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# SOME BASIC THREE-DIMENSIONAL INFLUENCE COEFFICIENTS FOR THE SURFACE ELEMENT METHOD


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## Abstract

Expressions are developed for the temperature rise and the heat flux vector at an arbitrary field point due to distributed volumetric, surface and line heat sources utilizing the integral form of the solution to Laplace's equation. The application of the integral solution to the problem of thermal constriction resistance of contact areas of arbitrary shape subjected to the boundary condition of the first, second or third kind is considered. The importance of influence coefficients in the efficient and accurate numerical solution of thermal constriction problems using the Surface Element Method is discussed. A list of basic three-dimensional solutions for several important geometric source areas is presented for future reference.


## Nomenclature

| | |
|---|---|
| A | = position; area |
| $\vec{a}_1, \vec{a}_2, \vec{a}_3$ | = unit vectors |
| a | = rectangular side |
| [illegible] | = position |
| $B(\cdot)$ | = complete elliptic integral |
| b | = rectangular side |
| C | = position |
| $C_{ij}$ | = influence coefficient |
| D | = position |
| $D(\cdot)$ | = complete elliptic integral |
| $d_1, d_2, d_3$ | = distances |
| $E(\cdot)$ | = complete elliptic integral of the second kind |
| $F(\cdot)$ | = incomplete elliptic integral of the first kind |
| $G_{ij}$ | = geometric coefficient |
| h | = contact conductance or film coefficient |
| $I_0(\cdot)$ | = modified Bessel function of the first kind of order zero |
| $J_0(\cdot), J_1(\cdot)$ | = Bessel functions of the first kind of order zero and one |
| i,j | = field and source point indices |
| $K_0(\cdot)$ | = modified Bessel function of the second kind of order zero |
| $K(\cdot)$ | = complete elliptic integral of the first kind |
| k | = thermal conductivity |
| $\ell$ | = length |
| m | = source strength per unit length |
| $m(\vec{s})$ | = source strength per unit length |
| P | = field point location; perimeter |
| $P_n(\cos\theta)$ | = Legendre polynomial of order n |
| Q | = heat flow rate; total source strength |
| q | = source strength per unit area |
| $q(\vec{s})$ | = source strength per unit area |
| $R_c$ | = constriction resistance |
| r | = radial coordinate |
| $\vec{r}$ | = field point position vector |
| $\vec{s}$ | = source point position vector |
| T | = temperature rise |
| $T(\vec{r})$ | = temperature rise at field point |
| $T_f$ | = external source temperature |
| $T_s$ | = source temperature |
| t | = dummy variable |
| V | = volume |
| w | = dummy variable |
| x,y,z | = Cartesian coordinates |

### Greek Symbols

| | |
|---|---|
| $\beta$ | = dummy variable |
| $\delta$ | = perpendicular in right triangle |
| $\zeta$ | = source point coordinate |

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$\eta$ = source point coordinate

$\theta$ = polar coordinate; dummy variable

$\kappa$ = modulus of complete elliptic integrals

$\mu$ = argument of Legendre polynomials

$\xi$ = source point coordinate

$\pi$ = pi

$\rho$ = volumetric source strength; polar coordinate

$\rho(\vec{s})$ = volumetric source strength

$\bar{\rho}$ = effective ring radius

$\Omega$ = right triangle influence parameter

$\omega_0$ = vertex angle of right triangle

$\nabla^2$ = Laplacian operator

## Introduction

During the past several years Yovanovich and his co-workers [1-7] have demonstrated that the Surface Element Method (SEM) or the Boundary Integral Equations Method (BIEM) are practical and efficient methods for obtaining numerical solutions to Laplace's equation. By means of these methods they determined the thermal constriction resistances of singly- and doubly-connected, planar contact areas of arbitrary shape on insulated, isotropic half-spaces. Isothermal contacts [7] as well as contacts subjected to a uniform flux distribution [1-5] have been considered. Recently Martin [6] has shown how the SEM can be applied to singly- and doubly-connected, planar contact areas subjected to the boundary condition of the third kind. He has demonstrated that numerical solutions for the boundary condition of the third kind will yield under certain conditions the two limiting cases: boundary condition of the first and second kinds. The numerical solutions have been found to be efficient and very accurate (errors less than 1% when compared with known exact solutions).

These techniques as applied to thermal constriction problems are based upon the integral form of the solution to Laplace's equation:

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \qquad (2)$$

The solution can be written as [8-12]

$$T(\vec{r}) = \frac{1}{2\pi k} \iint_A \frac{q(\vec{s})\,dA}{|\vec{r} - \vec{s}|} \qquad (2)$$

where $T(\vec{r})$ is the temperature excess related to some arbitrary ambient reference temperature, k is the thermal conductivity of the isotropic conductor, $q(\vec{s})$ is the heat flux distribution over the contact area of interest. The position vector to the arbitrary field point (x,y,z) is $\vec{r}$ while the position vector to an arbitrary source point $(\xi,\eta,\zeta)$ is denoted by $\vec{s}$. These position vectors are defined as

$$\vec{r} = x\vec{a}_1 + y\vec{a}_2 + z\vec{a}_3 \qquad (3)$$

$$\vec{s} = \xi\vec{a}_1 + \eta\vec{a}_2 + \zeta\vec{a}_3 \qquad (4)$$

where $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ are unit vectors.

The distance between an arbitrary source point $(\xi,\eta,\zeta)$ and a field point (x,y,z) as depicted in Figure 1 is

$$|\vec{r} - \vec{s}| = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} \qquad (5)$$

In its simplest form the SEM when applied to thermal constriction problems consists of dividing the contact area into a finite number, N, of surface elements, $A_i$, over each of which the heat flux, $q_i$, is assumed to be uniform. The centroid of a typical surface element is designated by coordinates $(x_i,y_i,z_i)$ and by the position vector $\vec{r}_i$ from the origin. The temperature excess $T_i$ (or simply temperature rise if the reference temperature is taken to be zero) at the centroid of the typical surface element $A_i$ is

$$T_i = \frac{1}{2\pi k} \iint_A \frac{q(\vec{s})\,dA}{|\vec{r}_i - \vec{s}|}$$

$$= \sum_{j=1}^{N} \frac{1}{2\pi k} \iint_{A_j} \frac{q_j\,dA_j}{|\vec{r}_i - \vec{s}|}$$

$$= \sum_{j=1}^{N} q_j \left[\frac{1}{2\pi k} \iint_{A_j} \frac{dA_j}{|\vec{r}_i - \vec{s}|}\right] \qquad (6)$$

Equation (6) can be conveniently expressed in the following manner:

$$T_i = \sum_{j=1}^{N} C_{ij}\, q_j \qquad (7)$$

where $C_{ij}\, q_j$ represents the temperature rise at the point $(x_i,y_i,z_i)$ due to the surface element $A_j$. $C_{ij}$ represents the temperature rise at the centroid of $A_i$ due to thermal sources of unit strength distributed over the surface element $A_j$. The influence coefficients $C_{ij}$ are known; they consist of the integrals in Equation (6).

