Journal of Heat Transfer

Technical Notes

This section contains shorter technical papers. These shorter papers will be subjected to the same review process as that for full papers.

On the Nondimensionalization of Constriction Resistance for Semi-infinite Heat Flux Tubes

K. J. Negus,[1] M. M. Yovanovich,[2] and J. V. Beck[3]

Nomenclature

- a = radius of circular contact; half-width of square contact
- A_c = contact area
- A_t = cylinder cross-sectional area
- b = radius of circular cylinder; half-width of square cylinder
- I_0 = polar second moment of area
- I_{RR} = radial second moment of area
- $J_0(\cdot)$ = Bessel function of first kind of order zero
- $J_1(\cdot)$ = Bessel function of first kind of order one
- k = thermal conductivity
- q = uniform heat flux on contact
- Q = total heat flow rate through contact
- r = polar coordinate
- R_c = constriction resistance
- $\bar{T}_c$ = average temperature across contact
- $\bar{T}(z=0)$ = average temperature across top face of cylinder
- x, y, z = Cartesian coordinates
- γ_n = positive real roots of $J_1(\gamma_n)=0$
- δ = characteristic dimension of length
- ϵ = relative contact size ($\equiv\sqrt{A_c/A_t}$)
- ψ = dimensionless thermal constriction resistance parameter ($\equiv k\sqrt{A_c}R_c$)
- ψ_∞ = constriction parameter for a contact on a half-space

Subscripts

- a, c, e = denote respectively a solution type as approximate, correlated, or exact

Superscripts

- c, s = denote respectively a circle or a square geometry

Introduction

The insulated semi-infinite cylinder with heat supplied uniformly through a coaxial contact area as shown in Fig. 1 is an important unit cell in the theory of contact resistance (Cooper et al., 1969). The three contact area/cylinder cross-sectional configurations shown in Fig. 2 are of interest not on-

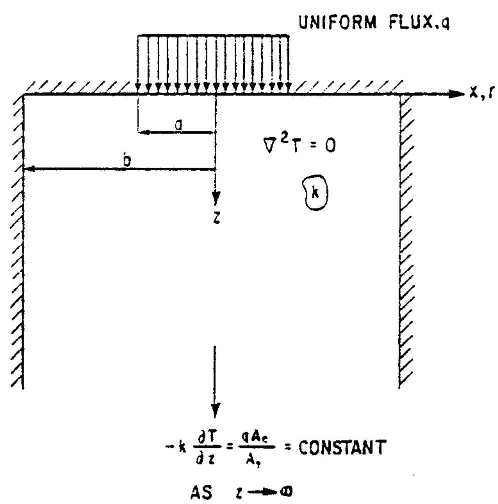

Fig. 1 Uniform heat flux on contact located centrally on a semi-infinite adiabatic cylinder

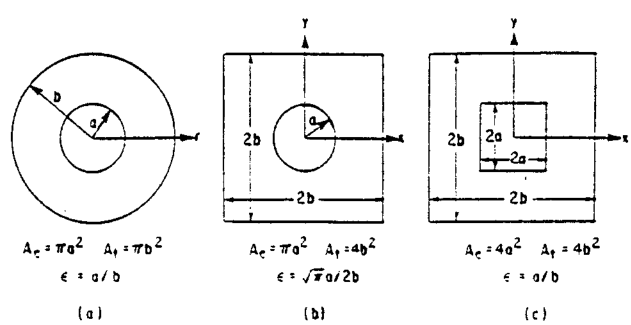

Fig. 2 Contact area/cylinder cross-sectional configurations under consideration

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ly in contact resistance but also in electronic component cooling (Antonetti and Yovanovich, 1984) and storage of radioactive wastes (Schankula, 1985). Because of the similarities that will be demonstrated for these three important cases, estimation of the thermal resistance of other configurations is also possible.

