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A Physical Approach to the Finite Difference Solution of the Conduction Equation in Orthogonal Curvilinear Coordinates

G. E. SCHNEIDER

Project Engineer

A. B. STRONG

Associate Professor

M. M. YOVANOVICH

Professor
Mem. ASME

Thermal Engineering Group,
Department of Mechanical Engineering,
University of Waterloo,
Waterloo, Ontario, Canada

A unified approach to the numerical solution of the transient heat conduction equation is presented. By formulating the numerical description of the heat conduction problem in a general orthogonal curvilinear coordinate system, advantages similar to those experienced in analytic solutions become available to the numerical analyst. Generalized finite difference coefficients are obtained by imposing a physical balance of the rates of heat flow, storage and generation on discrete curvilinear control volumes distributed spatially throughout the solution domain. This development is complemented by consideration of boundary condition application in general orthogonal coordinates which then permits the complete numerical description of conduction problems in any orthogonal coordinate system. Two references cited illustrate its successful usage on practical problems. The generalized derivation presented here has been shown to provide substantial flexibility, accuracy and economy of finite difference solutions when appropriate selection of the coordinate system is made.

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A Physical Approach to the Finite Difference Solution of the Conduction Equation in Orthogonal Curvilinear Coordinates

G. E. SCHNEIDER A. B. STRONG M. M. YOVANOVICH

ABSTRACT

A unified approach to the numerical solution of the transient heat conduction equation is presented. By formulating the numerical description of the heat conduction problem in a general orthogonal curvilinear coordinate system, advantages similar to those experienced in analytic solutions become available to the numerical analyst. Generalized finite difference coefficients are obtained by imposing a physical balance of the rates of heat flow, storage and generation on discrete curvilinear control volumes distributed spatially throughout the solution domain. This development is complemented by consideration of boundary condition application in general orthogonal coordinates which then permits the complete numerical description of conduction problems in any orthogonal coordinate system. Two references cited illustrate its successful usage on practical problems. The generalized derivation presented here has been shown to provide substantial flexibility, accuracy and economy of finite difference solutions when appropriate selection of the coordinate system is made.

NOMENCLATURE

A	area
B_1, B_2	constants, given by equation (27a) and (27b)
C_1, C_2, C_3, C_4, C_5, C_6, C_s	finite difference (f.d.) coefficients, equation (4)
C_p	specific heat
D	constant term in f.d. equation
e	internal energy per unit volume
g_1, g_2, g_3, g	metric or Lamé coefficients, equations (2) and (3)
h	convective heat transfer coefficient
i, j, k	nodal indices for f.d. discretization
P	heat generation rate per unit volume
q	heat flow rate per unit area
Q	heat flow rate
R	thermal resistance
s	length in curvilinear space
t	time
T	temperature
Ts	surface temperature on boundary
Tf	fluid temperature at convective boundary
u_1, u_2, u_3	generalized orthogonal curvilinear coordinates
V	volume
x, y, z	coordinates in cartesian frame
δ	inter-nodal spacing for accompanying arguments, equation (8)
Δ	increment in the accompanying argument
λ	thermal conductivity
ρ	density

INTRODUCTION

The advent of space exploration, nuclear power utilization, and measures for energy conservation has led to a vast number of complex heat conduction problems requiring solution. In addition to the increased number of engineering problems requiring attention, high accuracy is often demanded of the resultant solution, thus compounding the problem. The complex nature of the problem, which may be due to one or more of boundary conditions, geometry, or property variations, often excludes the possibility of finding an exact analytical solution. Thus the thermal designer often resorts to numerical methods to obtain a solution.

The finite difference method is a numerical solution precedure popular in the solution of heat transfer problems [1-6]. In this method, a spatial discretization process leads to a system of simultaneous algebraic equations which must be solved to determine the temperatures at discrete locations in the field, the nodal values. Determination of the coefficients multiplying the nodal temperatures in these algebraic equations is of considerable concern since the degree of approximation implied in obtaining these coefficients directly influences the accuracy and stability of the solution.

