Thermal Resistance of a Buried Cylinder With Constant Flux Boundary Condition

R. Thiyagarajan[1] **and M. M. Yovanovich**[2]

Nomenclature

- a = distance from ground surface to origin of bicylindrical
- g = metric coefficient
- k = thermal conductivity
- Q = total heat flow rate per unit length of pipe
- q = flux per unit area
- R = thermal resistance
- $R_{q=c}$ = constant flux resistance
- $R_{T=c}$ = constant temperature resistance
- r_0 = pipe radius
- T = temperature
- T_a = average surface temperature
- ψ = bicylindrical coordinate
- η = bicylindrical coordinate
- ω = distance from ground surface to pipe center

[1] Graduate Research Assistant, Thermal Engineering group, Department of Mechanical Engineering, University of Waterloo, Waterloo, Ont., Canada.

[2] Professor of Mechanical Engineering, Thermal Engineering group, Department of Mechanical Engineering, University of Waterloo, Waterloo, Ont., Canada. Mem. ASME.

[3] Number in brackets designate References at end of technical brief.

Contributed by the Heat Transfer Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscripts received by the Heat Transfer Division July 3, 1973.

Introduction

The determination of the thermal resistance for heat transfer between the surface of a buried pipe or a buried cylinder and ground level is important and has been calculated for the constant temperature boundary condition at the pipe surface [1–4].[3] The method used most often is based upon the superposition of infinite line source and sink solutions [1–3]. A solution based upon bicylindrical coordinates is also available [4]. A buried cable or a heating wire with uniform joulean heating will have a constant flux condition. However, in real situations, the actual boundary condition is not known exactly. It is felt that in most cases, it will be between the constant temperature and the constant flux condition. It becomes necessary then, to have the thermal resistance values corresponding to these two extreme cases. The thermal resistance to the constant flux condition is not available in conduction heat transfer texts, except reference [5], where a solution based upon bicylindrical coordinates is given. This solution in reference [5] is in error because the temperature and the thermal resistance go to zero, for the same heat transfer, as the pipe is displaced farther below ground level. This is obviously incorrect. The purpose of this technical brief, therefore, is to fill this gap and to point out the error in reference [5], so that we will have the thermal resistance values corresponding to the extreme cases, namely constant temperature and constant flux conditions. This brief deals with the exact solution based upon bicylindrical coordinates. The thermal resistance with a constant flux boundary condition is compared with the well-known expression for the constant temperature boundary condition.

Analysis

The governing differential equation for steady heat transfer through a homogeneous and isotropic medium of thermal conductivity k in bicylindrical coordinates is [4, 6]

$$\frac{\partial^2 T}{\partial \eta^2} + \frac{\partial^2 T}{\partial \psi^2} = 0 \qquad (1)$$

where η and ψ are the bicylindrical coordinates, Fig. 1. The surface of the ground is $\eta = 0$, and can have the temperature $T = 0$. The surface of the pipe $\eta = \eta_0$, has a constant flux boundary condition [4].

$$\left[\frac{k}{\sqrt{g_\eta}} \frac{\partial T}{\partial \eta}\right]_{\eta_0} = \frac{Q}{2\pi r_0} \tag{2}$$

The metric coefficient in the η-direction g_η, i.e., normal to the pipe surface, is given by the following expression [4, 6]:

$$g_\eta = \frac{a^2}{[\cos h\, \eta - \cos \psi]^2} \tag{3}$$

where a is the distance from the ground surface to the origin of the coordinate system, Fig. 1. By symmetry the other boundary conditions are $\partial T/\partial \psi = 0$ at $\psi = 0$ and π.

Equation (1) can be separated into two ordinary differential equations whose solutions depend upon trigonometric as well as hyperbolic functions. By the method of separation of variables, the solution to equation (1) satisfying the boundary conditions is

$$T(\eta, \psi) = \frac{Q}{\pi k}\left[\frac{\eta}{2} + \sum_{n=1}^{\infty} \frac{e^{-n\eta_0}}{n \cos h(n\eta_0)} \sin h\,(n\eta) \cos\,(n\psi)\right] \tag{4}$$

where η_0 is related to the pipe radius and the distance from ground surface to the pipe center line as follows [4]:

$$\omega = r_0 \cos h\, \eta_0 \tag{5}$$

Equation (4) differs from the solution given in reference [5], in that it does not contain the term $1/\sin h\eta_0$. The thermal resistance will be defined as the difference between the average pipe surface temperature and the ground surface temperature divided by the total heat flow rate. Therefore

$$R_{q=c} = \frac{T_a}{Q} \tag{6}$$

where

$$T_a = \frac{\int_0^\pi T\sqrt{[g_\psi]}_{\eta=\eta_0}\, d\psi}{\int_0^\pi [\sqrt{g_\psi}]_{\eta=\eta_0}\, d\psi} \tag{7}$$

with $g_\psi = g_\eta$ for the bicylindrical system [4, 6]. Upon substitution of equation (4) into (7) and evaluating, one obtains for the average pipe surface temperature

$$T_a = \frac{Q}{\pi k}\left[\frac{\eta_0}{2} + \sum_{n=1}^{\infty} \frac{e^{-2n\eta_0}}{n} \tan h\,(n\eta_0)\right] \tag{8}$$