In the past, several investigators have examined the problem of the thermal constriction resistance of a circular contact area on an insulated semi-infinite, coaxial circular cylinder as shown by Figs. 1 and 2(*a*). Recently the case shown in Fig. 2(*b*) of a circular contact on a square cylinder has been investigated for a small relative contact size by Beck (1979) and for any contact size by Negus and Yovanovich (1984). By using a new approximate technique developed by Beck (1979) and Negus and Yovanovich (1985), as well as double-infinite series solution to Laplace's equation in Cartesian coordinates (Mikic, 1966), the case of a square contact area on a square cylinder as shown in Fig. 2(*c*) can also be solved.

Previously the nondimensionalization technique employed for the results has varied with different configurations and researchers. However, the similarity of these three configurations in terms of their integrated parameter, the thermal constriction resistance, will only be seen when nondimensionalization is made using a characteristic dimension that best describes these geometries.

Nondimensionalization of Thermal Constriction Resistance

The thermal constriction resistance R_c for a finite contact area on a semi-infinite insulated cylinder is commonly defined as

$$R_c \equiv \frac{\bar{T}_c - \bar{T}(z=0)}{Q} \tag{1}$$

where $\bar{T}_c$ is the average temperature rise over the contact area, $\bar{T}(z=0)$ the average temperature rise of the cross section of the cylinder in the plane of the contact, and Q is the total heat flow rate across the contact area.

If heat is supplied to the cylinder for a sufficiently long time, R_c will reach a steady-state value. Nondimensionalization of R_c can be made by introducing a dimensionless thermal constriction parameter ψ defined as

$$\psi \equiv k\delta R_c \tag{2}$$

where k is the thermal conductivity of the cylinder and δ is any characteristic dimension with the units of length.

In the study of arbitrary single contacts on a half-space (Yovanovich et al., 1983, 1984), it was determined that for both steady-state and transient conditions, use of the square root of the contact area as the characteristic length gave similar results for many different geometries. Furthermore, Chow and Yovanovich (1982) have shown analytically that for single bodies or contacts in infinite or semi-infinite media, the constriction resistance R_c is inversely proportional to the square root of the contact area as a first approximation.

Although the problem considered in this work is not one of a single contact on a half-space, it will be demonstrated that the square root of the contact area is the characteristic dimension if the results for different cases are compared at a dimensionless contact size also based on the square root of the contact area. Thus the constriction parameter becomes

$$\psi \equiv k\sqrt{A_c}R_c \tag{3}$$

where A_c is the area of the contact through which a uniform heat flux is supplied.

For a given geometric configuration the constriction parameter is then a function only of the relative contact size. To compare results of different cases, a dimensionless relative contact size ϵ is defined as the square root of the ratio of contact area to cylinder cross-sectional area, or

$$\epsilon \equiv \sqrt{A_c/A_t} \tag{4}$$

where A_t is the area of the cylinder cross section as shown in Fig. 2.

Constriction Parameters

Case 1: Circle on Circle. For a circular contact area with uniform heat flux on a circular, insulated, semi-infinite, coaxial cylinder the constriction parameter is found analytically to be (Yovanovich, 1976a)

$$\psi_e^{cc} = \frac{16}{\pi\epsilon} \sum_{n=1}^{\infty} \frac{J_1^2(\gamma_n\epsilon)}{\gamma_n^3 J_0^2(\gamma_n)} \tag{5}$$

where the superscript *cc* indicates circle on circle, the subscript *e* indicates an exact solution, $J_0(\cdot)$ and $J_1(\cdot)$ are Bessel functions of the first kind of orders 0 and 1, and γ_n are the positive real roots of $J_1(\gamma_n)=0$.

Equation (5) is not valid for the limiting case of $\epsilon=0$ where (Yovanovich, 1976b)

$$\psi_e^{cc}(\epsilon=0) = 8/3\pi^{3/2} \tag{6}$$

From the work of Roess (1950), an approximate analytical expression for ψ^{cc} can be derived for a small relative contact size ϵ to give

$$\psi_a^{cc} \approx 0.47890 - 0.62446\epsilon + 0.11239\epsilon^3 \tag{7}$$

where the subscript *a* indicates an approximate analytical solution.