In the excellent work by Clausing [7], the three conventional coordinate systems are treated and their corresponding finite difference coefficients obtained. The major limitation of this analysis, however, lies in the restriction to the three conventional coordinate systems. In fact, in his and other difference analyses [1-6], not only is the restriction to the three conventional coordinate systems made, but for each of the three, a completely independent development has been required. Recently, a specialized coordinate system was applied successfully in a finite difference solution for a particular nonconventional problem [8, 9]; the generality used in the derivation of the coefficients provided one motivation for the present paper.

In addition, since the resultant solution accuracy is sensitive to the degree of spatial discretization, certain problems require that a great many algebraic equations be simultaneously solved. Even with the use of contemporary computational facilities, however, the number of equations cannot be indefinitely increased to obtain a prescribed accuracy while keeping the problem solution economically feasible. This is of particular concern in the analysis of mixed boundary value problems where very large gradients are experienced near boundary condition discontinuities. If one returns to analytical methods for guidance in overcoming these problems, it is noted that sometimes it is possible to set up a system of coordinates 'more natural' to the field of interest, in this case that of heat conduction, whose coordinate surfaces conform to the lines of flow and the potential surfaces [10]. In the solution of many of these problems, the nature of the resultant field is determined by the specification of its behavior at its bounding surfaces, by specifying the nature and position of its singularities, or by a combination of these two influences. The resultant field specification may often have a simple and tractable form in terms of these 'natural' coordinates [11] and thus the solution is easily obtained.

While many multi-dimensional problems can be reduced to problems dependent upon a single curvilinear coordinate, there still remains a wide variety of problems which cannot, but for which the flow of heat is predominantly unidirectional in nature. Where possible, analytic solutions to these problems are preferable to numerical solutions since the effect of various solution parameters can immediately be evaluated by examination of the functional form of the solution. Unfortunately, however, the scope of problems which lend themselves to such analytic solutions, whether approximate or exact, is limited to those having relatively simple problem specifications. The vast majority of two- and three-dimensional problems in conduction heat transfer have no known analytic solution due to irregular boundary geometries and/or inconvenient boundary conditions.

Convinced that advantages similar to those available when using the most appropriate coordinate system in an analytical solution are possible when performing a numerical solution, it is apparent that the need exists to examine the application of numerical solution techniques to general orthogonal curvilinear coordinate systems. This need has been satisfied in part by the work of Schneider, Strong and Yovanovich [12] in which generalized finite difference coefficients were obtained through a Taylor series approximation to the heat conduction equation. It is the aim of this work to recast the above formulation using an elemental heat balance to provide a generalized development of the finite difference coefficients. This will be done from physical considerations for use with any orthogonal curvilinear coordinate system in the numerical description of the heat conduction equation, by applying the first law of thermodynamics to a control volume centered about the current node of interest. Evaluation of the pertinent heat flow, storage, and generation rates will lead to expressions for the associated finite difference coefficients for use in numerical solutions. Successful application of these results is discussed in references [8] and [9].

It should be noted that in their work in recirculating convective flows a somewhat different approach was taken by Gosman et al [14] in that the finite difference procedure was formulated in generalized orthogonal coordinates. The starting point by these authors however was the integral form of the equations of motion which represents a significant difference from that of the present contribution.

PRELIMINARY REMARKS

In a general orthogonal curvilinear coordinate system, (u_1, u_2, u_3), the heat conduction equation can be written as [10].

$$\frac{\partial}{\partial u_1}\left[\frac{\lambda\sqrt{g}}{g_1}\frac{\partial T}{\partial u_1}\right] + \frac{\partial}{\partial u_2}\left[\frac{\lambda\sqrt{g}}{g_2}\frac{\partial T}{\partial u_2}\right] + \frac{\partial}{\partial u_3}\left[\frac{\lambda\sqrt{g}}{g_3}\frac{\partial T}{\partial u_3}\right] + P\sqrt{g} = \frac{\partial}{\partial t}\left(\sqrt{g}\,\rho C_p\,T\right) \tag{1}$$

where the metric coefficients relating the curvilinear system to the cartesian frame are defined by