Equation (7) can be written in more compact form using matrix notation:

$$\{T\} = [C]\,\{q\} \qquad (8)$$

For boundary conditions of the second kind, Equation (8) can be solved directly for $T_i$ because $q_j$ is known. For boundary conditions of the first kind, $T_i$ is known and Equation (8) must be solved for $q_j$. In matrix form we have

$$\{q\} = [C]^{-1}\,\{T\} \qquad (9)$$

where $[C]^{-1}$ is the inverse matrix of [C].

For contact areas subjected to the boundary condition of the third kind the solution to Equation (1) can be expressed as

$$T(\vec{r}) = \frac{1}{2\pi k} \iint_A h(\vec{s})[T_f - T(\vec{s})] \frac{dA}{|\vec{r}-\vec{s}|} \quad (10)$$

where $h(\vec{s})$ is the thermal contact conductance (or heat transfer coefficient in convective problems), and $T_f$ is the external source temperature.

For uniform $T_f$ and uniform h over each surface element Martin [6] has shown that the temperature rise at any point $(x_i, y_i, z_i)$ is given by

$$T_i = \frac{1}{2\pi k} \sum_{j=1}^{N} h_j(T_f - T_{sj}) \iint_{A_j} \frac{dA_j}{|\vec{r}_i - \vec{s}|} \quad (11)$$

where $T_{sj}$ is the temperature of the surface element $A_j$ over which the external source is applied.

Using matrix notation Equation (11) becomes

$$\frac{h}{k} [G] \{T_f - T_s\} = \{T_s\} \quad (12)$$

where

$$G_{ij} \equiv \frac{1}{2\pi} \iint_{A_j} \frac{dA_j}{|\vec{r}_i - \vec{s}|} = k\, C_{ij} \quad (13)$$

The geometric coefficients $G_{ij}$ are related to the previously discussed influence coefficients $C_{ij}$ by means of Equation (13).

Solving Equation (12) for the unknown contact area temperature rise $T_s$ we obtain

$$\left[[G] + \frac{k}{h}[I]\right] \{T_s\} = [G]\{T_f\} \quad (14)$$

or simply

$$[G'] \{T_s\} = \{T_f\} \quad (15)$$

In Equation (14) [I] is the identity matrix. Martin [6] has demonstrated that the general solution for the boundary condition of the third kind, Equations (11) and (14) reduces to the solution for the boundary condition of the fi rs t kind when $k/h \to 0$ and reduces to the solution for the boundary condition of the second kind when $k/h \to \infty$.

The thermal constriction resistance [1-7]

$$R_c \equiv \frac{\text{average contact temperature rise}}{\text{total heat flow rate}} = \frac{1}{A} \sum_{i=1}^{N} T_i\, dA_i \Big/ \sum_{j=1}^{N} q_j\, dA_j \quad (16)$$

For boundary conditions of the first kind, $T_i$ in Equation (16) is known and the unknown $q_j$ must be determined by means of Equation (9). On the other hand for boundary conditions of the second kind, $q_j$ in Equation (16) is known and the unknown $T_i$ can be determined by Equation (8).

When the boundary condition of the third kind is specified, both $T_i$ and $q_j$ in Equation (16) are unknown. The surface temperature $T_i$ is determined by means of Equation (14) and $q_j$ is determined from $q_j = h(T_f - T_i)$. In all cases the efficient and accurate solutions will depend upon analytical or numerical evaluation of the geometric and influence coefficients $G_{ij}$ and $C_{ij}$ respectively.

The influence coefficients appear in integral solutions of Laplace's equation in several different physical areas: Newtonian potential [8,11,16], electrostatics [9,10,12,19-21] and elastostatics [11,18] for example. Some special cases have been considered as examples in mathematical treatises [13-16].

The purpose of this paper is to compile a list of influence coefficients, noting their characteristics, and their applicability to a variety of thermal problems. This list should be useful to the thermal analyst who is interested in solving thermal constriction problems as they appear in several technological areas.

## Influence Coefficients for Point Sources

### Point Source

The temperature rise at an arbitrary point (x,y,z) due to a thermal source of strength dQ located at $(\xi, \eta, \zeta)$ as shown in Figure 1 is given by

$$4\pi k T(\vec{r}) = \frac{dQ}{|\vec{r} - \vec{s}|} \quad (17)$$

where the position vectors $\vec{r}$ and $\vec{s}$ are defined by Equations (3) and (4). It can be shown that Equation (17) is the solution to Equation (1).

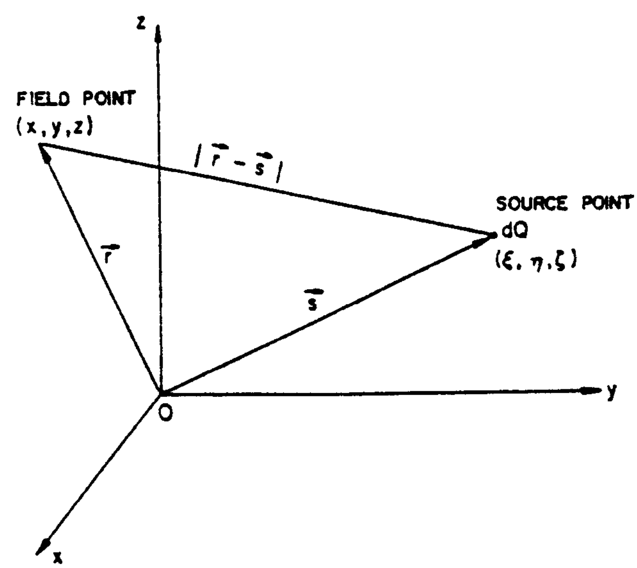


Fig. 1. Field point - source point schematic.