Finally, a correlation to four decimal places for $0 \le \epsilon \le 0.9$ has been provided from the accurate optimized image results of Negus and Yovanovich (1984) giving

$$\psi_c^{cc} = 0.47890 - 0.62498\epsilon + 0.11789\epsilon^3 - 0.000071\epsilon^5 + 0.02582\epsilon^7 \tag{8}$$

where the subscript *c* indicates a correlation of results.

Case 2: Circle on Square. For a circular contact with uniform flux on a square insulated semi-infinite cylinder, an exact solution recently derived by Sadhal (1984) gives

$$\psi_e^{cs} = \frac{2}{\pi^2\epsilon}\left\{ \sum_{n=1}^{\infty} \frac{J_1^2(2n\sqrt{\pi}\epsilon)}{n^3} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{J_1^2[2\sqrt{\pi}\epsilon(m^2+n^2)^{1/2}]}{(m^2+n^2)^{3/2}} \right\} \tag{9}$$

where $\epsilon \equiv \sqrt{A_c/A_t} = \sqrt{\pi}a/2b$ for the circle on a square.

Although equation (9) is exact, the convergence of the double-infinite summation is extremely slow, especially for small ϵ. At the limiting case of $\epsilon=0$, a circular contact on a half-space, $\psi_e^{cs}(\epsilon=0) = 8/3\pi^{3/2}$ as given in equation (6).

For efficient calculation of the constriction parameter for the circle on a square configuration, a correlation for $0 \le \epsilon \le 0.8$ exists from the accurate optimized image results of Negus and Yovanovich (1984). Note that $\epsilon > \sqrt{\pi}/2 \approx 0.886$ is impossible for a circle on a square configuration. The correlation is

$$\psi_c^{cs} = 0.47890 - 0.62055\epsilon + 0.11593\epsilon^3 + 0.006688\epsilon^5 + 0.04015\epsilon^7 \tag{10}$$

Negus and Yovanovich (1985) have developed an approximate expression for an elliptical contact on a nominally rectangular flux tube. For the limiting case of a circle on a square, their expression reduces to

$$\psi_a^{cs} \approx 0.47890 - 0.62075\epsilon + 0.1144\epsilon^3 \tag{11}$$

Although it is derived by the assumption of small contact

Table 1 Dimensionless constriction resistance parameters for each contact/cylinder configuration as a function of relative contact size

ϵ	ψ_c^{cc} (circle/circle)	ψ_c^{cs} (circle/square)	ψ_e^{ss} (square/square)
0	0.4789	0.4789	0.4732
0.1	0.4165	0.4170	0.4112
0.2	0.3548	0.3557	0.3500
0.3	0.2946	0.2959	0.2902
0.4	0.2365	0.2382	0.2327
0.5	0.1813	0.1836	0.1782
0.6	0.1301	0.1333	0.1277
0.7	0.0840	0.0887	0.0823
0.8	0.0447	0.0524	0.0437
0.9	0.0147	-	0.0143

size, equation (11) is found to be accurate to 0.5 percent for $0 \le \epsilon \le 0.5$.

Case 3: Square on Square. For a square contact with uniform flux on a square insulated semi-infinite cylinder, an exact solution for the constriction parameter is (Mikic, 1966)

$$\psi_e^{ss} = \frac{2}{\pi^3 \epsilon} \left\{ \sum_{m=1}^{\infty} \frac{\sin^2 m\pi\epsilon}{m^3} + \frac{1}{\pi^2 \epsilon^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin^2(m\pi\epsilon)\sin^2(n\pi\epsilon)}{m^2 n^2 \sqrt{m^2+n^2}} \right\} \qquad (12)$$

As with the circle on square situation, the double-infinite summation in equation (12) converges quite slowly, especially for very small values of ϵ. At the limiting value of $\epsilon = 0$, or a square contact on a half-space, the exact value of the constriction parameter is (Carslaw and Jaeger, 1959),

$$\psi_e^{ss}(\epsilon=0) = \frac{2}{\pi} \left\{ \ln[1+\sqrt{2}] + \frac{1}{3}(1-\sqrt{2}) \right\} \qquad (13)$$

By using the methods of Beck (1979) and Negus and Yovanovich (1985), an approximate analytical expression for the constriction parameter for small ϵ has been developed in this work as

$$\psi_a^{ss} \approx 0.47320 - 0.62075\epsilon + 0.1198\epsilon^3 \qquad (14)$$

This approximate expression is accurate to within 0.3 percent for $0 \le \epsilon \le 0.5$.