According to our definition of thermal resistance, the constant flux resistance per unit length of pipe is therefore:

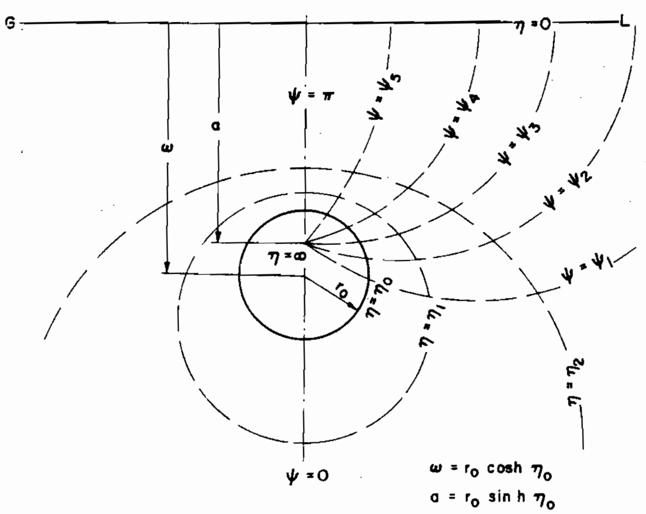

Fig. 1

Table 1

ω/r_0	η_0	$\frac{R_{q=c}}{R_{T=c}}$
1.001	0.04	20.21
1.005	0.10	9.07
1.010	0.14	6.44
1.050	0.31	2.97
1.1	0.44	2.18
1.2	0.62	1.66
1.3	0.76	1.44
1.4	0.87	1.32
1.5	0.96	1.25
1.6	1.05	1.20
1.7	1.12	1.16
1.8	1.19	1.14
1.9	1.26	1.11
2.0	1.32	1.10
3.0	1.76	1.03
4.0	2.06	1.015
5.0	2.29	1.009
10.0	2.99	1.002

$$R_{q=c} = \frac{\eta_0}{2\pi k} + \frac{1}{\pi k} \sum_{n=1}^{\infty} \frac{e^{-2n\eta_0}}{n} \tan h\,(n\eta_0) \tag{9}$$

This expression can now be compared with the constant temperature resistance expression [1–3]

$$R_{T=c} = \frac{1}{2\pi k} \ln\left[\frac{\omega}{r_0} + \sqrt{\left(\frac{\omega}{r_0}\right)^2 - 1}\right] \tag{10}$$

which can also be written as

$$R_{T=c} = \frac{\eta_0}{2\pi k} \tag{11}$$

The ratio of the constant flux resistance to the constant temperature resistance is simply:

$$\frac{R_{q=c}}{R_{T=c}} = 1 + \frac{2}{\eta_0} \sum_{n=1}^{\infty} \frac{e^{-2n\eta_0}}{n} \tan h\,(n\eta_0) \tag{12}$$

The ratio is shown in Table 1 for typical values of ω/r_0 and corresponding values of η_0.

Conclusion

An exact solution for the constant flux boundary condition is presented and the thermal resistance expression is compared with the well-known constant temperature expression. It is seen in equation (12) and Table 1, that the resistance for the constant flux boundary condition is equal to the constant temperature resistance plus an additional resistance which is due to the fact that more heat leaves the bottom portion of the pipe under constant flux conditions than under constant temperature conditions. This effect is strongly dependent upon η_0. If the pipe is buried deeper than 5 pipe radii, the difference between the two resistances is less than one percent. It can be seen that for all practical purposes a pipe which is buried at a depth of 3 or more pipe radii, can be treated as a pipe having constant temperature boundary conditions. Also for a pipe which is buried 1.5 radii or more, the difference in the resistances corresponding to the two extreme boundary conditions is 25 percent or less. Where there is a possibility, that the actual boundary condition may lie in between the two extreme cases, we now have the opportunity to use a realistic value, since we have solutions available for both cases.

References

1 Eckert, E. R. G., and Drake, R. M., *Heat and Mass Transfer*, Second ed., McGraw-Hill, New York, pp. 60–64.

2 Boelter, L. M. K., Cherry, V. H., Johnson, H. A., and Martinelli, R. C., *Heat Transfer Notes*, McGraw-Hill, New York, 1965, pp. 297–299.

3 Schenck, H., *Heat Transfer Engineering*, Prentice-Hall, 1959, pp. 35–46.

4 Yovanovich, M. M., *Advanced Heat Conduction*, ME 751 Notes; to be published.

5 Lebedev, N. N., Skalskaya, I. P., and Uflyand, Ya. S., *Problems in Mathematical Physics*, (translated from Russian), Prentice Hall, 1965.

6 Magnus, W., and Oberhettinger, F., *Formulas and Theorems for the Functions of Mathematical Physics*, Chelsea Publishing Co., New York, 1949, pp. 153.