$$g_i \equiv \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2, \quad i = 1, 2, 3 \tag{2}$$

and

$$g \equiv g_2 \cdot g_2 \cdot g_3 \tag{3}$$

The finite difference method is concerned with the approximation of equation (1) for use in a numerical computational scheme and reduces the problem from that of finding solutions to equation (1) to that of solving a system of simultaneous equations of the form

$$C_1 T_{i-1,j,k} + C_2 T_{i+1,j,k} + C_3 T_{i,j-1,k} + C_4 T_{i,j+1,k} + C_5 T_{i,j,k-1} + C_6 T_{i,j,k+1} + C_8 T_{i,j,k} + D = 0 \tag{4}$$

where the subscripts refer to locations within the discretized spatial domain. The constant term D contains information regarding the heat generation as well as temperatures from the time planes preceding the one under consideration. Equation (4) can be written once for each location of the discretized spatial domain yielding a system of simultaneous equations which require solution. In general, the coefficients of equation (4) must also be allowed to vary from location to location in the field. It is the basic problem, then, for any finite difference analysis to determine the coefficients and constants appearing in equation (4) for use with the particular coordinate system under consideration. An analysis is presented herein which examines this problem in a general fashion so that the results are applicable for any orthogonal coordinate system provided the associated metric coefficients are known. This is done by examining a finite volume element and applying the first law of thermodynamics.

GENERAL CONSIDERATIONS

Figure 1 illustrates a typical volume element in a general orthogonal curvilinear coordinate system having coordinate directions u_1, u_2 and u_3.

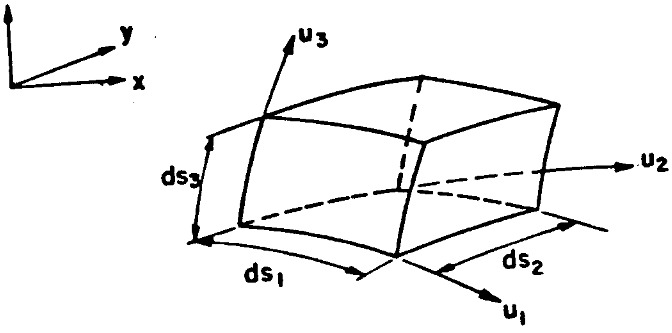

Fig. 1. Typical volume element in curvilinear coordinates

The physical dimensions of the volume element are related to the variation in the coordinate value through the metric coefficients by the relation [10]

$$ds_i = \sqrt{g_i}\, du_i, \quad i = 1, 2, 3 \tag{5}$$

Using (5), area elements are given by

$$dA_i = \sqrt{g_j\, g_k}\, du_j\, du_k \quad , \quad \begin{matrix} i,\ j,\ k = 1,\ 2,\ 3 \\ i \neq j \neq k \end{matrix} \tag{6}$$

where the convention has been used that the direction associated with the area element be normal to the plane in which it lies. Similarly, the element of volume is determined from

$$dV = \sqrt{g}\, du_1\, du_2\, du_3 \tag{7}$$

To faciliate the finite difference analysis for systems having non-uniform grid sizing, a nomenclature will be introduced. Considering the one-dimensional network shown in Figure 2, the control volume width surrounding node u_i is characterized by Δu_i where u_i is a generalized orthogonal curvilinear coordinate; the nodal spacing is denoted by $\delta_-(u_i)$ and $\delta_+(u_i)$ for the spacing between nodes u_{i-1} and u_i, and u_i and u_{i+1} respectively.