Comparison of Constriction Parameters

In Table 1 the values of the constriction resistance parameter, $\psi \equiv k\sqrt{A_c}R_c$, for each contact area/cylinder configuration are reported for relative contact sizes in the range $0 \le \epsilon \equiv \sqrt{A_c/A_t} \le 0.9$. For the square/square results the exact solution given by equation (12) was used. The correlations from optimized image results (equations (8) and (10)) were used to generate the values for the circle/circle and circle/square configurations. Comparison with exact solutions has shown these correlations to be accurate to the digits shown in Table 1.

Overall the constriction resistance parameters reported in Table 1 are very similar for each configuration at a given relative contact size, despite the fact that the actual temperature distributions can be quite different. At the limiting case of $\epsilon = 0$, a contact on a half-space, the constriction parameter of a circular contact is 1.2 percent higher than that of a square. For a relative contact size of $\epsilon = 0.5$, the constriction parameter for the circle/circle configuration differs from the circle/square by only 1.3 percent and the square/square by 1.7 percent. The circle/square and square/square results are 2.9 percent apart. Even at $\epsilon = 0.8$ where the actual temperature distributions are extremely different for the three different configurations, the maximum difference in their dimensionless constriction parameters is only 17 percent when the square root of contact area is chosen as the characteristic dimension. At the limiting case of $\epsilon = 1$, one-dimensional heat flow results, and thus the constriction resistance is zero.

For contact areas on a square cylinder, the approximate analysis of Negus and Yovanovich (1985) allows for some interesting generalizations. The approximate expression for ψ gives

$$\psi \approx \psi_\infty - 0.62075\epsilon + 1.4377 \left(\frac{2I_0 - 3I_{RR}}{A_c^2} \right) \epsilon^3 \qquad (15)$$

where $\psi \equiv k\sqrt{A_c}R_c$, $\epsilon \equiv \sqrt{A_c/A_t}$, ψ_∞ is the constriction parameter for the contact on a half-space, and I_0 and I_{RR} are second moments of area defined by Negus and Yovanovich (1985).

Equation (15) shows that if relative contact size ϵ is very small ($\epsilon << \epsilon^3$), the constriction parameter is given by the half-space result modified linearly by the relative contact size. Physically this linear term in ϵ represents the small perturbation to the temperature fields that is brought about by the placement of the adiabatic walls of the square cylinder in the vicinity of the contact area previously on a half-space. The next term in equation (15) is of order ϵ^3 and is a function of the contact shape. If ϵ is very large ($\epsilon \ge 0.6$) higher order terms in ϵ are required that cannot be easily evaluated by the approximate method of Negus and Yovanovich (1985). Note however that the two-term approximation $\psi \approx \psi_\infty - 0.62075\epsilon$ is accurate for engineering purposes to an ϵ of 0.3 where $\psi(\epsilon=0.3)/\psi_\infty \approx 0.61$ and the three-term version (equation (15)) is accurate to an ϵ of 0.6 where $\psi(\epsilon=0.6)/\psi_\infty \approx 0.27$.

Finally, simple observation of Table 1 leads to an engineering approximation for all three cases given as

$$\psi \approx 0.475 - 0.62\epsilon + 0.13\epsilon^3 \qquad (16)$$

where the maximum error with respect to Table 1 is less than 2 percent for $0 \le \epsilon \le 0.5$ and 4 percent for $0 \le \epsilon \le 0.7$. The success of this simple formula over a fairly wide range in ϵ further reinforces the assertion made here that the optimum characteristic length dimension for problems of this type is the square root of the contact area.