## Multiple Point Sources

Next consider the set of point sources $dQ_j$ located at $(\xi_j, \eta_j, \zeta_j)$ with position vectors $s_j$ as shown in Figure 2. The total temperature rise at the field point (x,y,z) due to n point sources is obtained by the superposition of the effects of the individual point sources acting alone. Therefore,

$$4\pi kT(\vec{r}) = \frac{dQ_1}{|\vec{r}-\vec{s}_1|} + \frac{dQ_2}{|\vec{r}-\vec{s}_2|} + \frac{dQ_3}{|\vec{r}-\vec{s}_3|} + \ldots + \frac{dQ_n}{|\vec{r}-\vec{s}_n|}$$

$$= \sum_{j=1}^{n} \frac{dQ_j}{|\vec{r}-\vec{s}_j|} \qquad (18)$$

When the point sources can be modelled as a uniform flux $q_j$ emitted by a differential area $dA_j$, Equation (18) becomes

$$T(x,y,z) = \frac{1}{4\pi k} \sum_{j=1}^{n} \frac{2q_j\, dA_j}{|\vec{r}-\vec{s}_j|} \qquad (19)$$

The factor of 2 in Equation (19) appears because heat is emitted from both sides of the differential area. This was taken into account when the integral solution, Equation (2), was developed for contact areas situated on half-spaces.

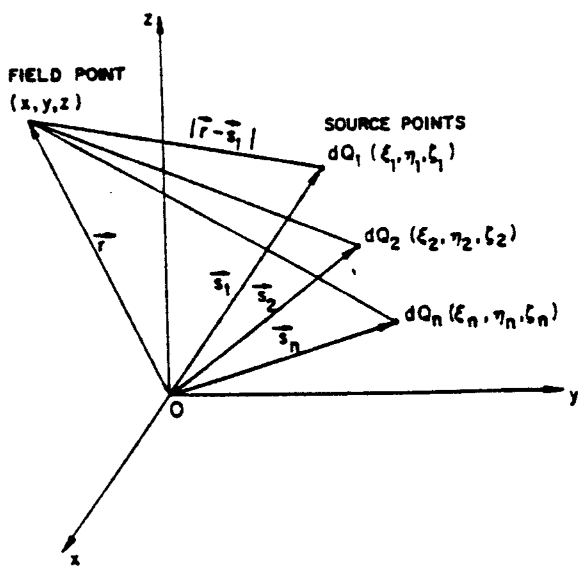


Fig. 2. Field point - multiple source point schematic.

## Distributed Sources

We next consider the temperature rise due to an arbitrary volumetric distribution of thermal sources $\rho$ as depicted in Figure 3. The volume V can be approximated as the sum of n cubes; the typical cube has volume $\Delta V_j$ and source density $\rho_j$. The temperature rise at the arbitrary point (x,y,z) can be expressed as

$$4\pi kT(\vec{r}) = \lim_{\substack{n\to\infty \\ \Delta V_j \to 0}} \sum_{j=1}^{n} \frac{\rho_j\, \Delta V_j}{|\vec{r}-\vec{s}_j|} \qquad (20)$$

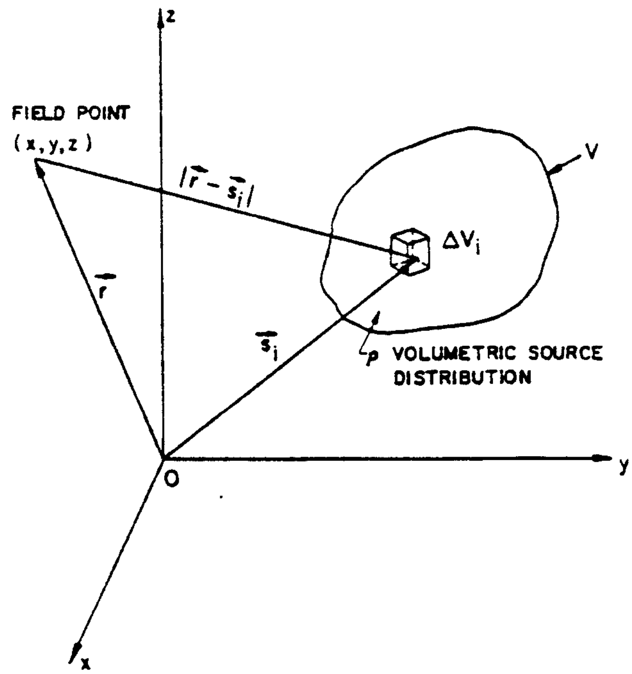


Fig. 3. Field point - volumetric source point configuration.

where $\vec{s}_j$ is the position vector from the common origin to the typical volume element. In the limit the summation becomes a volume integral; therefore,

$$4\pi kT(\vec{r}) = \iiint_V \frac{\rho(\vec{s})\, dV}{|\vec{r} - \vec{s}|} \qquad (21)$$

In Cartesian coordinates Equation (21) becomes

$$4\pi kT(x,y,z) = \iiint_V \frac{\rho(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \qquad (22)$$

The integrand is a function of six variables (x,y,z) and $\xi,\eta,\zeta$). Integration eliminates the dummy variables $(\xi,\eta,\zeta)$, therefore the temperature rise is a function of (x,y,z) only.

## Flux Distribution

The spatial flux distribution $\vec{q}$ is often of interest to the thermal analyst. It may be determined at any point (x,y,z) by means of Fourier's rate equation

$$\vec{q} = -k \text{ grad } T = -k\, \nabla T \qquad (23)$$

Substitution of Equation (21) into Equation (23) yields

$$\vec{q}(\vec{r}) = -k\nabla \left[\frac{1}{4\pi k} \iiint_V \frac{\rho(\vec{s})\, dV}{|\vec{r} - \vec{s}|}\right] \qquad (24)$$

Using Leibnitz's rule for the differentiation of an integral we obtain

$$\vec{q}(\vec{r}) = -\frac{1}{4\pi}\iiint_V \nabla\left[\frac{\rho(\vec{s})}{|\vec{r}-\vec{s}|}\right] dV \qquad (25)$$

But,

$$\nabla\left[\frac{\rho(\vec{s})}{|\vec{r}-\vec{s}|}\right] = -\rho(\vec{s})\,\nabla\left[\frac{1}{|\vec{r}-\vec{s}|}\right] \qquad (26)$$

It can be shown that

$$\nabla\left[\frac{1}{|\vec{r}-\vec{s}|}\right] = \frac{(\vec{r}-\vec{s})}{|\vec{r}-\vec{s}|^3} \qquad (27)$$

Combining Equations (26) and (27) with Equation (25) gives for the flux vector

$$\vec{q}(\vec{r}) = \frac{1}{4\pi}\iiint_V \frac{\rho(\vec{s})(\vec{r}-\vec{s})dV}{|\vec{r}-\vec{s}|^3} \qquad (28)$$