The δ's of figure 2 are related to the Δ's by the relation

$$\delta_-(u_i) = \frac{\Delta u_{i-1} + \Delta u_i}{2} \ , \ \delta_+(u_i) = \frac{\Delta u_{i+1} + \Delta u_i}{2} \tag{8}$$

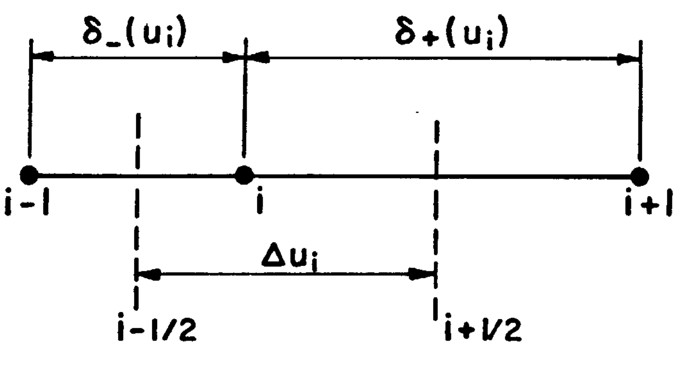

Fig. 2. Definition of spatial discretization nomenclature

Extension of this nomenclature to other coordinate directions is taken directly but it should be noted that Δu_i and $\delta_{\mp}(u_i)$ correspond to changes in the orthogonal coordinate u_i and may not reflect directly the physical distances involves.

CONTROL VOLUME ENERGY BALANCE

A finite difference representation of the heat conduction equation can be determined by examination of a finite control volume and applying the first law of thermodynamics. In words, an energy balance applied to a control volume (C.V.) as shown in figure 3 can be stated as

[Net rate of heat conduction into C.V.] +
[Rate of heat generation within C.V.] =
[Rate of change of internal energy within C.V.] (9)

Referring to the figure this can be written

$$Q_{i-\frac{1}{2},j,k} - Q_{i+\frac{1}{2},j,k} + Q_{i,j-\frac{1}{2},k} - Q_{i,j+\frac{1}{2},k} + Q_{i,j,k-\frac{1}{2}} - Q_{i,j,k+\frac{1}{2}} + \iiint_{C.V.} P\, dV = \frac{\partial}{\partial t} \iiint_{C.V.} \rho C_p T\, dV \tag{10}$$

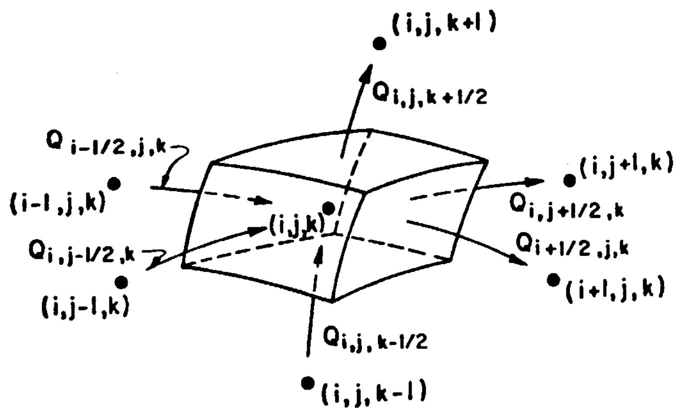

Fig. 3. Internodal heat flow rates

An evaluation of each term of the above equation (10) is given below:

1) Conduction

Considering a typical conductive heat transfer rate, for the u_1 - direction we can write

$$Q_{i+\frac{1}{2},j,k} = \iint_{A_{i+\frac{1}{2}}} q_{i+\frac{1}{2},j,k}\, dA \tag{11}$$

In equation (11) above the local flux on the surface $A_{i+\frac{1}{2}}$ is given by

$$q_{i+\frac{1}{2},j,k} = \left. \frac{-\lambda}{\sqrt{g_1}} \frac{\partial T}{\partial u_1} \right|_{i+\frac{1}{2}} \tag{12}$$

where the unknown derivative, $\partial T/\partial u_1$, must be approximated. Using a first central difference quotient, we write

$$\left.\frac{\partial T}{\partial u_1}\right|_{i+\frac{1}{2}} = \frac{T_{i+1,j,k} - T_{i,j,k}}{\delta_+(u_1)} + 0\ [\delta_+^2(u_1)] \qquad (13)$$