Conclusions

From the observations made with three configurations of contact areas on insulated semi-infinite cylinders, nondimensionalization of constriction resistance by the square root of the contact area produces strikingly similar results for all configurations at any given relative contact size. Although analytical justification is not presently available as with the contact on a half-space, it seems apparent from the results obtained that the optimum characteristic length that describes conduction problems of this type is the square root of the contact area.

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imilarity Solution for Non-Darcy Free Convection om a Nonisothermal Curved Surface in a Fluid-Saturated Porous Medium

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Nomenclature

- c = empirical constant associated with porous inertia
- f = dimensionless stream function
- g = acceleration due to gravity
- Gr = modified Grashof number, see equation (12*a*)
- K = permeability
- m = exponent associated with body shape
- n = exponent associated with wall temperature
- Nu_x = local Nusselt number
- r = function representing body shape
- Ra_x = local Rayleigh number, see equation (8*b*)
- T = temperature
- u, v = Darcian velocity components
- x, y = boundary layer coordinates
- z = vertical distance measured from the lower stagnation point
- α = equivalent thermal diffusivity
- β = coefficient of thermal expansion
- η = similarity variable, see equation (7*c*)
- θ = dimensionless temperature
- ν = kinematic viscosity
- ψ = stream function

Subscripts

- e = boundary layer edge
- r = reference
- w = wall

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Introduction

Considerable attention has been directed to heat and fluid flow within fluid-saturated porous media because of its importance in geophysical and engineering applications such as geothermal energy conversion, thermal insulation of buildings, and packed-bed reactors. (Cheng, 1978). In most previous studies, either on free (e.g., Cheng and Minkowycz, 1977; Merkin, 1979; Nakayama and Koyama, 1987) or combined (e.g., Cheng, 1977; Minkowycz et al., 1985; Nakayama and Koyama, 1987) convection, boundary layer treatments based on Darcy's law were employed. It is, however, well known that the non-Darcy flow situation prevails when the Reynolds number based on the pore diameter and characteristic velocity becomes large (Forchheimer, 1901; Bear, 1972). Fand et al. (1986) experimentally showed deviations from the Darcy law. Forchheimer (1901) proposed a velocity square term in addition to the Darcy term to account for the inertia effects on the pressure drop, as the fluid makes its way through the porous media. This pioneering work was followed by many proposals for mathematically describing non-Darcy flows (e.g., Ergun, 1952; Ward, 1969).

An attempt to obtain a similarity solution for non-Darcian free convective flow over a vertical flat plate was first made by Plumb and Huenefeld (1981) using the model proposed by Ergun (1952). The same model was employed by Vasantha et al. (1986) for a vertical frustum of a cone and by Lai and Kulacki (1981) for a horizontal flat surface to investigate combined effects of the Darcy term and the inertia term. The limiting condition where the Darcy term is negligible, namely Forchheimer flow, was treated by Bejan and Poulikakos (1984) and Ingham (1986). So far, only a limited number of similarity solutions have been reported for simple flow configurations.

In this study, we shall investigate non-Darcy free convective flows using the Ergun model. It will be shown that there exists a certain family of body shape geometries and corresponding wall temperature distributions, which permit similarity solutions. The effects of inertia and geometric shape on the velocity and temperature fields are investigated and the corresponding heat transfer characteristics are discussed in detail.

Governing Equations and Transformation

Figure 1 depicts the physical model and its boundary layer coordinates (x, y). The coordinate z is the vertical distance measured from the lower stagnation point. The body under consideration is two dimensional and its geometry is described by r as a function of x (or z). The body surface may be nonisothermal, and its temperature $T_w(x)$ exceeds the ambient temperature T_e everywhere. Thus, there is an upward convective fluid movement as a result of buoyancy force.

The governing equations for non-Darcy free convective flow, namely, the continuity equation, the Forchheimer equation with the Boussinesq approximation, and the energy equation can be written by exploiting the usual boundary layer approximations as