Using Cartesian coordinates the three flux components may be written as

$$q_x(x,y,z) = \frac{1}{4\pi}\iiint_V \frac{\rho(\xi,\eta,\zeta)(x-\xi)d\xi\, d\eta\, d\zeta}{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{3/2}} \qquad (29)$$

$$q_y(x,y,z) = \frac{1}{4\pi}\iiint_V \frac{\rho(\xi,\eta,\zeta)(y-\eta)d\xi\, d\eta\, d\zeta}{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{3/2}} \qquad (30)$$

$$q_z(x,y,z) = \frac{1}{4\pi}\iiint_V \frac{\rho(\xi,\eta,\zeta)(z-\zeta)d\xi\, d\eta\, d\zeta}{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{3/2}} \qquad (31)$$

Two methods are available for obtaining the flux distribution: 1) calculate the temperature rise by means of Equation (22), then determine the flux by means of Fourier's rate equation; 2) calculate the flux directly by means of Equation (28).

## Influence Coefficients for Distributed Sources

### Surface and Line Sources

For analytical and computational purposes we may regard surface source distributions and line source distributions as special cases of volume source distributions.

For example the volume source distribution, Equation (21), reduces to the surface source distribution

$$4\pi k T(\vec{r}) = \iint_A \frac{q(\vec{s})\,dA}{|\vec{r}-\vec{s}|} \qquad (32)$$

where $q(\vec{s})$ is the surface strength per unit area; and to the line source distribution

$$4\pi k T(\vec{r}) = \int_\ell \frac{m(\vec{s})\,d\ell}{|\vec{r}-\vec{s}|} \qquad (33)$$

where $m(\vec{s})$ is the line source strength per unit length.

These concepts will be used to develop additional solutions in the subsequent sections.

### Uniform Finite Line Source

Consider a finite line source of length 2a, strength m watts/unit length and the total strength is Q = 2ma watts. By means of Equation (33) the temperature rise at any field point $(\rho,z)$, Figure 4, is

$$T(\rho,z) = \frac{1}{4\pi k}\cdot\frac{Q}{2a}\int_{\zeta=-a}^{\zeta=+a}\frac{d\zeta}{\sqrt{\rho^2+(z-\zeta)^2}}$$

$$= \frac{Q}{8\pi ka}\ln\frac{(z+a)+r_1}{(z-a)+r_2} \qquad (34)$$

where $r_1$ and $r_2$ are the distances from the ends of the finite line source to the point $(\rho,z)$. The isothermal surfaces are confocal rotational (prolate) ellipsoids.

The heat flux components can be determined by means of Fourier's rate equation. Thus

$$q_z = -k\frac{\partial T}{\partial z} = \frac{Q}{8\pi a}\frac{1}{\rho}(\sin\alpha_2 - \sin\alpha_1)$$

$$= \frac{Q}{8\pi a}\left(\frac{1}{r_2}-\frac{1}{r_1}\right) \qquad (35)$$

and

$$q_\rho = -k\frac{\partial T}{\partial \rho} = \frac{Q}{8\pi a}\frac{1}{\rho}(\cos\alpha_1 - \cos\alpha_2) \qquad (36)$$

The isothermal surfaces have semi-major and minor axes b and c respectively, Figure 4. Also $a = \sqrt{b^2-c^2}$. If we select the point M on the isothermal ellipsoidal surface where z=0, $r_1=r_2=b$, then

$$T = \frac{Q}{8\pi ka}\ln\frac{b+a}{b-a}$$

$$= \frac{Q}{8\pi k\sqrt{b^2-c^2}}\ln\frac{b+\sqrt{b^2-c^2}}{b-\sqrt{b^2-c^2}} \qquad (37)$$

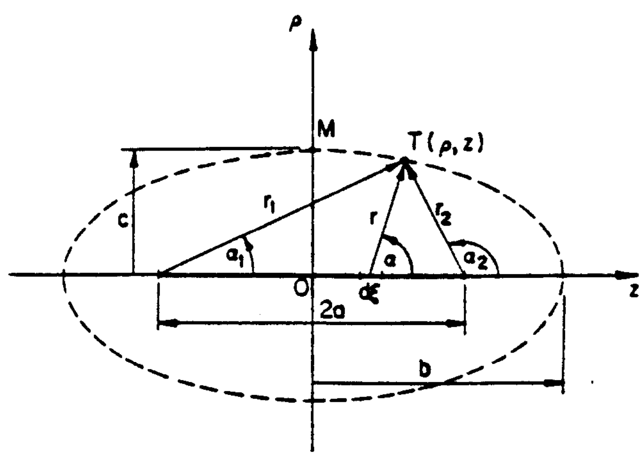


Fig. 4. Finite line source temperature rise field configuration.

The influence coefficient for a finite line source can be determined from Equation (37).

## Uniform Circular Ring Source

Consider the case of a total source strength Q uniformly distributed over a circular ring of radius a. The line source strength is $m = Q/2\pi a$ watts per unit length. Using Equation (33) the temperature rise at an arbitrary field point $(\rho,z)$, Figure 5, is therefore

$$T(\rho,z) = \frac{m}{4\pi k}\int_{\psi=0}^{\psi=2\pi}\frac{a\,d\psi}{r}$$

$$= \frac{m}{4\pi k}\int_{\psi=0}^{\psi=2\pi}\frac{a\,d\psi}{\sqrt{(\rho-a\cos\psi)^2+(a\sin\psi)^2+z^2}} \qquad (38)$$

The following transformation:

$$\cos\psi = 2\sin^2 t - 1, \qquad d\psi = -2\,dt$$

reduces Equation (38) to the following integral:

$$T(\rho,z) = \frac{Q}{4\pi k}\cdot\frac{2}{\pi}\frac{1}{\sqrt{(\rho+a)^2+z^2}}\int_0^{\pi/2}\frac{dt}{\sqrt{1-\kappa^2\sin^2 t}}$$

$$= \frac{Q}{4\pi k}\cdot\frac{2}{\pi}\frac{1}{\sqrt{(\rho+a)^2+z^2}}K(\kappa) \qquad (39)$$

where $K(\kappa)$ is the complete elliptic integral of the first kind of modulus $\kappa$ where

$$\kappa^2 = \frac{4\rho a}{(\rho+a)^2+z^2} \qquad (40)$$

Along the axis $\rho = 0$, $\kappa^2 = 0$ and $K(0) = \pi/2$; therefore,

$$T(o,z) = \frac{Q}{4\pi k}\cdot\frac{1}{\sqrt{a^2+z^2}} \qquad (41)$$

This result can be obtained directly by a simple integration of Equation (38).