Using (13) and neglecting the terms of order $\delta_+^2(u_1)$ and higher, equation (11) can be written

$$Q_{i+\frac{1}{2},\ j,k} = \int_{u_3(k-\frac{1}{2})}^{u_3(k+\frac{1}{2})} \int_{u_2(j-\frac{1}{2})}^{u_2(j+\frac{1}{2})} -\left[\frac{\lambda\sqrt{g}}{g_1}\right]_{i+\frac{1}{2}} \left[\frac{T_{i+1,j,k} - T_{i,j,k}}{\delta_+(u_1)}\right] du_2\ du_3 \qquad (14)$$

having multiplied both numerator and denominator by $\sqrt{g_1}$ to obtain the term in the square brackets. Under the assumption that the gradient of equation (13) is uniform over the area $A_{i+\frac{1}{2}}$, the integration of equation (14) can be carried out in full. Further, defining a resistance in the usual fashion,

$$R_{i+\frac{1}{2},j,k} \equiv \frac{T_{i,j,k} - T_{i+1,j,k}}{Q_{i+\frac{1}{2},j,k}} \qquad (15)$$

We can write

$$\left[R_{i+\frac{1}{2},j,k}\right]^{-1} = \int_{u_3}\int_{u_2} \left[\frac{\lambda\sqrt{g}}{g_1}\right]_{i+\frac{1}{2}} \frac{du_2\ du_3}{\delta_+(u_1)} \qquad (16)$$

where the limits of integration are those indicated in equation (14). Expressions similar to equation (16) can be written for each of the three coordinate directions, in both the positive and negative directions.

2) Internal Heat Generation

This term is easily determine to be exactly

$$\iiint_{C.V.} P\ dV = \left[\int_{u_3(k-\frac{1}{2})}^{u_3(k+\frac{1}{2})} \int_{u_2(j-\frac{1}{2})}^{u_2(j+\frac{1}{2})} \int_{u_1(i-\frac{1}{2})}^{u_1(i+\frac{1}{2})} \times\ P\sqrt{g}\ du_1\ du_2\ du_3\right] \qquad (17)$$

3) Internal Energy

The last term of equation (10) involves the time derivatives of temperature. A central difference or forward difference approximation here requires an input of temperatures taken from a time-plane in the future, whose values are not yet known. Moreover, these are not of concern in the solution of the current time-plane temperatures. Therefore, unless a fully implicit forward difference solution with respect to time is attempted which is not considered practical with present computational storage requirement limitations, for present purposes the best approximation to time derivatives is considered to be the implicit backward difference quotient. The rate of change of internal energy is thus approximated by

$$\frac{\partial e}{\partial t} = \left[\frac{T_{i,j,k} - T^\circ_{i,j,k}}{\Delta t}\right] \int_{u_3}\int_{u_2}\int_{u_1} \rho C_p\sqrt{g}\ du_1\ du_2\ du_3 \qquad (18)$$

Using equations (14), (16), (17) and (18) in the energy balance (10) and comparing with equation (4)

$$C_1\ T_{i-1,j,k} + C_2\ T_{i+1,j,k} + C_3\ T_{i,j-1,k} + C_4\ T_{i,j+1,k} + C_5\ T_{i,j,k-1} + C_6\ T_{i,j,k+1} + C_s\ T_{i,j,k} + D = 0 \qquad (4)$$

leads to the definitions

$$\begin{aligned}
C_{1,2} &= \int_{u_3}\int_{u_2} \left[\frac{k\sqrt{g}}{g_1}\right]_{i\mp\frac{1}{2}} \frac{du_2\ du_3}{\delta_\mp(u_1)} \\
C_{3,4} &= \int_{u_3}\int_{u_1} \left[\frac{k\sqrt{g}}{g_2}\right]_{j\mp\frac{1}{2}} \frac{du_1\ du_3}{\delta_\mp(u_2)} \qquad (19) \\
C_{5,6} &= \int_{u_1}\int_{u_2} \left[\frac{k\sqrt{g}}{g_3}\right]_{k\mp\frac{1}{2}} \frac{du_1\ du_2}{\delta_\mp(u_3)} \\
C_s &= -\left[\sum_{n=1}^{6} C_n + \iiint_{C.V.} \left[\frac{\rho\ C_p\sqrt{g}}{\Delta t}\right] du_1\ du_2\ du_3\right]
\end{aligned}$$

$$\text{and } D = \iiint_{C.V.} \left[P + \frac{\rho\ C_p\ T^\circ_{i,j,k}}{\Delta T}\right] \sqrt{g}\ du_1\ du_2\ du_3$$

which are the desired coefficients. The expressions above are valid for every nodal location and control volume within the spatial domain of the problem provided the surfaces of the control volume do not contact physical boundaries of the real system. These coefficients can easily be evaluated for the common orthogonal curvilinear coordinate systems [13]. These control volumes require special treatment and are discussed below.

