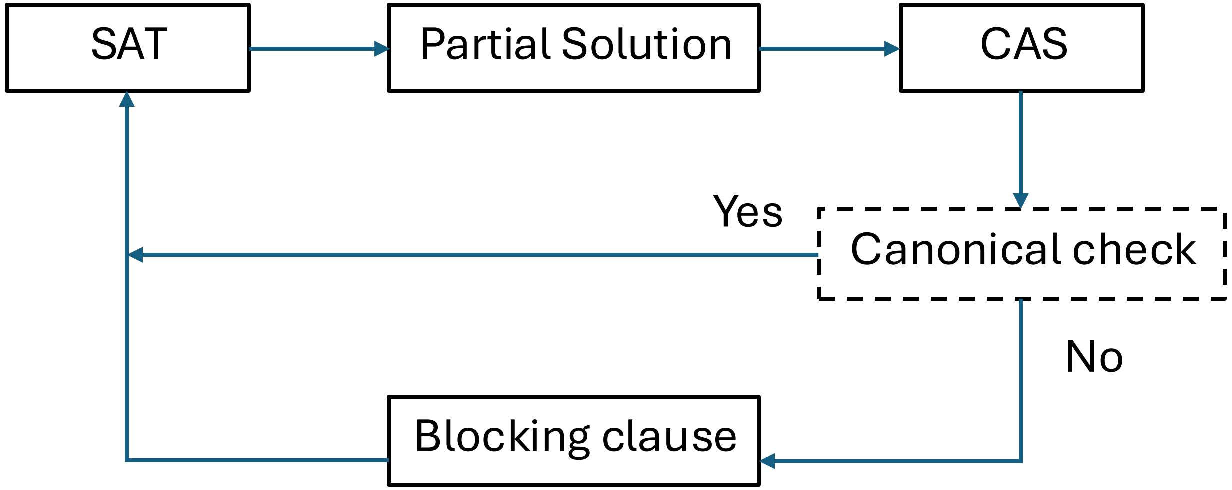
Non-canonical

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Canonical

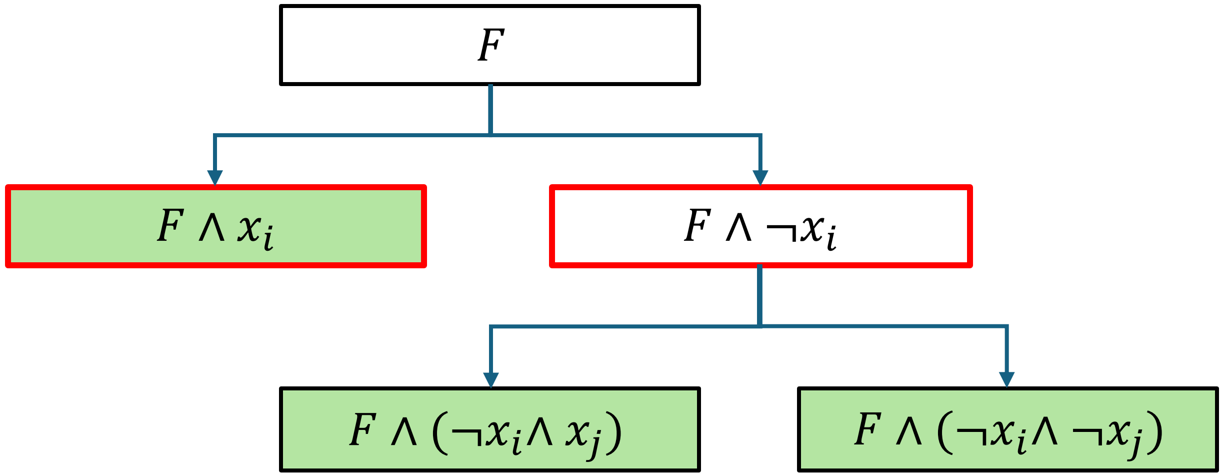
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Blocks all extensions of non-canonical graphs.
- Provides an orders of magnitude speed up on Ramsey problems.



Parallelisation through Cube and Conquer

- Solving a Boolean formula F is equivalent to solving $F \wedge x_i$ and $F \wedge \neg x_i$, for some x_i appearing in F.
- Rank variables and choose the variable which creates the most balanced tree.
- Initial cost to rank variables.



Solving and Verifying *R(3,8)*

- Originally solved in 1992 By McKay and Min by recursively generating bigger Ramsey graphs and removing non-canonical graphs.
- 695 million graph extensions and 5.2 million distinct canonical graphs generated.
- However, errors can occur in any computer assisted proof.
- We use a DRAT proof checker to verify the clauses learnt by the SAT solver and use custom code to verify that the clauses learnt by the CAS block non-canonical graphs.
- We solved in 57 hours and generated a 31Gb proof file. This verified in 89 hours.

Solving and Verifying *R(3,9)*

- Existence of a (3,9)-graph on 36 vertices implies the existence of a (3,8)-graph on 27 vertices and 80 edges, Graver and Yackel 1968.
- Using a series of lemmas on (3,7)-graphs and computational non-existence proofs, Grinstead and Roberts 1982 showed no (3,8;27;80)-graphs exist.
- We solved and verified an encoding of (3,8;27;80)-graph using cube and conquer with the SAT+CAS.
- In total, we generated 13.7k cubes and used a total computational time of 245 days to solve and verify. The total size of the proof files was 1.1TB.
- With parallelisation, this corresponded to 2 days real time.

Conclusion and future

- Utilised theoretical results and SAT+CAS to automatically solve and verify $R(3,8)$ and $R(3,9)$.

<u>Future</u>

- Applying a SAT+CAS directly to $R(3,10)$ is infeasible due to space requirements. A series of theoretical results, similar to the $R(3,9)$ result, reduces the requirements.
- Although significantly harder problems, $R(4,5)$ and $R(4,6)$ are feasible. Theoretical results would also help.
- Experiments with additional constraints – degree constraints, maximal p-clique free etc.
- Apply to Ramsey variants

Thank you