In the plane of the ring source $z = 0$, the temperature rise is

$$T(\rho,o) = \frac{Q}{4\pi k}\cdot\frac{2}{\pi}\cdot\frac{1}{(\rho+a)}K(\kappa) \qquad (42)$$

where $\kappa^2 = 4\rho a/(\rho+a)^2$.

Alternate expressions are available [9,12,16] for the temperature rise at any field point $(r,\theta)$ where $z = r\cos\theta$, $\rho = r\sin\theta$ and $\mu = \cos\theta$, Figure 6,

$$T(r,\theta) = \frac{Q}{4\pi ka}\left[1 - \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\mu) + \frac{1.3}{2.4}\left(\frac{r}{a}\right)^4 P_4(\mu) - \ldots\right] \qquad (43)$$

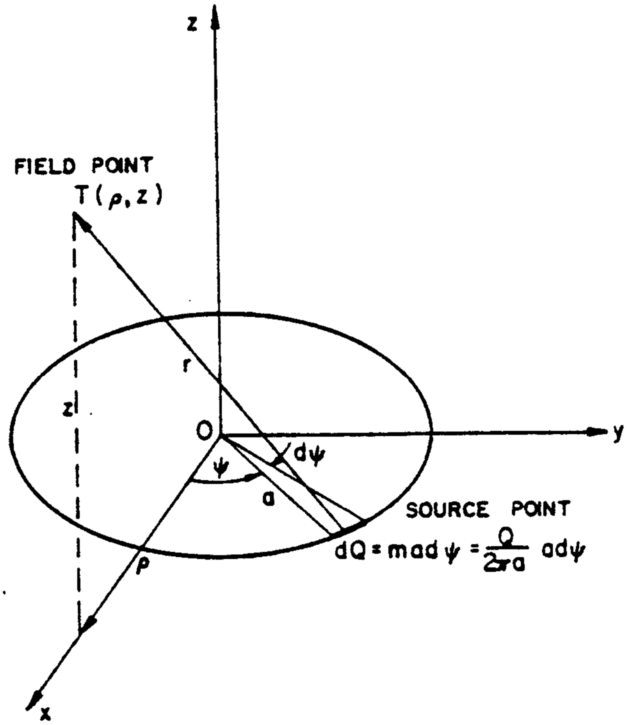


Fig. 5. Circular ring source configuration.

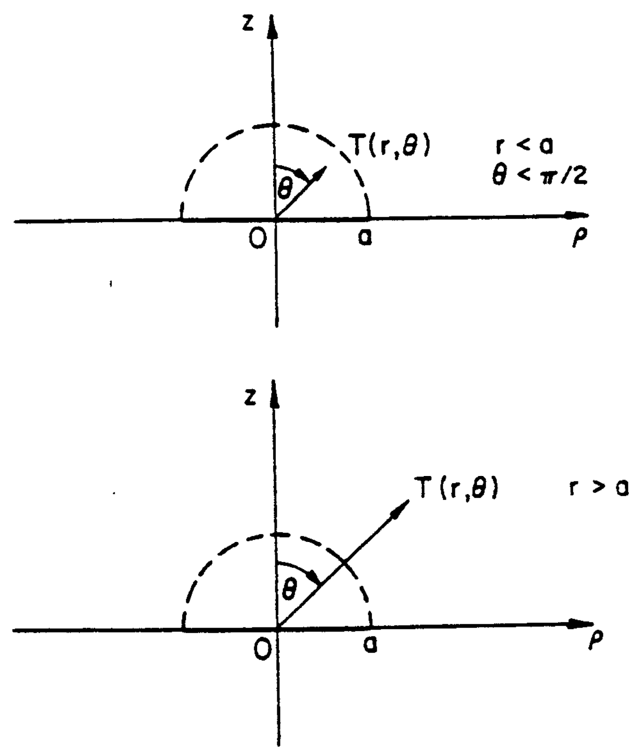


Fig. 6. Inner and outer field points for ring and circular sources.

for r < a, or

$$T(r,\theta) = \frac{Q}{4\pi ka}\left[\frac{a}{r} - \frac{1}{2}\left(\frac{a}{r}\right)^3 P_2(\mu) + \frac{1.3}{2.4}\left(\frac{a}{r}\right)^5 P_4(\mu) - \ldots\right] \quad (44)$$

for r > a. $P_2(\mu)$, $P_4(\mu)$, etc. are the even order Legendre polynomials. It can be seen from Equation (44) that when $a/r \to 0$, $T(r,\theta) \to Q/4\pi kr$ independent of $\theta$; the ring appears to be a point source of strength Q located at the origin.

Other alternate expressions are presented in the text by Budak et al [14],

$$T(\rho,z) = \frac{2Q}{\pi k}\int_0^\infty K_o(\omega a)I_o(\omega\rho)\cos\omega z\, d\omega \quad \text{for } \rho < a \quad (45)$$

$$T(\rho,z) = \frac{2Q}{\pi k}\int_0^\infty I_o(\omega a)K_o(\omega\rho)\cos\omega z\, d\omega \quad \text{for } \rho > a \quad (46)$$

where $I_o(\cdot)$ and $K_o(\cdot)$ are modified Bessel functions of the first and second kind, respectively, of order zero.

Uniform Source Distribution Over a Circular Area

Suppose that heat is supplied uniformly over the circular area $0 \le \rho < a$ in the plane z = 0. The total source strength $Q = q\, 2\pi a^2$ results in a temperature rise at the field point $P(\rho,z)$ which is [17]

$$T(\rho,z) = \frac{qa}{k}\int_0^\infty e^{-w|z|} J_o(w\rho)\, J_1(wa)\, \frac{dw}{w} \quad (47)$$

where $J_o(\cdot)$ and $J_1(\cdot)$ are Bessel functions of the first kind of order zero and unity, and w is a dummy variable.

Along the axis $\rho = 0$, $|z| \ge 0$, the temperature rise can be determined directly by means of the following double integration:

$$4\pi kT(o,z) = 2q\int_0^{2\pi}\int_0^a \frac{\rho\, d\rho\, d\theta}{\sqrt{\rho^2 + z^2}}$$

$$= 4\pi q\left[\sqrt{a^2 + z^2} - z\right] \quad (48)$$

In the contact plane z = 0, the temperature rise for internal points $0 \le \rho \le a$ is given [1]:

$$T(\rho) = \frac{2}{\pi}\,\frac{qa}{k}\,E(\kappa) \quad (49)$$

where $E(\cdot)$ is the complete elliptic integral of the second kind,

$$E(\kappa) = \int_0^{\pi/2}\sqrt{1 - \kappa^2\sin^2 t}\, dt \quad (50)$$

of modulus $\kappa = \rho/a$ and $0 \le \kappa \le 1$.