BOUNDARY CONDITION APPLICATION

The considerations required for boundary nodes having the various types of boundary conditions will be illustrated here by example. Following the reasoning of Clausing [7], the nodes will be located at the center of the control volume and the grid arranged so that where the physical boundaries follow the coordinate surfaces, the control volume boundaries will be made coincident with the physical boundaries.

1) Adiabatic Boundaries

As an example of an adiabatic boundary consider

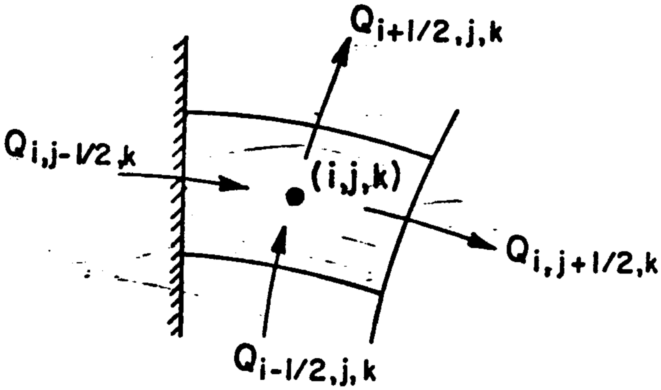

Fig. 4. Adiabatic boundary

the control volume of figure 4. This boundary condition is perhaps the simplest to apply. From the figure $Q_{i,j-\frac{1}{2},k} = 0$. The most direct method of obtaining this assignment is to set $C_3 = 0$ for all such boundary nodes. This is the only change required since by definition of C_s

$$C_s = -\left[\sum_{n=1}^{6} C_n + \iiint_{C.V.} \left[\frac{\rho\, C_p \sqrt{g}}{\Delta t}\right] du_1\, du_2\, du_3\right] \quad (20)$$

the influence of C_3 is felt on C_s by a change in the summation of the coefficients.

2. Specified Flux Boundary

Since the physical approach taken here is based on a control volume energy balance, this condition is also easily incorporated. For a flux distribution as shown in figure 5, the total heat flow rate entering the control volume through the surface can be determined by direct integration

$$Q_{i-\frac{1}{2},j,k} = \iint_{\Delta A} q\, dA \quad (21)$$

It should be noted that the terms of equation (21) are <u>not</u> dependent on the nodal temperatures associated with this control volume. Indeed, this term is really a rate of energy addition independent of the temperature field and as such 'represents a heat source'. In our finite difference equations for these control volumes this is represented by

$$D = \iiint_{C.V.} \left[P + \frac{\rho\, C_p\, T^\circ}{\Delta t}\right]\sqrt{g}\, du_1\, du_2\, du_3 + \iint_{\Delta A} q\,(u_2,\, u_3)\, \sqrt{g_2\, g_3}\, du_2\, du_3 \quad (22)$$

For these control volumes, then, equation (22) will account for the heat flow across the surface at the boundary and to include the conductive heat transfer expressed through the coefficient C_1 would be a redundant specification. Hence we must also make the assignment

$$C_1 = 0 \quad (23)$$

for these control volumes.

3) Specified Temperature Boundary

The expression of the heat flow rate crossing a temperature specified boundary typified by that of figure 6 can be handled in a manner similar to that of an internal control volume. Where the differential of temperature with respect to u_1 was formerly approximated by a central difference quotient about $(i+\frac{1}{2},j,k)$, the best estimate now available will be a first forward difference. Carrying this modification through the analysis leads to the simple modification of C_2 to

$$C_2 = \int_{u_3}\int_{u_2} \left[\frac{k\sqrt{g}}{g_1}\right]_{i+\frac{1}{2}} \frac{du_2\, du_3}{\Delta u_1/2} \quad (24)$$

which resembles its former value. The appropriate replacement for $T_{i+1,j,k}$ for use with this C_2 is $Ts_{j,k}$, the boundary surface temperature at (j,k).

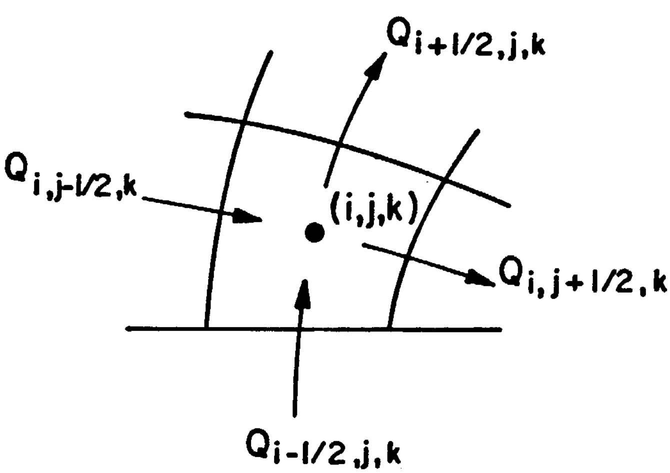

Fig. 5. Flux prescribed boundary

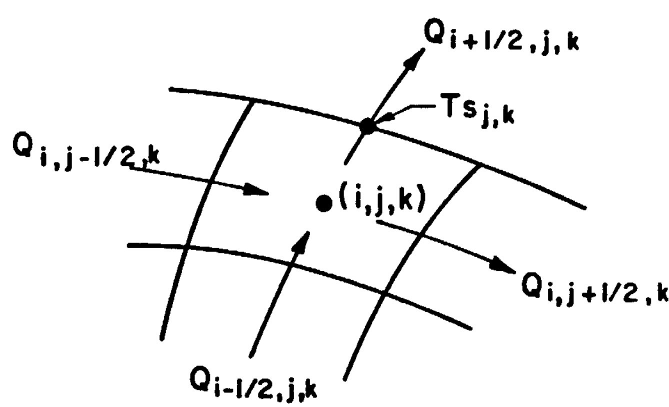

Fig. 6. Temperature prescribed boundary

4) Convective Boundary

This boundary condition presents the most difficulty to our analysis but when examined once, can be applied easily thereafter using the results. Consider the control volume of figure 7. The heat flow rate $Q_{i-\frac{1}{2},j,k}$ can be expressed for a uniform grid by

$$Q_{i-\frac{1}{2},j,k} = 2C_1 \left[Ts_{j,k} - T_{i,j,k} \right] \qquad (25)$$

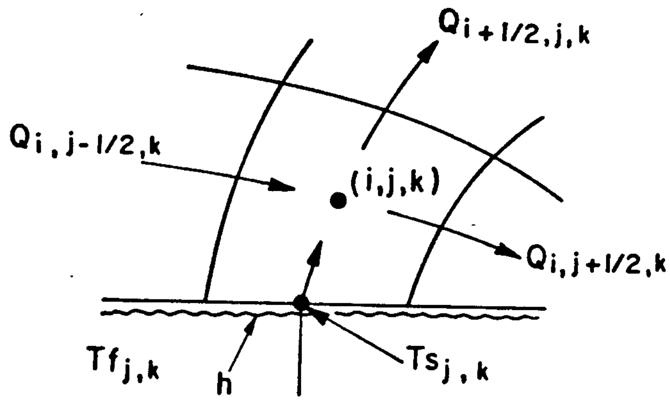

Fig. 7. Convective boundary

This same heat flow rate can also be given by

$$Q_{i-\frac{1}{2},j,k} = \left[Tf_{j,k} - Ts_{j,k} \right] \iint_{\Delta A} \left[h \sqrt{g_2 \, g_3} \right]_{i-\frac{1}{2}} du_2 \, du_3 \qquad (26)$$