External to the contact area, the temperature rise is [1]

$$T(\rho) = \frac{2}{\pi}\,\frac{qa}{k}\,\kappa\, B(\kappa) \quad (51)$$

where $\kappa = a/\rho \le 1$, and

$$B(\kappa) = K(\kappa) - D(\kappa) \text{ and } D(\kappa) = [K(\kappa) - E(\kappa)]/\kappa^2 \quad (52)$$

In Equations (52), $K(\cdot)$ and $E(\cdot)$ are complete elliptic integrals of the first and second kind respectively. $B(\cdot)$ is also a complete elliptic integral, $B(1) = 1$ and $B(0) = \pi/4$.

Martin [6] has developed the following polynomial approximation for $B(\kappa)$:

$$B(\kappa) = 0.7854 + 0.1072\kappa^2 + 0.081749\kappa^{6.8154} + 0.024619\kappa^{48.47} \quad (53)$$

with a maximum error of 0.10% when $\kappa = 1$.

For arbitrary field points $(\rho,z)$ or $(r,\theta)$, Figure 6, where $\rho = r\sin\theta$, $z = r\cos\theta$, we have [12,16]

$$T(r,\theta) = \frac{qa}{k}\sum_{n=1}^\infty A_{2n}\left(\frac{a}{r}\right)^{2n-1} P_{2n-2}(\cos\theta) \quad (54)$$

for r > a, and

$$T(r,\theta) = \frac{qa}{k}\left[1 - \left(\frac{r}{a}\right)\cos\theta + \sum_{n=1}^\infty A_{2n}\left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\theta)\right] \quad (55)$$

for r < a. In both equations we have

$$A_2 = \frac{1}{2}$$

$$A_{2n} = (-1)^{n+1}\frac{1.3.5.\ldots.(2n-3)}{2.4.6.\ldots.(2n)} \quad (56)$$

$$A_{2n+1} = 0$$

It is seen from an examination of Equation (48) and Equation (51) with Equation (53) that the temperature rise produced by a circular area with uniformly distributed sources approaches 99% of the point source temperature rise $T = Q/4\pi kr$ within 1.5 diameters.

Uniform Source Distribution Over a Circular Annulus

Consider the circular annulus having radii a, b with a < b lying in the z = 0 plane, Figure 7. The total source strength is $Q = q\, 2\pi(b^2 - a^2)$. By means of Equation (32) using polar coordinates $(\rho,\vartheta,z)$, the temperature rise at any point $P(\rho,z)$ is [6]

$$T(\rho,z) = \frac{2}{\pi}\,\frac{q}{k}\int_a^b \frac{\beta\, K(\kappa)\, d\beta}{\sqrt{(\rho+\beta)^2 + z^2}} \quad (57)$$

where $K(\cdot)$ is the complete elliptic integral of the first kind with modulus

$$\kappa^2 = \frac{4\rho\beta}{(\rho+\beta)^2 + z^2} \quad (58)$$

and $\beta$ is a dummy variable.

A closed form expression for the integral in Equation (57) is presently not available. One can however obtain expressions for the temperature rise along the axis and in the plane of the annulus by means of the superposition principle. Yovanovich [1] superposed a uniformly distributed source ($+q$ over $0 \le \rho \le b$) and a uniformly distributed sink ($-q$ over $0 \le \rho \le a < b$) to give the circular annulus solution. He obtained for the temperature rise along the axis $\rho = 0$, $z \ge 0$

$$T(o,z) = \frac{q}{k}\left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right] \qquad (59)$$

and for points in the contact plane $z = 0$, he gave three expressions:

1) $0 \le \rho < a < b$

$$T(\rho) = \frac{2}{\pi}\frac{qb}{k}E\left(\frac{\rho}{b}\right) - \frac{2}{\pi}\frac{qa}{k}E\left(\frac{\rho}{a}\right) \qquad (60)$$

2) $a \le \rho \le b$

$$T(\rho) = \frac{2}{\pi}\frac{qb}{k}E\left(\frac{\rho}{b}\right) - \frac{2}{\pi}\frac{qa}{k}\kappa\, B(\kappa) \qquad (61)$$

3) $\rho \ge b$

$$T(\rho) = \frac{2}{\pi}\frac{qb}{k}\frac{b}{\rho}B\left(\frac{b}{\rho}\right) - \frac{2}{\pi}\frac{qa}{k}\frac{a}{\rho}B\left(\frac{a}{\rho}\right) \qquad (62)$$

where $E(\cdot)$ and $B(\cdot)$ are complete elliptic integrals defined by Equations (50) and (52) respectively.

Alternate expressions for the temperature rise at an arbitrary point $P(\rho,z)$ or $P(r,\theta)$, Figure 6, can be obtained by the superposition of the circular source solutions, Equations (54)-(56). The region above or below a circular annulus can be separated into three zones corresponding to $r < a$, $a < r < b$ and $r > b$. The temperature rise within each zone can be developed from Equations (54)-(56). They are

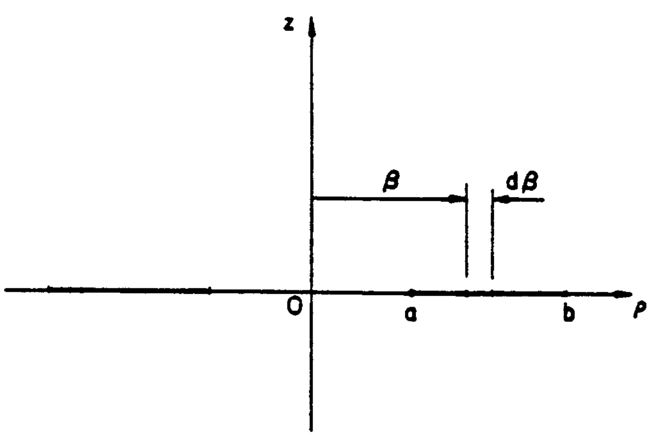


Fig. 7. Circular annulus source configuration.