where $Tf_{j,k}$ is the fluid temperature adjacent to the boundary at (j,k). Defining two new coefficients B_1 and B_2 by

$$B_1 = 2C_1 \qquad (27a)$$

and

$$B_2 = \iint_{\Delta A} \left[h \sqrt{g_2 \, g_3} \right]_{i-\frac{1}{2}} du_2 \, du_3 \qquad (27b)$$

the surface temperature, $Ts_{j,k}$, can be found by solving the simultaneous equations (25) and (26), and is

$$Ts_{j,k} = \frac{B_1 \, T_{i,j,k} + B_2 \, Tf_{j,k}}{B_1 + B_2} \qquad (28)$$

from which

$$\left[Ts_{j,k} - T_{i,j,k} \right] = \frac{B_2}{B_1 + B_2} \left[Tf_{j,k} - T_{i,j,k} \right] \qquad (29)$$

Using equations (25) and (29) the heat transfer rate crossing this control volume surface can be determined from

$$Q_{i-\frac{1}{2},j,k} = \frac{B_1 \, B_2}{(B_1 + B_2)} \left[Tf_{j,k} - T_{i,j,k} \right] \qquad (30)$$

It can be seen, then, that account of the convective boundary heat transfer can be made by assigning C_1 a new value for these control volumes, given by

$$C_1 = \frac{B_1 \, B_2}{(B_1 + B_2)}$$

where the appropriate value to be used for $T_{i-1,j,k}$ is $Tf_{j,k}$.

DISCUSSION AND CONCLUSIONS

General expressions for the coefficients in the finite difference representation of the heat conduction equation have been derived in this work. It is now possible, knowing the metric coefficients for a coordinate system, to quickly determine the corresponding finite difference coefficients.

The physical approach is appropriate for use in curvilinear systems since the heat flow areas are evaluated at the location of interest and as such are exact areas. Since rapidly changing cross-sectional area is a trait common to many of the non-conventional coordinate systems, the accurate account of this effect will be instrumental in reducing the number of nodal points required to achieve a prescribed accuracy.

The expressions of equation (19) appear complex and the obvious question must be answered as to whether their use is warranted in preference to the relatively simple form obtained for the conventional systems. However, in consideration of present solution techniques a single computation of each coefficient for each nodal location in the field will represent in general a small cost when compared to the cost of solution. A single computation of each coefficient will suffice since they can be effectively stored in a matrix having few zero elements. In consideration of the advantage possible by their use in compatible problems, use of the expressions of equation (19) after integration is certainly warranted.

The results of this paper have been used successfully by Strong, Schneider and Yovanovich [8] in the solution of steady-state heat transfer from a circular disk conducting heat to a semi-infinite body. A variety of arbitrary disk heat flux distributions were examined and the results reported. The extremely short computing times reported in all cases certainly support the hypothesis of this work, that considerable advantages can be gained by employing curvilinear coordinates where appropriate in the solution of heat conduction problems.

The results presented herein have also been used successfully by Schneider, Strong and Yovanovich [9] in their examination of the same geometry for transient heat flow. The analysis considered several disk flux distributions (also the isothermal disk) for which a complete transient solution was sought. The success of this application also supports the claim to utility of the results of this work.

In conclusion, computational time and effort for a finite difference heat conduction analysis can be greatly reduced without sacrificing solution accuracy by utilizing the 'most natural' coordinate system for the particular problem of interest. By combining numerical methods with the most appropriate coordinate system, many problems posing great difficulty to both the theoretician and the numerical analyst can be attacked with greater ease. It has been the object in this work to provide the means by which these gains can be achieved. In providing general expressions for the finite difference coefficients for any orthogonal curvilinear coordinate system, the engineer and researcher can quickly assemble a solution program to most efficiently analyze his particular problem. In addition the generalized results are now available so that as new coordinate systems become available their associated finite difference coefficients can be determined with a minimum of effort. This generalization will undoubtedly be invaluable to researchers in the future.

ACKNOWLEDGEMENTS

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