1) $r < a$

$$T(r,\theta) = \frac{qb}{k}\left[1 - \frac{r}{b}\cos\theta + \sum_{n=1}^{\infty} A_{2n}\left(\frac{r}{b}\right)^{2n} P_{2n}(\cos\theta)\right] - \frac{qa}{k}\left[1 - \frac{r}{a}\cos\theta + \sum_{n=1}^{\infty} A_{2n}\left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\theta)\right] \qquad (63)$$

2) $a < r < b$

$$T(r,\theta) = \frac{qb}{k}\left[1 - \frac{r}{b}\cos\theta + \sum_{n=1}^{\infty} A_{2n}\left(\frac{r}{b}\right)^{2n} P_{2n}(\cos\theta)\right] - \frac{qa}{k}\sum_{n=1}^{\infty} A_{2n}\left(\frac{a}{r}\right)^{2n-1} P_{2n-2}(\cos\theta) \qquad (64)$$

3) $r > b$

$$T(r,\theta) = \frac{qb}{k}\sum_{n=1}^{\infty} A_{2n}\left(\frac{b}{r}\right)^{2n-1} P_{2n-2}(\cos\theta) - \frac{qa}{k}\sum_{n=1}^{\infty} A_{2n}\left(\frac{a}{r}\right)^{2n-1} P_{2n-2}(\cos\theta) \qquad (65)$$

Martin [6] has shown that the temperature rise at any point $P(\rho,z)$ can be computed with an error less than 1% when the circular annulus is replaced by an equivalent ring source. The strength per unit length of the ring is obtained from

$$Q = 2\pi\bar{\rho}m = q\, 2\pi(b^2-a^2) \qquad (66)$$

If the effective ring radius is chosen to be

$$\bar{\rho} = (a+b)/2 \qquad (67)$$

Then the strength per unit length becomes

$$m = 2(b-a)q \qquad (68)$$

When these equations are substituted into Equation (39), we obtain the following equivalent ring expression:

$$T(\rho,z) = \frac{q(b-a)}{\pi k}\sqrt{\frac{\bar{\rho}}{\rho}}\,\kappa\, K(\kappa) \qquad (60)$$

where the modulus of $K(\cdot)$ is defined as

$$\kappa^2 = \frac{4\bar{\rho}\rho}{(\bar{\rho}+\rho)^2 + z^2} \qquad (70)$$

The difference between the exact solution given by Equations (57) and (58) and the approximate solution given by Equations (69) and (70) will be less than 1% provided the arbitrary point does not lie within the volume $(2a-b) \le \rho \le (2b-a)$, $0 \le z \le 3(b-a)$ adjacent to the circuluar annulus.

## Uniform Source Distribution Over a Right Triangular Area

Suppose a uniform heat source is distributed over the right triangle with vertices $A(0,0)$, $B(\delta,0)$ and $C(\delta,\delta\tan\omega_0)$ with the vertex angle at A denoted by $\omega_0$ and the perpendicular from A to B designated $\delta$. The triangular area lies in the $z = 0$ plane and has the total strength $Q = q\delta^2 \tan\omega_0$.

The temperature rise at the point P(0,0,z) located directly above the vertex A is obtained from

$$T(0,0,z) = \frac{q}{4\pi k} \int_0^{\omega_0} \int_0^{\rho_0} \frac{2\rho \, d\rho \, d\theta}{\sqrt{\rho^2 + z^2}} \qquad (71)$$

where $\rho_0 = \delta/\cos\theta$. The first integration with respect to $\rho$ gives

$$T(0,0,z) = \frac{q}{2\pi k} \left\{ \int_0^{\omega_0} \sqrt{\delta^2 + z^2 - z^2 \sin^2\theta} \, \frac{d\theta}{\cos\theta} - z\,\omega_0 \right\} \qquad (72)$$

Yovanovich [22] has obtained the following closed form expression for Equation (72):

$$\frac{2\pi k T}{q\delta} = \frac{1}{2} \ln \left[ \frac{\sqrt{1 + \zeta^2 \cos^2\omega_0} + \sin\omega_0}{\sqrt{1 + \zeta^2 \cos^2\omega_0} - \sin\omega_0} \right] + \zeta \sin^{-1}\left[ \frac{\zeta \sin\omega_0}{\sqrt{1 + \zeta^2}} \right] - \zeta\,\omega_0 \qquad (73)$$

with $\zeta = z/\delta$. The function on the right hand side of Equation (73) is defined to be $\Omega(\zeta, \omega_0)$, therefore Equation (73) can be written as

$$T = \frac{q\delta}{2\pi k} \, \Omega(\zeta, \omega_0) \qquad (74)$$

When $z = 0$, the omega function reduces to the following equivalent expressions:

$$\Omega(0, \omega_0) = \ln \tan\left(\frac{\pi}{4} + \frac{\omega_0}{2}\right)$$

$$= \frac{1}{2} \ln \left[ \frac{1 + \sin\omega_0}{1 - \sin\omega_0} \right]$$

$$= F(\omega_0, 1) \qquad (75)$$

where $F(\omega_0, 1)$ is the incomplete elliptic integral of the first kind of unit modulus,

$$F(\omega_0, 1) = \int_0^{\omega_0} \frac{d\theta}{\sqrt{1 - \sin^2\theta}} \qquad (76)$$

If $\omega_0 \le 85°$ and $\zeta \ge P/\delta$ where the perimeter P of the triangular area is

$$P = \delta[1 + \tan\omega_0 + \sqrt{1 + \tan^2\omega_0}\,] \qquad (77)$$

the exact expression, Equation (73), can be approximated by

$$\frac{2\pi k T}{q\delta} = \frac{A}{r_0} = \frac{1.5 \tan\omega_0}{\sqrt{4 + 9\zeta^2 + \tan^2\omega_0}} \qquad (78)$$

where A is the area of the triangle and $r_0$ is the distance from the centroid of the triangle to the field point (0,0,z). The error is less than 1.0% provided $\omega_0 < 85°$ and $\zeta > P/\delta$.

By the superposition principle, Equation (74) can be used to obtain the temperature rise at points which are located directly above any vertex of any arbitrary triangular area.

## Uniform Source Distribution Over a Rectangular Area

A rectangular area lying in the first quadrant ($x > 0$, $y > 0$) has corners located at $A(x_1, y_1)$, $B(x_2, y_1)$, $C(x_2, y_2)$, $D(x_1, y_2)$ with a uniform surface source density q. The temperature rise at the field point P(0,0,z) is obtained by means of Equation (32) where the integration is taken over both faces of the rectangle. Thus,

$$\frac{2\pi k T}{q}(0,0,z) = x_1 \ln \left[ \frac{y_1 + \sqrt{x_1^2 + y_1^2 + z^2}}{y_2 + \sqrt{x_1^2 + y_2^2 + z^2}} \right] + x_2 \ln \left[ \frac{y_2 + \sqrt{x_2^2 + y_2^2 + z^2}}{y_1 + \sqrt{x_2^2 + y_1^2 + z^2}} \right]$$
$$+ y_1 \ln \left[ \frac{x_1 + \sqrt{x_1^2 + y_1^2 + z^2}}{x_2 + \sqrt{x_2^2 + y_1^2 + z^2}} \right] + y_2 \ln \left[ \frac{x_2 + \sqrt{x_2^2 + y_2^2 + z^2}}{x_1 + \sqrt{x_1^2 + y_2^2 + z^2}} \right]$$
$$+ z \arctan \left[ \frac{x_1 y_1}{z\sqrt{x_1^2 + y_1^2 + z^2}} \right] + z \arctan \left[ \frac{x_2 y_2}{z\sqrt{x_2^2 + y_2^2 + z^2}} \right]$$
$$- z \arctan \left[ \frac{x_1 y_2}{z\sqrt{x_1^2 + y_2^2 + z^2}} \right] - z \arctan \left[ \frac{x_2 y_1}{z\sqrt{x_2^2 + y_1^2 + z^2}} \right] \qquad (79)$$

When the field point lies in the plane of the rectangle, Equation (79) becomes

$$\frac{2\pi k T}{q}(0,0,0) = x_1 \ln \left[ \frac{y_1 + \sqrt{x_1^2 + y_1^2}}{y_2 + \sqrt{x_1^2 + y_2^2}} \right]$$

$$= x_2 \ln \left[ \frac{y_2 + \sqrt{x_2^2 + y_2^2}}{y_1 + \sqrt{x_2^2 + y_1^2}} \right]$$

$$+ y_1 \ln \left[\frac{x_1 + \sqrt{x_1^2 + y_1^2}}{x_2 + \sqrt{x_2^2 + y_2^2}}\right]$$

$$+ y_2 \ln \left[\frac{x_2 + \sqrt{x_2^2 + y_2^2}}{x_1 + \sqrt{x_1^2 + y_2^2}}\right] \quad (80)$$

For the field point at the center of the rectangular area with sides $a = (x_2 - x_1)$, $b = (y_2 - y_1)$, Equation (80) reduces to

$$\frac{2\pi kT}{q} = a \ln \left(\frac{b + \sqrt{a^2 + b^2}}{-b + \sqrt{a^2 + b^2}}\right) + b \ln \left(\frac{a + \sqrt{a^2 + b^2}}{-a + \sqrt{a^2 + b^2}}\right) \quad (81)$$

An alternate expression can be developed for Equation (81) using the right triangular area solution, Equation (74) with $\zeta = 0$. By symmetry there are two sets of four identical triangles with

$$\delta_1 = \frac{a}{2}, \ \omega_1 = \tan^{-1} \frac{b}{a}, \ \delta_2 = \frac{b}{2}, \ \omega_2 = \tan^{-1} \frac{a}{b} \quad (82)$$

Superposing solutions we obtain for the temperature rise at the centroid of the rectangular area (axb),

$$\frac{\pi kT}{q} = a\Omega(0, \tan^{-1} \frac{b}{a}) + b\Omega(0, \tan^{-1} \frac{a}{b}) \quad (83)$$

which is equivalent to Equation (81).

For the field point which lies directly above the vertex A, Kellogg [8] gives the following simple expression for the temperature rise:

$$\frac{2\pi kT}{q} = b \ln \frac{a + d_3}{d_2} + a \ln \frac{b + d_3}{d_1} - z \arctan \frac{ab}{zd_3} \quad (84)$$

with $z = PA$, $d_1 = PB$, $d_2 = PC$ and $d_3 = PD$.

The temperature fields produced by a square source and a circular source having the same total area are similar for field points which are not too close to the source. For example a 2 by 2 square source will raise the temperature of a field point which lies in the plane of the source at a distance of 2 units from the centroid,

$$\frac{kT}{q\sqrt{A}} = 0.1652 \quad (85)$$

where A is the area of the square.

A circular source of radius $a = 2/\sqrt{\pi}$ will raise the temperature of the same field point, according to Equation (51) with $\kappa = 1/\sqrt{\pi}$,

$$\frac{kT}{q\sqrt{A}} = 0.1664 \quad (86)$$

The difference between the square source and the equivalent circular source is less than 1% for the relatively near field point considered above. For field points whose distance from the centroid of the square source $r_0$ is equal to or greater than the square root of the area (one side only), i.e., $r_0/\sqrt{A} \geq 1$, its temperature rise can be computed with negligible error using Equations (51) and (53) for an equivalent circular source.

## Uniform Source Distribution Over An Infinite Strip

Consider an infinite strip of width $-a \leq x \leq a$, lying in the $y = 0$ plane with uniform surface density q. The temperature rise at any point P(x,y) can be obtained by means of the following integral:

$$4\pi kT(x,y) = 2q \int_{-a}^{a} \int_{-\infty}^{\infty} \frac{d\xi \, d\zeta}{\sqrt{(x-\xi)^2 + y^2 + \zeta^2}}$$

$$= -2q \int_{-a}^{a} \ln \sqrt{(x-\xi)^2 + y^2} \, d\xi \quad (87)$$

A second integration yields [14]

$$\frac{2\pi kT(x,y)}{q} = 2a - y \arctan\left[\frac{2ay}{x^2 + y^2 - a^2}\right]$$

$$- \frac{(a-x)}{2} \ln[y^2 + (a-x)^2]$$

$$- \frac{(a+x)}{2} \ln[y^2 + (a+x)^2] \quad (88)$$

## Summary and Conclusions

Several basic integral solutions of Laplace's equation for point sources and distributed sources have been considered. Expressions for the temperature rise at arbitrary field points due to a finite line source, a circular ring source, a circular source, a circular annular source, a rectangular source and a right triangular source have been developed. Alternate expressions are also presented for certain geometries. Temperature rise expressions along the axis and in the plane of the source are also developed and discussed.

The fundamental solutions presented in this paper form the basis of the Surface Element Method which is an efficient and accurate numerical technique for the solution of Laplace's equation with complex boundary conditions. Martin [6] has demonstrated the power of this numerical method when applied to certain important thermal constriction problems such as singly- and doubly-connetted, planar contact areas subjected to the boundary condition of the third kind with uniform and non-uniform contact conductance. In his solutions he employed several of the basic solutions given here.

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